Escape times across a golden Cantorus and the evolution of approximant islands

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Goal

We consider the Chirikov standard map $M_k : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ as a paradigm of twist area-preserving maps (twist APM) of the cylinder. It is given by

$$M_k : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + \bar{y} \\ y + \frac{k}{2\pi}\sin(2\pi x) \end{pmatrix}, \tag{1}$$

We study escape rates across the golden Cantorus, that is, through the remnant Cantor set after the breakdown of the rotational invariant circle (RIC) with golden rotation number $\omega = (\sqrt{5}-1)/2$.

In other words:

we investigate the transport properties for values of the parameter k larger than but close to Greene's constant $k_G \approx 0.971635406$.

Summary on renormalization for invariant curves I

The finer self-similar structure near the Cantorus is revealed by investigating the Greene-MacKay renormalization operator $\mathcal{R}$:

- Consider a twist APM F having a RIC with rotation number $\rho \in [0, 1)$. Denote by $\bar{F}$ its lift to the plane.
- Consider the sequence $\{p_n/q_n\}_n$ of best approximants of ρ.
  For $\rho =$ golden these are quotients of successive Fibonacci numbers.
- Then, one considers a sequence of maps of the form $\Lambda \bar{F}^{q_n} R^{p_n} \Lambda^{-1}$, where $R(x, y) = (x - 1, y)$ and Λ is a change of variables that zoom in regions of the phase space chosen according to the relative positions of periodic orbits whose periods correspond to two consecutive best approximants of ρ (further details later).
- To guarantee the renormalized map to be defined on the cylinder, $\mathcal{R}$ is defined on commuting pairs (U, T) of orientation preserving diffeomorphisms of the plane.

MacKay, R.S., A renormalisation approach to invariant circles in area-preserving maps, Phys. D 7(1-3), 283–300, 1983. Order in chaos (Los Alamos, N.M., 1982).

Summary on renormalization for invariant curves II

- That is, for $m \in \mathbb{Z}$ one defines

$$\mathcal{R}_m(U, T) = \Lambda(T, T^m U)\Lambda^{-1}$$

  where $\Lambda(A, B)\Lambda' = (\Lambda A \Lambda', \Lambda B \Lambda')$ for arbitrary scalings (Λ, Λ').
- Then, to investigate the phase space structure near the RIC one considers the commuting pair $(U, T) = (R, \bar{F})$ and perform iterates under a suitable sequence of renormalisation operators $\mathcal{R}_m$.
- The choice of m's depends on arithmetic properties of $\rho \in [0, 1)$. If

$$\rho = [l_0, l_1, l_2, \ldots] = \frac{1}{l_0 + \frac{1}{l_1 + \ldots}}$$

  the best approximants of ρ are p_k/q_k where $p_0 = 0$, $q_0 = p_1 = 1$, $q_1 = l_0$ and $p_k = l_{k-1}p_{k-1} + p_{k-2}$, $q_k = l_{k-1}q_{k-1} + q_{k-2}$, for $k \geq 2$. It follows from properties of continued fraction expansions that

$$\mathcal{R}_{l_j} \cdots \mathcal{R}_{l_0}(R, \bar{F}) = \Lambda_{j+1}(\bar{F}^{q_j} R^{p_j},\ \bar{F}^{q_{j+1}} R^{p_{j+1}})\Lambda_{j+1}^{-1},$$

  where Λ_{j+1} is the composition of successive shifted scalings.

Dynamics of $\mathcal{R}_1$: known facts

It follows that to study the phase space structure close to golden Cantorus one is reduced to consider iterates of M_k under $\mathcal{R}_1$. The most relevant part of the phase space of $\mathcal{R}_1$ is characterised by the existence of two fixed points:

- The *trivial* fixed point R_T which is an integrable linear shear

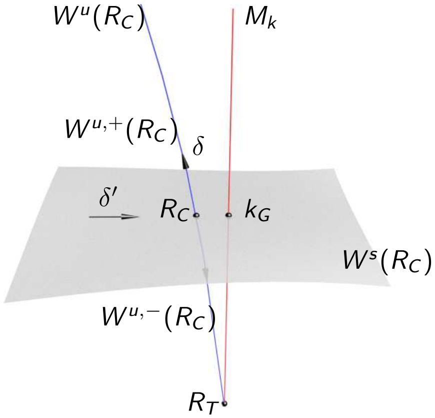
  $$R_T(x, y) = (x + (\omega + 1)y + \omega, y). \quad \leftarrow \text{Atracting fixed point}$$

  All the periodic orbits in the phase space of R_T are parabolic ($\tau = 2$).

