Geometry of double resonances of 4D symplectic maps and diffusive properties.

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The map

In this presentation we consider the 4D Froeschlé-like symplectic family of maps:

$$T_\delta : \begin{pmatrix} \psi_1 \\ \psi_2 \\ J_1 \\ J_2 \end{pmatrix} \to \begin{pmatrix} \bar{\psi_1} \\ \bar{\psi_2} \\ \bar{J_1} \\ \bar{J_2} \end{pmatrix} = \begin{pmatrix} \psi_1 + \delta(\bar{J_1} + a_2 \bar{J_2}) \\ \psi_2 + \delta(a_2 \bar{J_1} + a_3 \bar{J_2}) \\ J_1 - \delta \sin(\psi_1) \\ J_2 - \delta \epsilon \sin(\psi_2) \end{pmatrix}$$

where $a_2, a_3, \epsilon, \delta$ are real parameters.

Note that it is related to the time-δ map of the 2-dof Hamiltonian

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$

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2. Some geometrical aspects of the Hamiltonian H: normally hyperbolic invariant cylinders (NHICs), transversality of the invariant manifolds of the NHIC.
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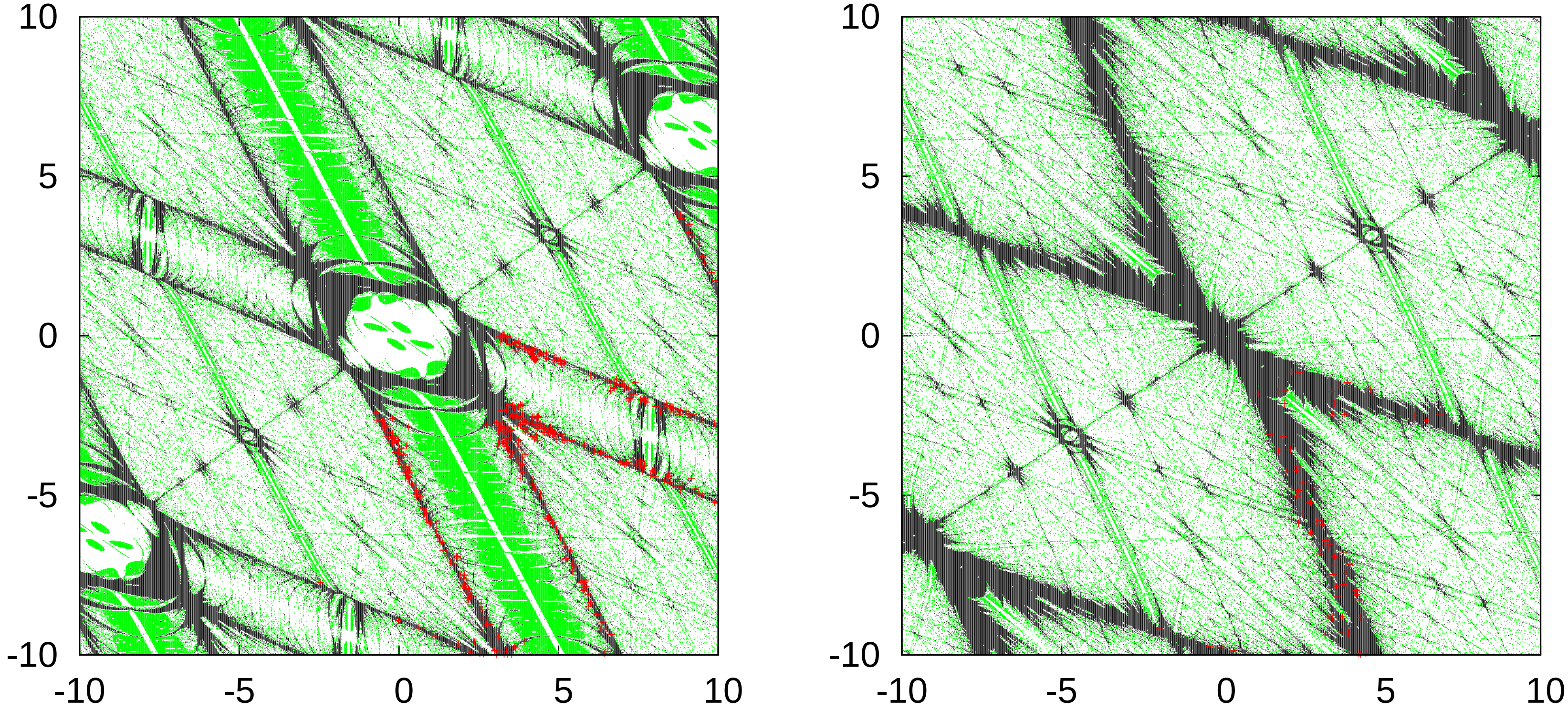
4. Diffusion in phase space: some numerical simulations for T_δ.

Why this map T_δ?
Some comments on the derivation

Action space: double resonances

Diffusion along phase space takes place basically along single resonances but multiple resonances play a key role in an explanation of the Arnold diffusion (e.g. Nekhoroshev theory – upper bounds on the rate of diffusion).



$\delta = 0.5$, $\epsilon = 0.5$, $a_2 = 0.5$ and $a_3 = 1.25$. Lyap. exp. (megno): **black** $\rightarrow$ chaotic, green $\rightarrow$ weakly chaotic, white $\rightarrow$ regular. Red: Iterates of the point $(0, 0, 4.5, -5.25)$ in a slice of width 5×10^{-3} around $\psi_1 = \psi_2 = 0$ (left plot) and $\psi_1 = \psi_2 = \pi$ (right plot). Total number of iterates=10^{12}.

Doubly resonant elliptic fixed points

Our goal is to study the role of double resonances in the (Arnold) diffusion.
The 4D symplectic map is derived as a model for the dynamics at a double resonance unfolding from a totally elliptic fixed point.

Let $F_{\boldsymbol{\delta}}$ be a 2-parameter family of analytic symplectic 4D maps,
$\boldsymbol{\delta} = (\delta_1, \delta_2) \in \mathbb{R}^2$ small enough (unfolding) parameter.
such that:
$F_{\boldsymbol{\delta}}(\mathbf{0}) = \mathbf{0}$ totally elliptic fixed point (for all $\boldsymbol{\delta}$),
Spec$(DF_{\boldsymbol{\delta}})(\mathbf{0}) = \{\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $\alpha_k \in (0, 1/2)$,
$k = 1, 2$. Assume also that the eigenvalues are simple, i.e., $\alpha_1 \neq \alpha_2$.

Concretely, we shall consider

$$\alpha_j = p_j/q_j + \delta_j, \qquad p_1, p_2, q_1, q_2 \in \mathbb{N}$$

meaning that we unfold a doubly resonant fixed point.

Birkhoff NF and the set of resonances

The local dynamics of $F_{\boldsymbol{\delta}}$ around $\mathbf{0} \in \mathbb{R}^4$ can be described by adding the effect of the unfolding to the Birkhoff NF of $F_{\mathbf{0}}$.

The Birkhoff NF structure is determined by the set of resonances

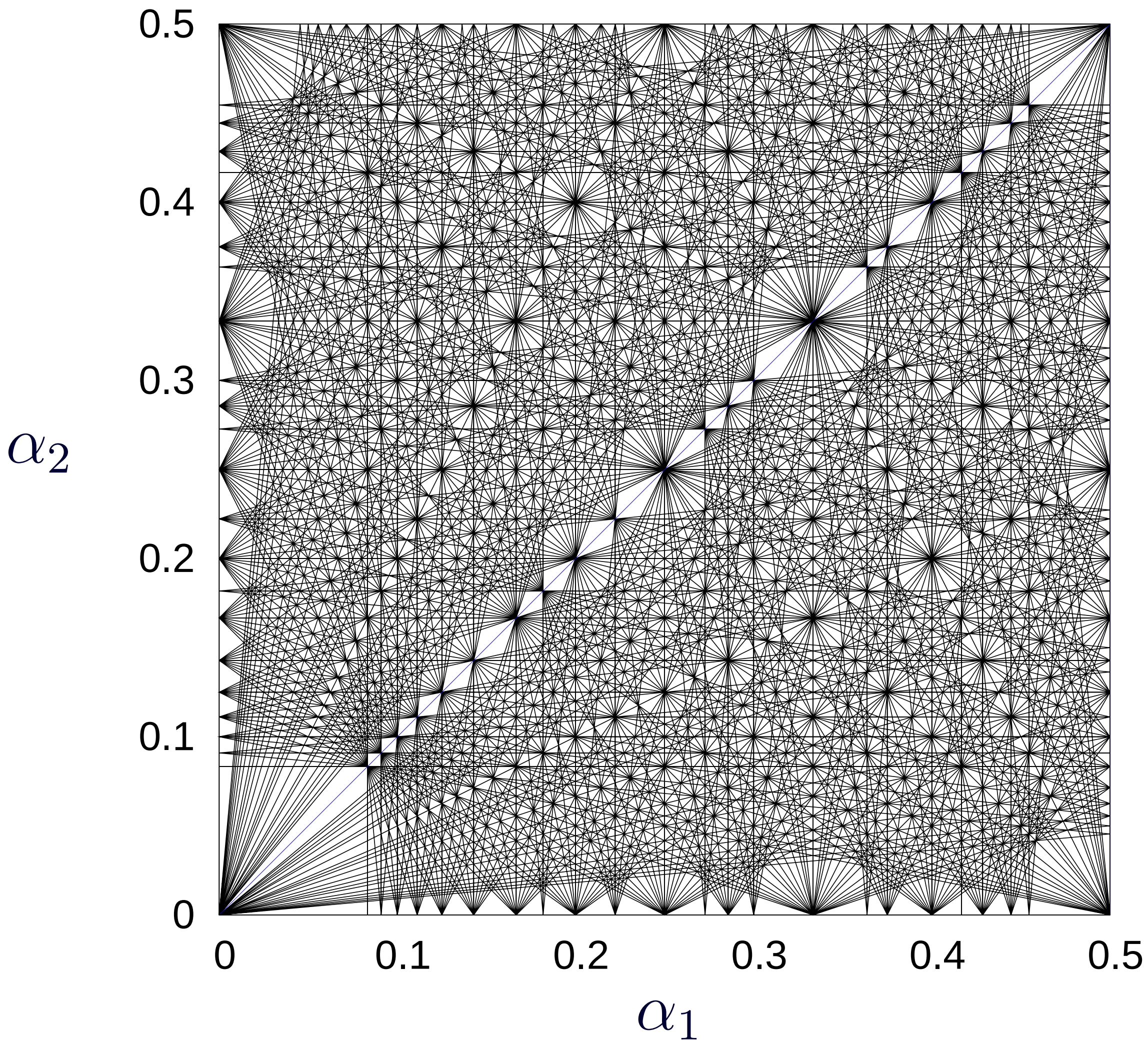
$$\Gamma = \{\, (k_1, k_2) : k_1\alpha_1 + k_2\alpha_2 = 0 \pmod 1 \,\} \subset \mathbb{Z}^2 \,.$$

→ $(0, 0)$ *trivial (or unavoidable)* resonance.

→ $\mathbf{r} = (k_1, k_2) \in \Gamma$ is *a resonance of order* $|\mathbf{r}| = |k_1| + |k_2|$.

The fixed point is doubly resonant if $\delta_1 = \delta_2 = 0$. Then Γ is a two-dimensional lattice.

Frequency space: Resonant lines with $r \leq 12$



Takens NF

$F_{\boldsymbol{\delta}}$ symplectic 4D maps ($\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$), $\boldsymbol{\delta} \in \mathbb{R}^2$ small enough, $F_{\boldsymbol{\delta}}(\mathbf{0}) = \mathbf{0}$, Spec=$\{\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $k = 1, 2$.

$$\alpha_1 \neq \alpha_2 \Longrightarrow DF_{\mathbf{0}}(0) \sim \Lambda_0 = \begin{pmatrix} R_{2\pi\alpha_1} & 0 \\ 0 & R_{2\pi\alpha_2} \end{pmatrix}$$

A canonical change of variables reduces $F_{\boldsymbol{\delta}}$ to BNF $N_{\boldsymbol{\delta}}$:

$$N_{\boldsymbol{\delta}} \circ \Lambda_{\mathbf{0}} = \Lambda_{\mathbf{0}} N_{\boldsymbol{\delta}}.$$

Since $DN_{\mathbf{0}}(\mathbf{0}) = \Lambda_{\mathbf{0}}$ the map $\Lambda_{\mathbf{0}}^{-1} N_{\boldsymbol{\delta}}$ is tangent to the identity

$\Longrightarrow$ it can be formally interpolated (in a compact domain around $\mathbf{0}$) by a (Hamiltonian) vector field:

$$N_{\boldsymbol{\delta}} = \Lambda_{\mathbf{0}} \Phi^1_{H_{\boldsymbol{\delta}}} + \text{ exp. small error}$$

Interpolating Hamiltonian

Moreover $H_{\boldsymbol{\delta}}$ is $\Lambda_{\mathbf{0}}$-invariant ($H_{\boldsymbol{\delta}} = H_{\boldsymbol{\delta}} \circ \Lambda_{\mathbf{0}}$) $\Longrightarrow N_{\boldsymbol{\delta}}^j = \Lambda_{\mathbf{0}}^j \Phi_{H_{\boldsymbol{\delta}}}^j$ for all $j \in \mathbb{N}$

$\Longrightarrow$ study the flow of $H_{\boldsymbol{\delta}}$ instead of iterations of $N_{\boldsymbol{\delta}}$.

To obtain $H_{\boldsymbol{\delta}}$:

→ Complex vbles ($z_k = x_k + iy_k,\ \bar{z}_k = x_k - iy_k$), $\Lambda_{\mathbf{0}} = \mathrm{diag}(\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2)$.

→ $z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$ resonant $\Longleftrightarrow \Lambda_{\mathbf{0}}$-invariant $\Longleftrightarrow (j-k, l-m) \in \Gamma$.

Then $H_{\boldsymbol{\delta}}$ is a sum of res. monomials: $H_{\boldsymbol{\delta}} = \sum_{\substack{(j-k,l-m)\in\Gamma \\ j,k,l,m\geq 0}} h_{jklm}(\boldsymbol{\delta}) z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$

In Poincaré vbles ($I_j = \frac{|z_j|^2}{2},\ \varphi_j = \arg z_j$):

$$H_{\boldsymbol{\delta}} = \sum_{\substack{(k_1,k_2)\in\Gamma \\ p,q\geq 0}} a_{k_1 k_2 pq}(\boldsymbol{\delta}) I_1^{p+k_1/2} I_2^{q+|k_2|/2} \cos(k_1\varphi_1 + k_2\varphi_2 + b_{k_1 k_2 pq})$$

Q: Dominant terms of $H_{\boldsymbol{\delta}}$? Arithmetic properties of Γ depending on (α_1, α_2).

Minimal generators of Γ: primary resonances

Recall that Γ is a 2-dimensional lattice.