- The *critical* fixed point R_C: a map having a critical golden RIC. It is a saddle fixed point with a single unstable eigenvalue $\delta \approx 1.62795$. All elliptic "approximant" orbits of R_C have the same trace, $\tau = \tau^\star \approx 0.999644$.

"Local picture" of the skeleton of $\mathcal{R}_1$ and the orbit of M_k

Partially conjectured



R_T trivial fixed point (attracting)

R_C critical fixed point (saddle)

Unstable eigenvalue: $\delta \approx 1.62795$

Dominant attracting rate in $W^s(R_C)$: $\delta' \approx -0.610830$

Arioli G. and Koch H., The critical renormalization fixed point for commuting pairs of area-preserving maps, Comm. Math. Phys. 295(2), 415–429, 2010.

Koch, H., On hyperbolicity in the renormalization of near-critical area-preserving maps, Discrete Contin. Dyn. Syst. 36(12), 7029–7056, 2016.

Computing iterates of M_k under $\mathcal{R}_1$

1. Compute the following orbits and points:

   P_e^j – Elliptic (or reflection-hyperbolic) p_j/q_j-point on $\{x = 1/2\}$.

   Q_e^j – If j is odd/even, iterate of P_e^j closest to the right/left of it.

   P_e^{j+1} – Analogous to P_e^j with p_{j+1}/q_{j+1} as rotation number.

   L_h^j, R_h^j – The left/right p_j/q_j-hyperbolic points closest to P_e^j.

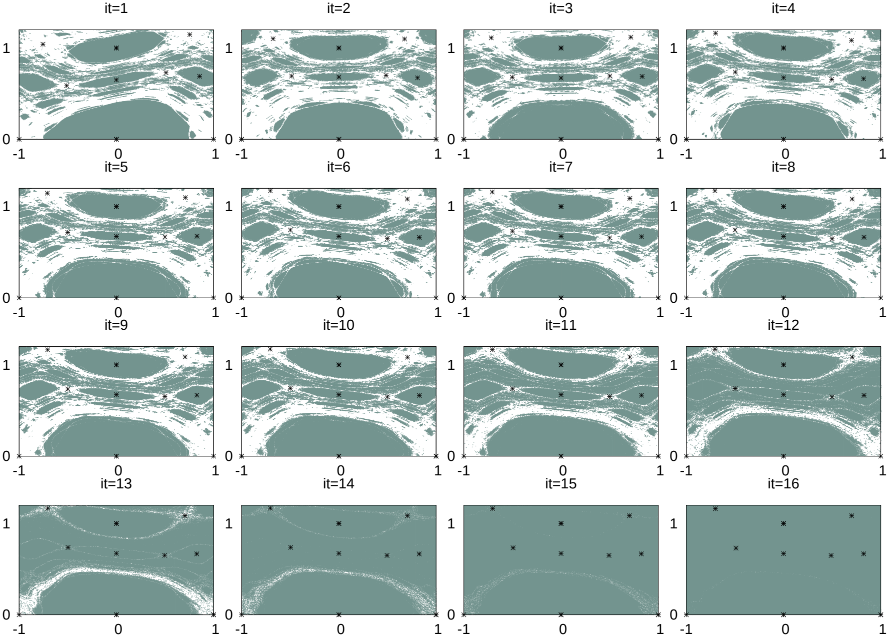
2. Let $p^{(j)}(\xi) = s_1^{(j)}\xi + s_2^{(j)}\xi^2 + s_3^{(j)}\xi^3$ be the cubic interpolating polynomial of the 4 points $L_h^j, P_e^j, R_h^j, Q_e^j$, after moving their abscissas -0.5, that is, in such a way that the x-coordinate of P_e^j is 0.

3. Let $d_x^{(j)} = \max\left(|\pi_1(P_e^j - L_h^j)|, |\pi_1(P_e^j - R_h^j)|\right)$, and $d_y^{(j)} = |\pi_2(P_e^j - P_e^{j+1})|$, where π_1 and π_2 are the projections onto the first and second variable.