Consider $\mathbf{r}_0 \in \Gamma$ a smallest non-trivial element, and $\mathbf{r}_1 \in \Gamma$ any of the smallest elements independent from $\mathbf{r}_0$
$\Longrightarrow \mathbf{r}_0$ and $\mathbf{r}_1$ generate Γ (provided $\alpha_1 \neq \alpha_2$).

If $|\mathbf{r}_0| \geq 5$ the fixed point is called weakly resonant and otherwise it is strongly resonant.

We consider the **double weakly resonant case** $5 \leq |\mathbf{r}_0| < |\mathbf{r}_1|$. Up to order $|\mathbf{r}_0| - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Moreover, up to order $|\mathbf{r}_1| - 1$ is also integrable.

Weak double resonances: a truncated model

Recall: $F_{\boldsymbol{\delta}}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $\alpha_k = p_k/q_k + \delta_k$ for $k = 1, 2$, $\delta = \|\boldsymbol{\delta}\|$ small.

Takens NF: $H_{\boldsymbol{\delta}} = \sum_{(k_1,k_2)\in\Gamma} a_{k_1k_2pq}(\boldsymbol{\delta}) I_1^{p+k_1/2} I_2^{q+|k_2|/2} \cos(k_1\varphi_1 + k_2\varphi_2 + b_{k_1k_2pq})$

Let $\mathbf{r_0} = (k_1, k_2)$ and $\mathbf{r_1} = (m_1, m_2)$ be minimal generators of Γ.

Adapting variables $\psi_1 = k_1\varphi_1 + k_2\varphi_2, \quad \psi_2 = m_1\varphi_1 + m_2\varphi_2,$

$$I_1 = k_1 J_1 + m_1 J_2, \quad I_2 = k_2 J_1 + m_2 J_2$$

to the double resonance (this is a symplectic change) one gets

$$\boxed{H_{\boldsymbol{\delta}} = H_0(\boldsymbol{J}, \boldsymbol{\delta}) + H_1(\boldsymbol{J}, \psi_1, \boldsymbol{\delta}) + H_2(J_1, J_2, \psi_1, \psi_2, \boldsymbol{\delta}) + \mathcal{O}_{|\mathbf{r}_1|+1}(\boldsymbol{z})}$$

$$\begin{aligned} H_0 &= A_{00}(J_1, J_2, \boldsymbol{\delta}), \\ H_1 &= \sum_{l_1=1}^{[|\mathbf{r}_1|/|\mathbf{r}_0|]} I_1^{l_1|k_1|/2} I_2^{l_1|k_2|/2} A_{l_10}(J_1, J_2, \boldsymbol{\delta}) \cos\big(l_1\psi_1 + B_{l_10}(J_1, J_2, \boldsymbol{\delta})\big), \\ H_2 &= I_1^{|m_1|/2} I_2^{|m_2|/2} A_{01}(0, 0, \boldsymbol{\delta}) \cos\big(\psi_2 + B_{01}(0, 0, \boldsymbol{\delta})\big). \end{aligned}$$

Localizing around the double resonance

In a neighbourhood of the origin

$$H_0 = c_1\delta J_1 + c_2\delta J_2 + a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta^5)$$

→ inv. $\mathbb{T}^2$ at $J_1 = \delta r_1$, $J_2 = \delta r_2 \Rightarrow$ inv. $\mathbb{T}^2$ for the NF system if $I_1, I_2 > 0$.

Then $J_k = \delta r_k + \delta^{|\mathbf{r}_0|/4}\tilde{J}_k$ and $H = \delta^{|\mathbf{r}_0|/2}\tilde{H}$ gives

$$\begin{aligned}
H_0(J_1, J_2, \boldsymbol{\delta}) &= a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta^{|\mathbf{r}_0|/4}) \,, \\
H_1(J_1, J_2, \psi_1, \boldsymbol{\delta}) &= \sum_{l_1=1}^{[|\mathbf{r}_1|/|\mathbf{r}_0|]} \delta^{(l_1-1)|\mathbf{r}_0|/2} \tilde{A}_{l_1 0}(J_1, J_2, \boldsymbol{\delta}) \cos\big(l_1\psi_1 + \tilde{B}_{l_1 0}(J_1, J_2, \boldsymbol{\delta})\big) \,, \\
H_2(J_1, J_2, \psi_1, \psi_2, \boldsymbol{\delta}) &= \delta^{(|\mathbf{r}_1|-|\mathbf{r}_0|)/2} a_{01} \cos\big(\psi_2 + b_{01}\big) \,.
\end{aligned}$$

Furthermore, if $|\mathbf{r}_1| < 2|\mathbf{r}_0|$ (different but similar order resonances) then

$$H_1(J_1, J_2, \psi_1, \boldsymbol{\delta}) = \left(a_{10} + \delta^{|\mathbf{r}_0|/4-1}\hat{A}_{10}(J_1, J_2, \boldsymbol{\delta})\right) \cos\psi_1$$

$\longrightarrow$ No other harmonics in H_1 appear!

Some geometrical aspects of the Hamiltonian H:
normally hyperbolic invariant cylinders, transversality of the invariant manifolds of the NHIC.

Analysis of the truncated Hamiltonian model

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$

→ For the moment $\epsilon \sim \delta^{(|\mathbf{r}_1| - |\mathbf{r}_0|)/2}$ will be considered as a small parameter.

→ 4 fixed points: Let $d = a_3 - a_2^2$. If $\nu = \epsilon d > 0$ and $|\epsilon|$ small enough

$p_1 = (0, 0, 0, 0)$ – EE, $p_2 = (0, \pi, 0, 0)$ – EH

$p_3 = (\pi, 0, 0, 0)$ – HE, $p_4 = (\pi, \pi, 0, 0)$ – HH

→ For $0 < |\epsilon| << 1$ the system has two normally hyperbolic cylinders.

The “main” NHIC

For $\epsilon = 0$:

- The cylinder $\Pi_1 = \{\psi_1 = \pi \pmod{2\pi}, J_1 + a_2 J_2 = 0\}$ is a 2D NHIM.
- It is foliated by periodic orbits $C_h = \Pi_1 \cap \{H = h\}$.
- $W^{u/s}(\Pi_1)$ are 3D and $W^u(\Pi_1) = W^s(\Pi_1)$.

By Fenichel's theory (normal hyperbolicity) it persists for $0 < \epsilon << 1$.

- There exists $\Pi_{1,\epsilon}$ $\mathcal{O}(\epsilon)$-close to Π_1.
- $\exists\, W^{u/s}(\Pi_{1,\epsilon})$ $\mathcal{O}(\epsilon)$-close to $W^{u/s}(\Pi_1)$.
- For a_2, a_3 fixed, it can be proved that: [a]
  1. $W^u(\Pi_{1,\epsilon})$ intersects $W^s(\Pi_{1,\epsilon})$ along two lines of homoclinic points.
  2. The intersection is transversal (except for some discrete values of J_2).

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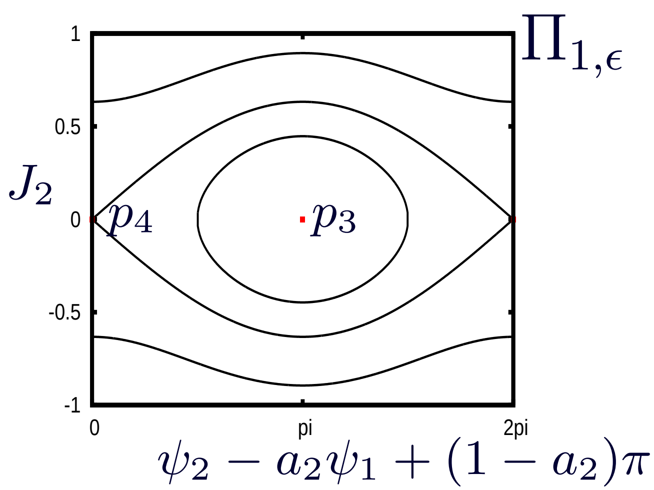
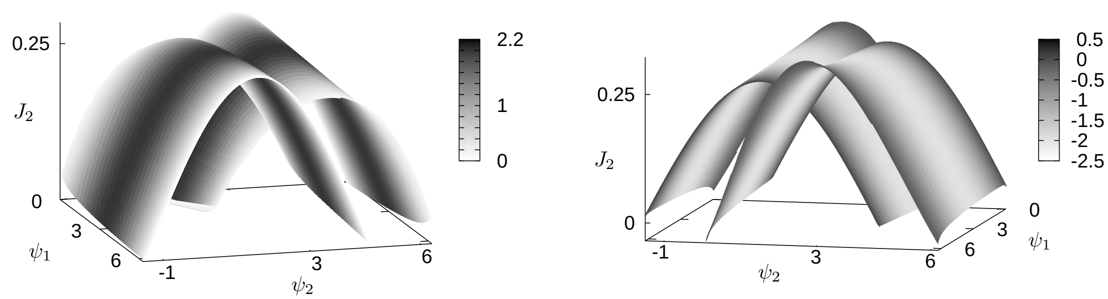
[a] V.Gelfreich, C.Simó and AV, *Dynamics of 4D symplectic maps near a double resonance*, Physica D 243(1), 2013.

Transversality inside the level of energy

Inside the level of energy $\{H = h\}$ the invariant manifolds of the periodic orbit $\Pi_{1,\epsilon} \cap \{H = h\}$ intersect transversally $\Leftrightarrow h \neq 1 \pm \epsilon \cos \pi a_2 + \mathcal{O}(\epsilon^2)$. The angle is $\mathcal{O}(\sqrt{\epsilon})$.



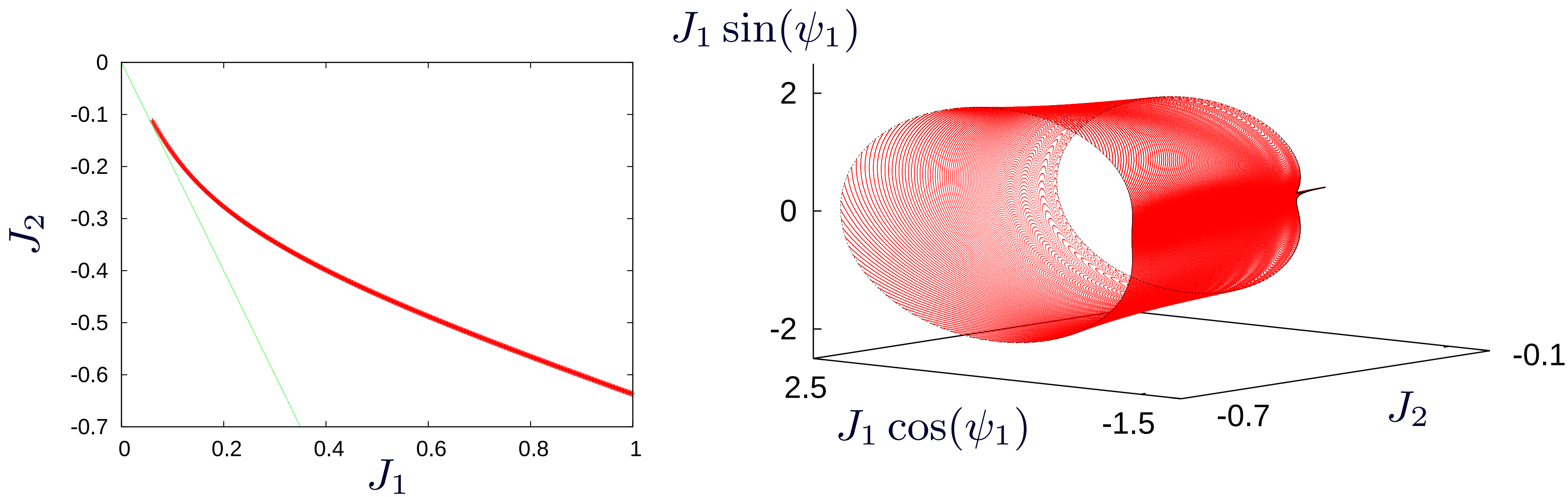


$\epsilon = 0.1$, $a_2 = 0.25$, $d = 0.5$ and $h = 1 + \epsilon$

Remark: The cylinder $\Pi_{1,\epsilon}$ is not analytic. One expects $\Pi_{1,\epsilon} \in \mathcal{C}^r$, being $r = \tilde{\lambda}_1/\tilde{\lambda}_2$ the quotient of the normal and tangent maximal Lyapunov exponents. In this case, they are determined by the unstable eigenvalues of the HH fixed point, and $r = \mathcal{O}(1/\sqrt{\epsilon})$.

The 2nd NHIC Π_2

Another NHIC Π_2 exists for $0 < \epsilon << 1$: if $\dot{\psi}_2 = a_2 J_1 + a_3 J_2 \approx 0$ then ψ_2 becomes a slow variable. If $\dot{\psi}_1 = J_1 + a_2 J_2$ is large enough, then one can apply an averaging step to remove the dependence on ψ_1. For a fixed J_1 value, the averaged Hamiltonian has a saddle (and ψ_1 is cyclic). [a]



Left: Intersection of Π_2 with $\psi_1 = \psi_2 = \pi$. Green line: $J_1 + a_2 J_2 = 0$.
Right: 3D representation of the periodic orbits of Π_2. Parameters: $\epsilon = 0.05$, $a_2 = 0.5$, $a_3 = 1.25$

[a] V. Kaloshin and K. Zhang, *Arnold diffusion for smooth convex systems of two and a half degrees of freedom*, Nonlinearity 28, 2015.

Some geometrical aspects of the map T_δ: splitting of the separatrices of the hyperbolic-hyperbolic (HH) fixed point.

A discrete model for double resonances

Truncated NF: $H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$

$$T_\delta : \begin{pmatrix} \psi_1 \\ \psi_2 \\ J_1 \\ J_2 \end{pmatrix} \to \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{J}_1 \\ \bar{J}_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2) \\ \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2) \\ J_1 - \delta \sin(\psi_1) \\ J_2 - \delta \epsilon \sin(\psi_2) \end{pmatrix}$$