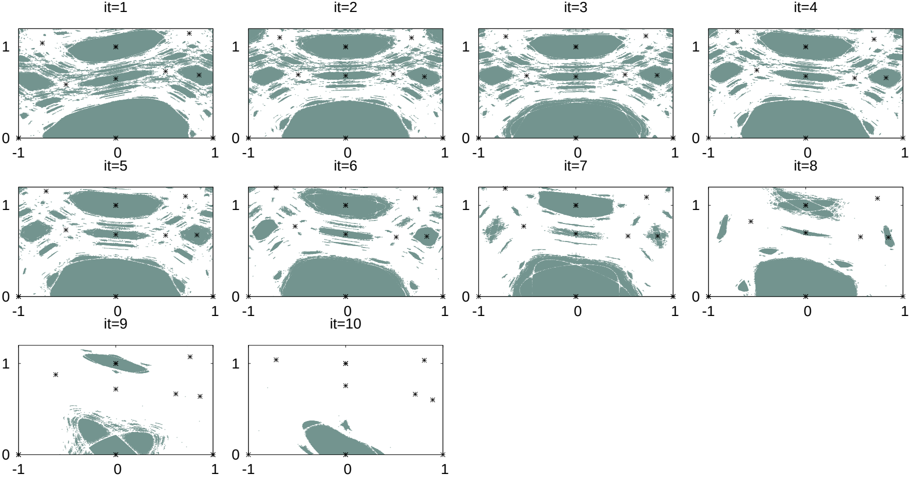
Then, we use the scaling (we remove j dependences):

$$\Lambda_j : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} d_x\xi + 1/2 \\ d_y\eta + \pi_2(P_e^j) + s_1 d_x\xi + s_2(d_x\xi)^2 + s_3(d_x\xi)^3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Renormalization iterates $\mathcal{R}_1^{it}(M_k)$, $k = 0.9716 < k_G$



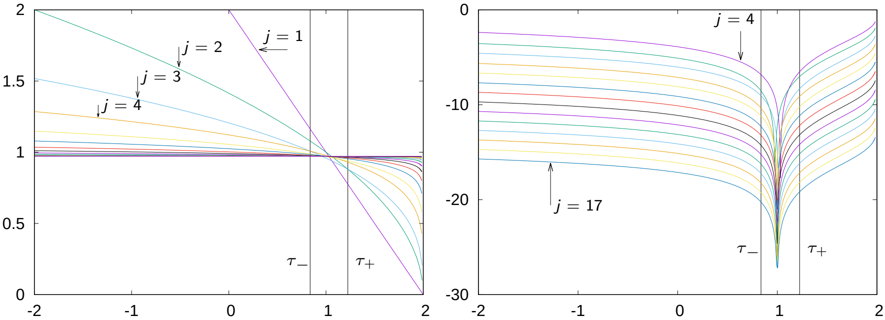
Renormalization iterates $\mathcal{R}_1^{it}(M_k)$, $k = 0.98 > k_G$



In the plots of this and of the previous slide, we considered an equispaced 512×512 grid and we indicate in light grey those pixels whose center is considered regular by approximating the maximal Lyapunov exponent. In black, we show the positions of the 0/1, 1/2 and 2/3-periodic orbits of $\mathcal{R}_1^j[M_k]$.

Estimate of k_G: trace sequences

For a given trace $\tau \in [-2, 2]$ and for a given j, we compute the parameter $k = k_j(\tau)$ for which $\mathrm{trace}(DM_k^{q_j}(P_e^j)) = \tau$.



We represent, as a function of $\tau \in [-2, 2)$, $k_j(\tau)$ for $j = 1, \ldots, 17$ (left) and $\tilde{k}(\tau) = \log_\delta(k_j(\tau) - k_G)$ for $j = 4, \ldots, 17$ (right).

In the plot, $\tau_- = \tau_{3,4}$ and $\tau_+ = \tau_{2,3}$ where, for $1 \leq j < l$, we denote as $\tau_{j,l}$ the value of the trace such that $k_j(\tau_{j,l}) = k_l(\tau_{j,l})$.

Estimate of k_G: trace sequences conjecture

Numerically supported conjecture:

For all $\tau \in [-2, 2)$, the sequence $\{k_j(\tau)\}_j$ converges to k_G geometrically. For $\tau = \tau^\star$ the rate of convergence is δ', and δ otherwise. Furthermore,

1. If $\tau \in [-2, \tau_-)$ the sequence $\{k_j(\tau)\}_j$ is strictly decreasing,
2. If $\tau \in (\tau_+, 2)$ the sequence $\{k_j(\tau)\}_j$ is strictly increasing,
3. For $\tau \in [\tau_-, \tau_+] \setminus \{\tau^\star\}$, if $\tau < \tau^\star$ the sequence $\{k_j(\tau)\}_j$ is eventually decreasing and, if $\tau > \tau^\star$, it is eventually increasing.
4. For $\tau = \tau^\star$ the sequence $\{k_j(\tau)\}_j$ alternates around k_G. Moreover the sequence of pairs of points

$$\{(\tau_{j-1,j}, k_j(\tau_{j-1,j})), (\tau_{j,j+1}, k_j(\tau_{j,j+1}))\}_j,$$

defines domains around the limit point $(\tau^\star, k_G)$ that scale as $1/\delta'$ in τ and as δ/δ' in k.