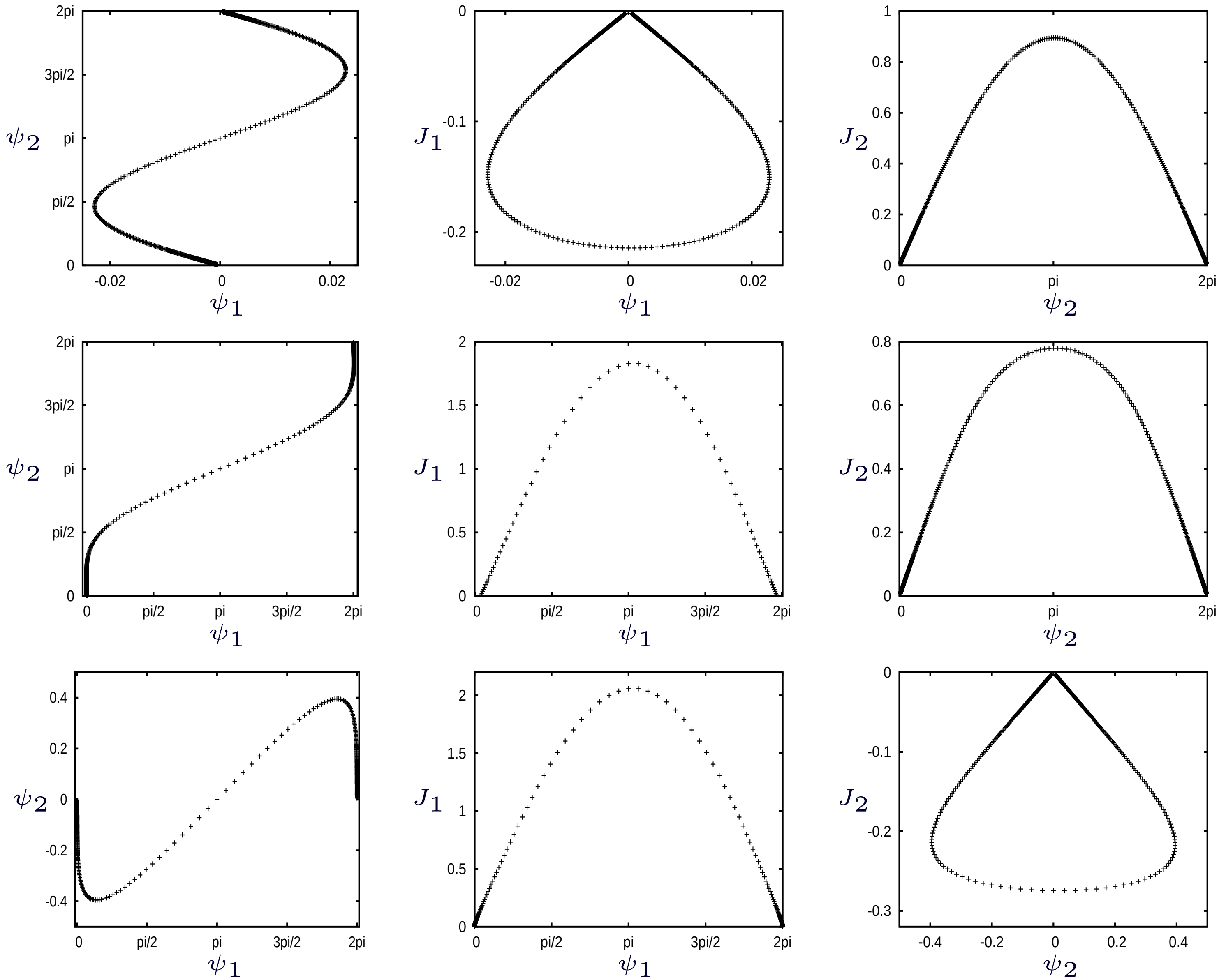
Phase space structure similar to H (but the homoclinic trajectories split!):

- 4 fixed points: HH, HE, EH, EE.
- NHICs $\Pi_{1,2}$ for H, that depend on ϵ, persist for $|\delta| << 1$.

  (but the discrete dynamics inside the NHICs resembles the Chirikov standard map dynamics).
- Reversible: $R_1 = (-\psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2)$,
  $R_2 = (2\pi - \psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2)$ and $R_3 = (2\pi - \psi_1, -\psi_2, \bar{J}_1, \bar{J}_2)$.

Trajectories homoclinic to the HH fixed point

Reversibilities $\Rightarrow T$ has 6 primary homoclinic trajectories.



"Pendulum" separatrix in Π_1

$\delta = 0.1$

$\epsilon = 0.1$

$a_2 = \frac{1}{4}$

$a_3 = \frac{9}{16}$

Splitting of 2D invariant manifolds of the HH f.p.

Let p_h be the homoclinic point on Σ_{R_k}, $k = 1, 2, 3$. We compute the volume of a $4D$ parallelotope defined by two pairs of vectors tangent to W^u and W^s:
$G(s_1, s_2)$ - the (local) parameterisation.

1. Consider the vectors (tangent to W^u):

$$\tilde{v}_1 = (\partial G/\partial s_1)(s_1^h, s_2^h), \quad \tilde{v}_2 = (\partial G/\partial s_2)(s_1^h, s_2^h).$$

2. Transport these vectors under T to p_h and consider, by the reversibility,

$$\tilde{v}_3 = R_k(\tilde{v}_1^{p_h}), \quad \tilde{v}_4 = R_k(\tilde{v}_2^{p_h}).$$

3. Finally, normalize them $v_j = \tilde{v}_j^{p_h} / \|\tilde{v}_j^{p_h}\|, \ j = 1, \ldots, 4$ and define

$$\boxed{V = \text{det}(v_1, v_2, v_3, v_4)}$$

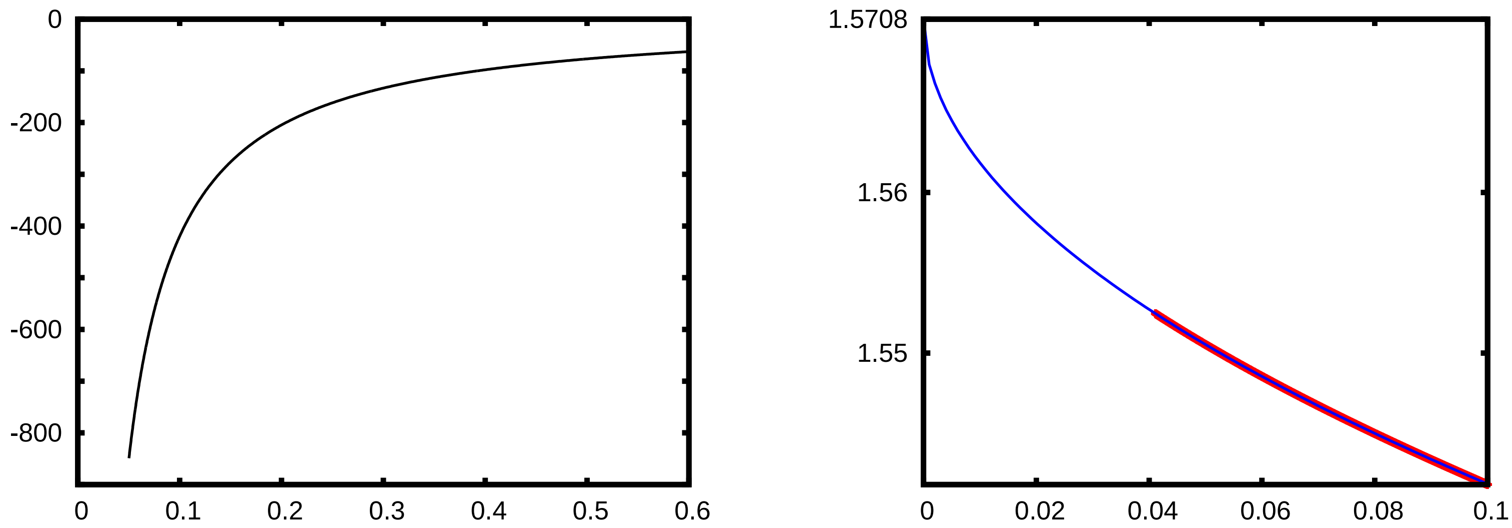
Asymptotic behaviour of V in Σ_{R_1}

For a fixed ϵ, a_2 and a_3 parameters we study the behaviour as $\delta \to 0$.

At p_h in $\Sigma_{R_1} = \text{Fix}(R_1)$ (homoclinic trajectory on Π_1, which depends on δ):

$$V \sim A\mu_2^B \mathrm{e}^{-2\pi\,\mathrm{Im}\,\hat{\tau}_2/\mu_2},$$

where $\mu_2 = \log \tilde{\lambda}_2$, $A, B \in \mathbb{R}$ and $\hat{\tau}_2 = i\pi/2 + \mathcal{O}(\sqrt{\epsilon})$.



$\epsilon = 0.1$, $a_2 = 0.25$, $a_3 = 0.5625$. Left: $\log V$ vs. δ. Right: $\mathrm{Im}\,\hat{\tau}_2$ vs. ϵ.

→ Similar for p_h in $\Sigma_{R_k} = \text{Fix}(R_k)$, $k=2,3$ (with $\hat{\tau}_k$ from the corresp. limit vector field).

Diffusion in phase space: some numerical simulations for T_δ.

Different time-scales

Problem:

To measure the (Arnold) diffusion we need to separate the slowest variable, i.e. the one which evolves in the largest time-scale that it is expected to be exponentially large in δ, from the other faster variables and measure its variation.

An option could be to transform (in different points of the phase space!) the system to a NF up to a suitable order to obtain an approximating interpolating Hamiltonian. $\leftarrow$ **Too expensive!**

We construct a suitable observable $E(x)$, an "energy" of $x \in \mathbb{R}^4$, as follows.

The "energy" $E(x)$

Consider $x_B \in \mathbb{R}^4$ fixed and $\gamma(s) = x_B + sv$, $v = x - x_B$, $s \in [0, 1]$, then define

$$E(x) = \int_0^1 \nabla E(\gamma(s))v\, ds$$

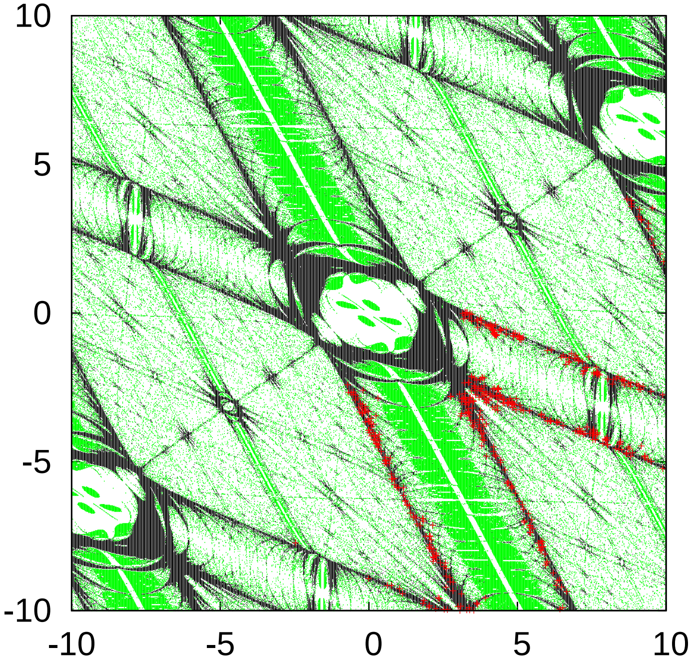
To evaluate $\nabla E(x_0)$:

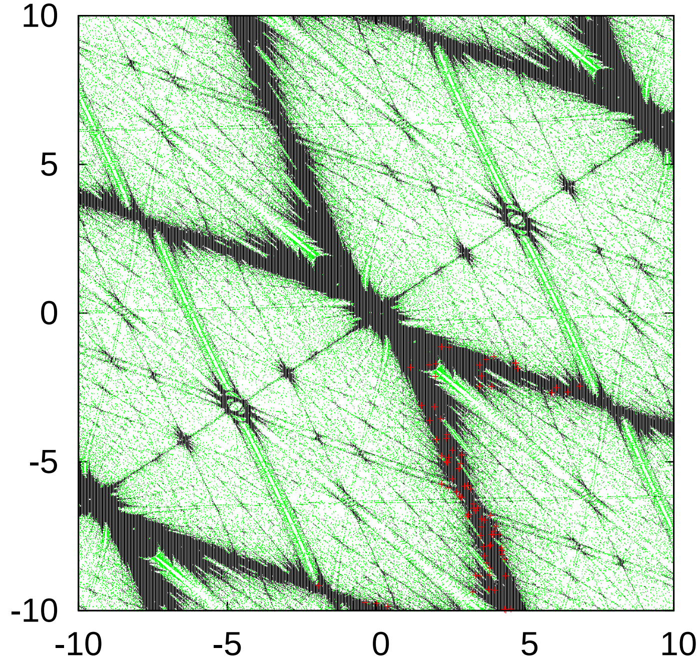
1. Fix N and consider $(k\delta, T_\delta^k(x_0))$, $-N \leq k \leq N$.
2. Construct the interpolating polynomial $p_N(t)$ of degree $2N$.
3. Then $X_N : x_0 \mapsto p'_N(0)$ defines a vector field.
4. Since T_δ is symplectic we require X_N to be Hamiltonian with energy E. Then $q' = \partial E/\partial p$, $p' = -\partial E/\partial q$ implies $\nabla E = (-p', q')$, where $x = (q, p) = (\psi_1, \psi_2, J_1, J_2)$.

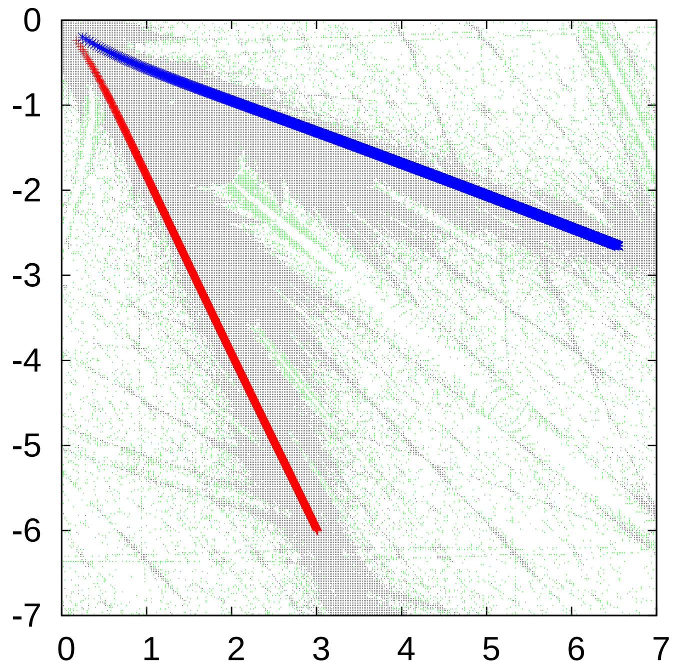
$E(x)$ remains almost constant for large enough number of iterates of F_δ

It varies roughly $\mathcal{O}(10^{-6})$ in $10^5/10^6$ iterates for the parameters used in simulations.

Transition between two resonances







Why iterates turn to the right?

Do they reach the HH fixed point?

Bottom left plot:

The NHICs Π_1 and Π_2 for T_δ

$\delta = 0.5$, $\epsilon = 0.5$, $a_2 = 0.5$ and $a_3 = 1.25$.