$\longrightarrow$ It leads to the following method to estimate k_G...

Estimate of k_G: fast convergence sequences

To estimate k_G we compute (alternatively) the sequences

$$\{\tau_{j-1,j}\}_{j\geq 2} \xrightarrow{\delta'} \tau^\star \qquad \text{and} \qquad \{k_j(\tau_{j-1,j})\}_{j\geq 2} \xrightarrow{\delta'/\delta} k_G$$

Remark: The right sequence converges faster than the sequence $\{k_j(\tau)\}_j$ for fixed τ.

We compute the pairs $(\tau_{j,j+1}, k_j(\tau_{j,j+1}))$ for $1 \leq j \leq 35$. We perform computations with 50 decimal digits arithmetics. Using the Aitken's acceleration method we get the first 21 digits of k_G and the first 12 digits of $\tau^\star$, that is, we obtain the approximations

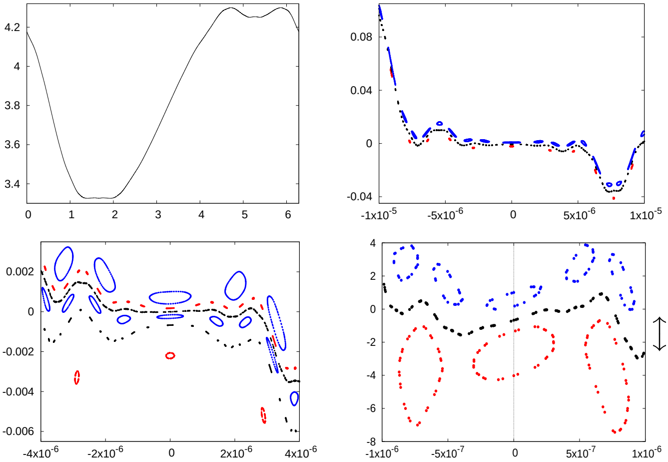
$$\tau^\star = 0.999644540920\ldots \quad \text{and} \quad k_G = 0.971635406047502179389\ldots$$

The estimates of the inverses of the rates of convergence of the sequences are -0.610830 ($\approx \delta'$) and -2.6651429 ($\approx \delta'/\delta$), respectively.

The "approximant" islands

Islands with rotation number = best approximant of the golden frequency.

M_k expressed in $(x, y) \in [0, 2\pi]^2$



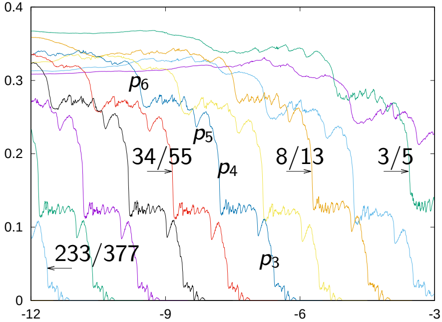
We display $x - \pi$ and $y - p_3(x)$, where $p_3(x)$ is a cubic polynomial fit of the approximate golden RIC

$\updownarrow \times 10^{-22}$

Illustrations for the value k_G obtained. In black the "critical RIC". Last plot: red and blue islands with $\rho = \rho_{46}$ (period$=$ 2971215073) and $\rho = \rho_{47}$ (period$=$ 4807526976) resp., black points with $\rho-$golden $\approx -0.123E - 19$.