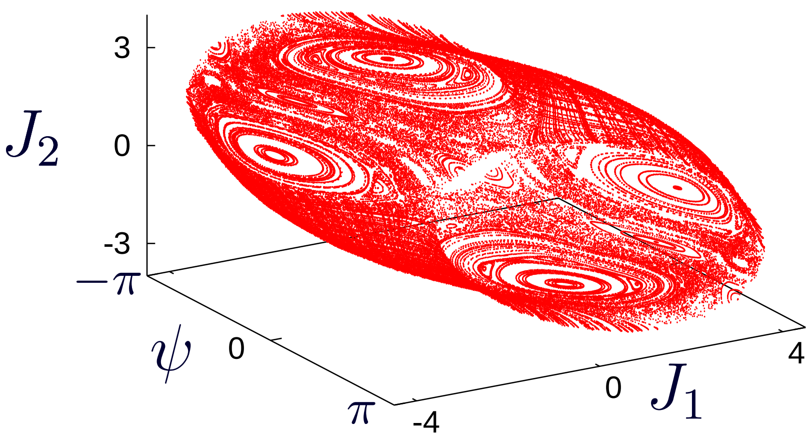
A suitable "2D Poincaré map"

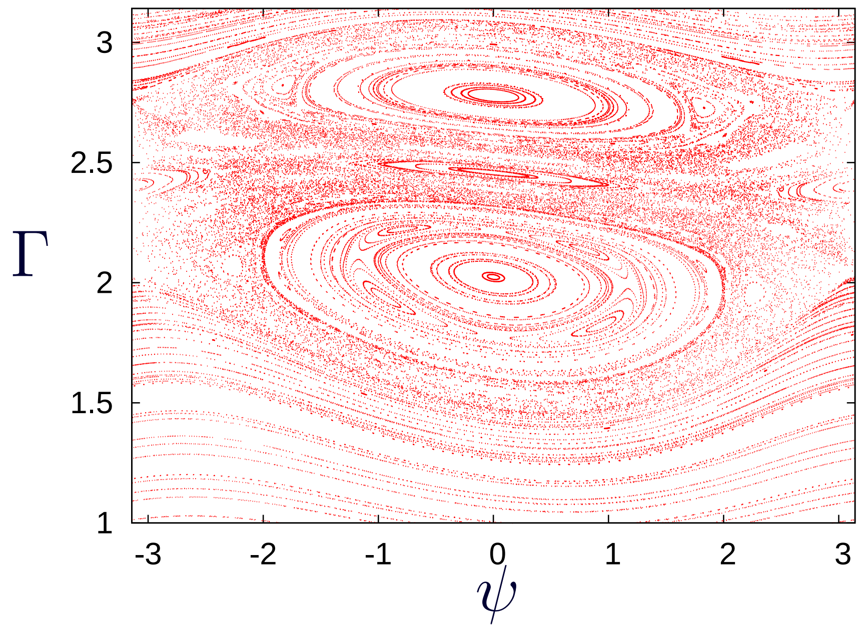
Consider the 3D hyperplane $\Sigma = \{\psi_1 = \psi_2\}$.

At least $m \approx 10^5$ iterates of T_δ stay (aprox.) on $\{E = h\}$.

Then $\Sigma_T = \Sigma \cap \{E = h\}$ defines a "Poincaré section" for T_δ ($< m$ iterates).

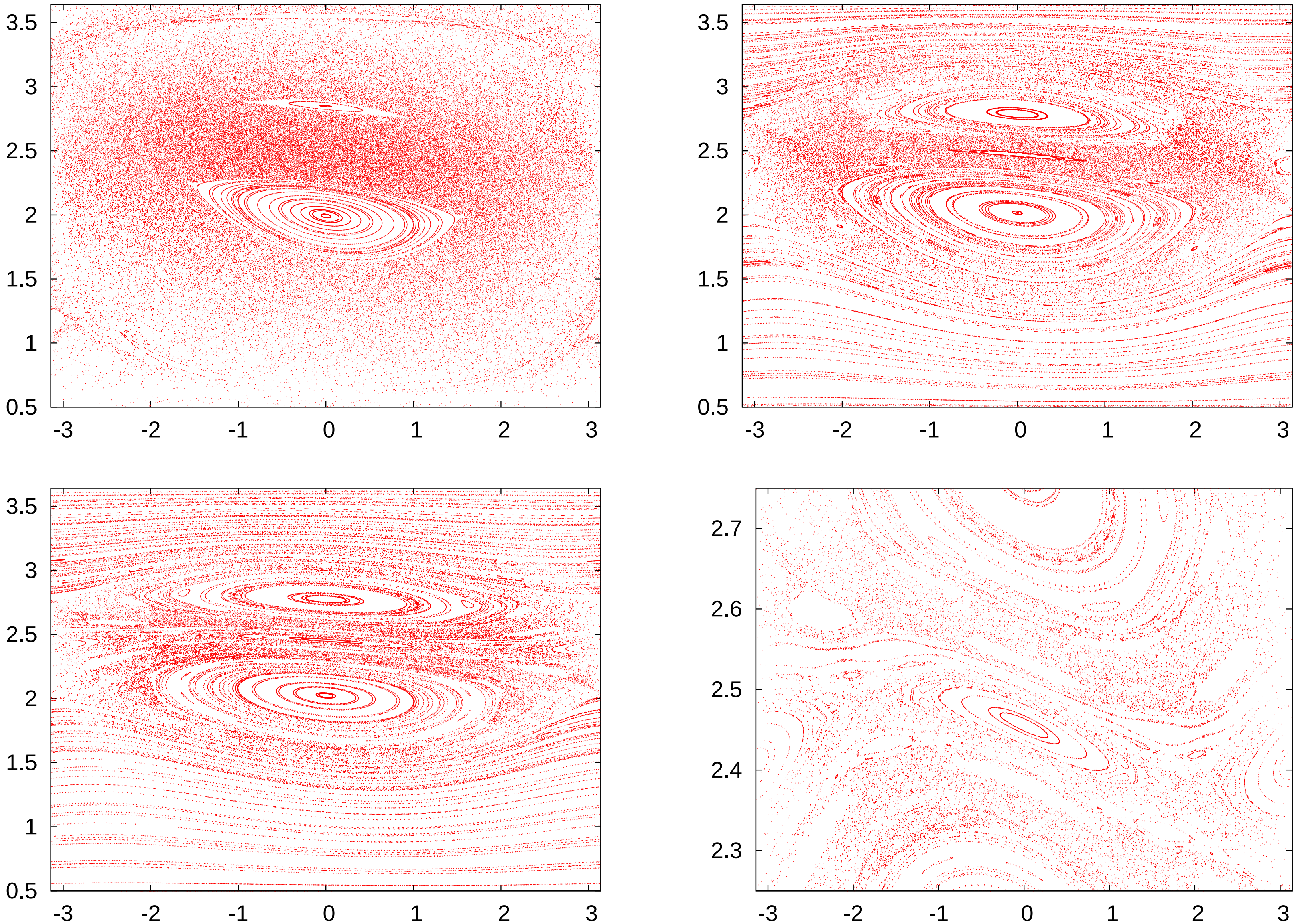
Iterates will jump the section! $\Rightarrow$ Use the vector field X_N to project on Σ_T.





We represent 500 iterates of the map $P_T : \Sigma_T \to \Sigma_T$ for 400 different initial points in $\Sigma \cap \{E = h\}$, with $h = 5$ ($x_B = HH$). $\Gamma = \arg(J_1 + iJ_2)$.

Different “energy levels”: “last ric”



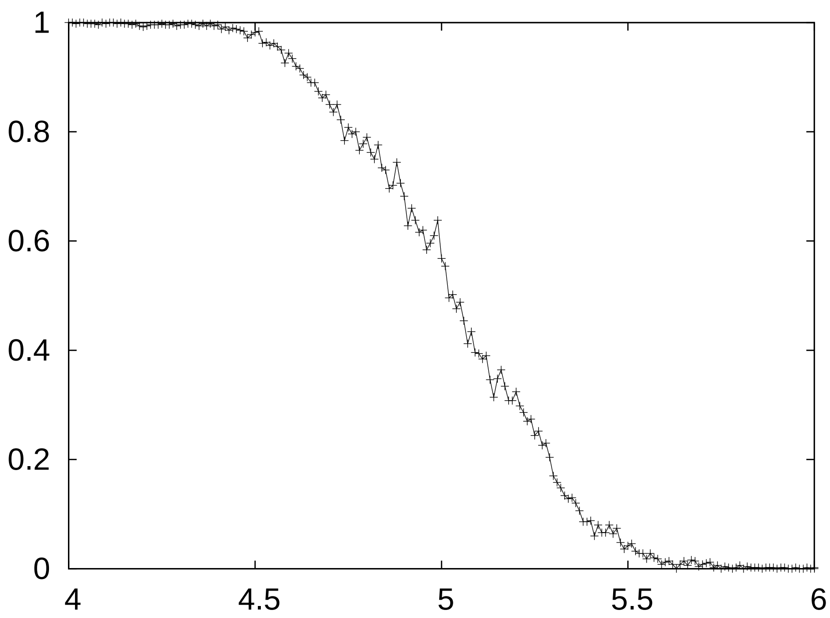
$h = 0.2$ (top left), $h = 3$ (top right) and $h = 5.7$ (bottom left and right (magnification)).

Some comments

1. Last plots explain why the iterates shown (of an initial point in a neighbourhood of Π_1) “turn to the right part” of the NHIC Π_2: the smaller the angle between the two resonance the higher the “energy” h_c to have destruction of the invariant tori between the NHIC.
2. The iterates of an initial point move along a single resonance until they reach an “energy” $h < h_c$ where they can travel from Π_1 to Π_2 (and viceversa) “inside $\{E = h\}$” in a relatively small number of iterates. These two motions represent completely different time-scales.
3. Finally, we note that $h_c \searrow 0$ as $\epsilon \searrow 0$. That is, for resonances of large different order the iterates should travel following Π_1 until they are close enough to HH to be able to cross to Π_2. Recall that the parameter ϵ destroys integrability of the Hamiltonian flow obtained when $\delta \searrow 0$.

A final experiment: probability of passage

For $h = 4(10^{-2})6$ we consider 500 random initial points with $-10^{-2} \leq \psi, \Gamma \leq 10^{-2} \subset \Sigma \cap \{E = h\}$ (close to Π_1 in the chaotic zone). For each point we perform a maximum number of 10^5 iterates of T_δ. If they reach a distance $d < 0.5$ from the point $\Pi_2 \cap \Sigma \cap \{E = h\}$ (the results are similar for $d < 0.25$) we say that a passage from Π_1 to Π_2 takes place.



Ratio of points arriving at a distance d from $\Pi_2 \cap \Sigma$.

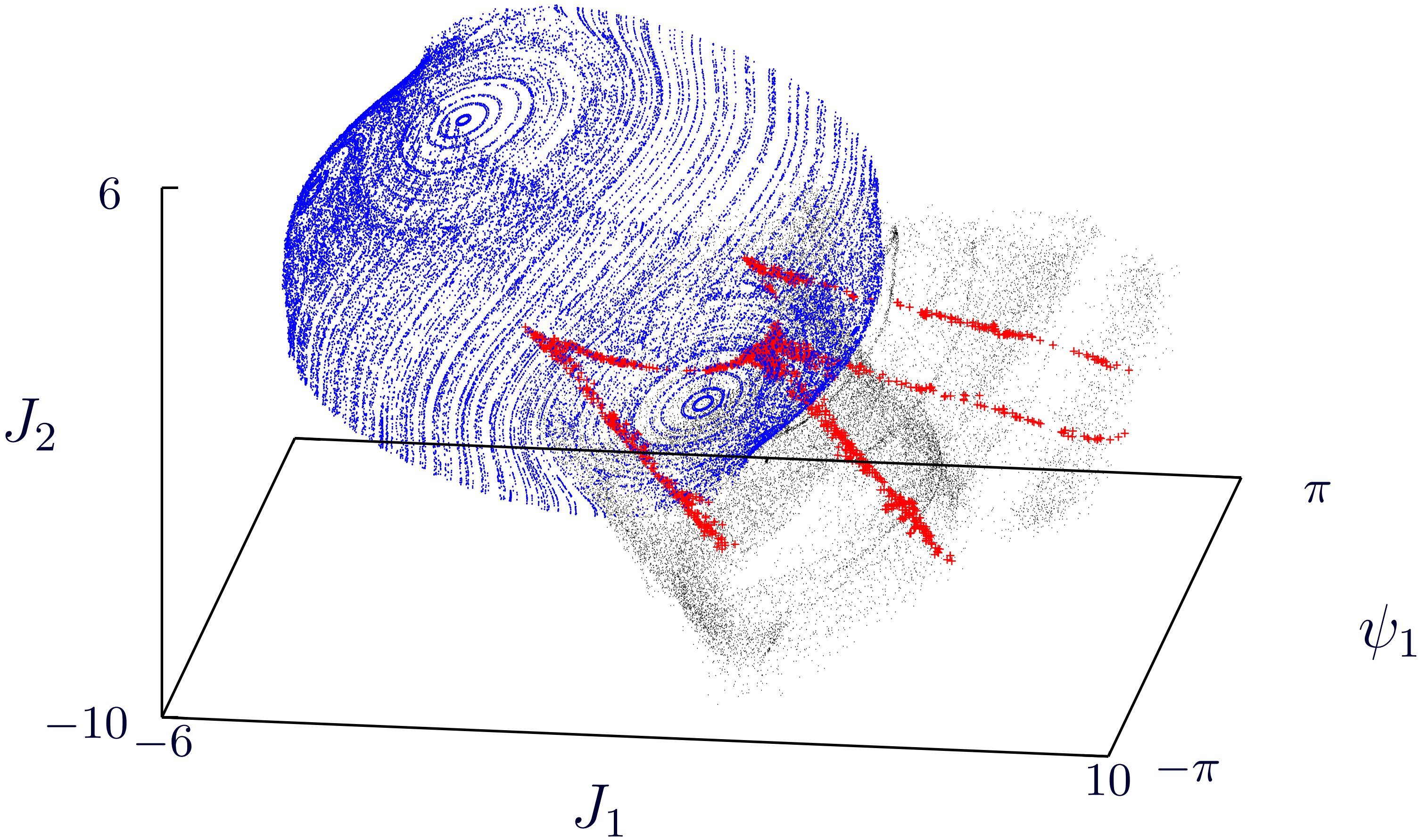
We have seen in the “Poincaré sections” that for $h \approx 5.6$ there is the last “r.i.c”.

Future directions

- Analyse the diffusion properties (obtain quantitative data from massive numerical simulations, and relate it with the geometry at the simple/double resonances). Also compute the local Diffusion coefficients accurately and how they change for different values of h.
  It is important to perform simulations with smaller values of δ to better separate time-scales and be able to explain phenomena related to Arnold diffusion properly.
- Clarify the situations where a different truncated model is obtained, and study the strong doubly resonant cases. Consider $|\epsilon|$ in a non-perturbative regime (e.g. two resonances of equal order).
- Study the *non-definite case*: In particular, for $|\epsilon|$ large the EE fixed point can suffer a Hamiltonian-Hopf bifurcation (complex instability). Determine the role of the invariant manifolds of the complex-saddle point. [a]

[a] Work in progress in collaboration with E.Fontich.

A final plot



Thanks for your attention!!