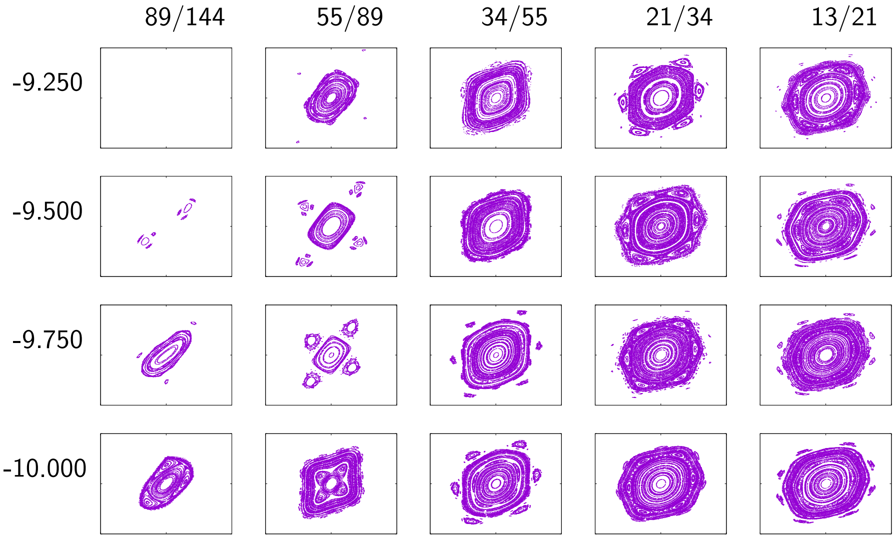
The size of the "aproximating" islands

For $k > k_G$ we consider $\texttt{npts} = 800 \times 800$ equispaced points $(\xi, \eta) \in Q = [-1, 1] \times [-0.6, 0.6]$. A point is considered to be "in the island" if it stays in Q for, at least, 10^5 iterates of M_k.
Let $\mu_j = \#\{\text{points "in the island"}\}/\texttt{npts}$.



We display $\tilde{\mu}_j = \mu_j/(d_x^{(j)} d_y^{(j)})$, that is, the scaled areas of "approximant" islands, for $j = 3, \ldots, 13$, as a function of $\tilde{k} = \log_\delta(k - k_G)$ for $j = 3, \ldots, 13$.
The jumps p_i related to the breakdown of the invariant curve surrounding the islands of period i inside the "approximant" islands.

Evolution of the "approximant" islands

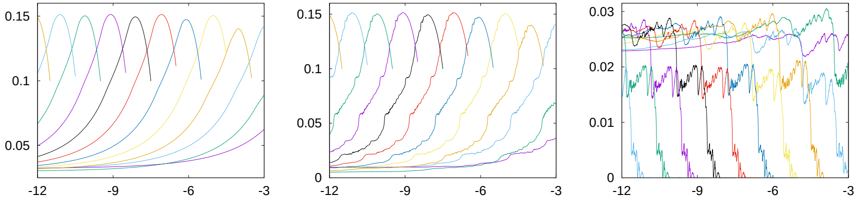


Shape of the "approximant" islands with $\rho = 13/21, 21/34, 34/55, 55/89$ and $89/144$ in the interval $\tilde{k} \in [-10, -9)$.

The Mather's ΔW for the "approximant" islands

The gaps of a Cantorus allow orbits to leak across it, but the time to cross them can be very large, specially for parameters just after the breakdown.

Transport properties accross the Cantorus (for $k > k_G$) are related to Mather's ΔW_j, that is, to the flux (the area per iterate that crosses a turnstile defined by the jth approximant pair of periodic orbits).



We display $\Delta\tilde{W}_j$ (left), $\Delta\tilde{W}_j^{nc}$ (center) and $\Delta\tilde{W}_j^{c}$ (right) as a function of $\tilde{k}$. The left and center plots display the curves for $3 \leq j \leq 14$ but the curve for $j = 14$ is not shown in the right plot.

We use the same grid of initial conditions as for the computation of $\tilde{\mu}_j$ before.
Subscript c/nc means confined/not confined in the island.
We display scaled quantities: $\Delta\tilde{W}_j = \Delta W_j/(d_x^{(j)} d_y^{(j)})$.

Escape rates: renormalization theory

Given $(x_0, y_0) \in \mathbb{S}^1 \times (0,1)$ denote by $(x_n, y_n) = M_k^n(x_0, y_0)$. Let $n = n(x_0, y_0)$ be the number of iterates for which either $y_n > y^{(u)}$ or $y_n < y^{(l)}$ for the first time (we take $y^{(l)} = 0$ and $y^{(u)} = 1$ in the computations). If $n = n(x_0, y_0) < \infty$ we say that (x_0, y_0) escapes across one of the golden Cantori in n iterates. Denote by $\langle N_k \rangle$ the mean escape rate for a given k.

For any irrational rotational number ω, Mather's ΔW_ω is obtained as the limit of ΔW_j. It follows from renormalisation theory that

$$\Delta W_\omega(k_G + \Delta k/\delta) \approx \Delta W_\omega(k_G + \Delta k)/(\alpha\beta), \quad \alpha, \beta \text{ limit domain scalings}$$

Hence, there exists a 1-periodic universal function $U(x) = U(x+1)$ s.t.

$$\Delta W_\omega(k_G + \Delta k) \approx A(\Delta k)^B U(\log_\delta(\Delta k)), \qquad \text{where } B = \log_\delta(\alpha\beta).$$

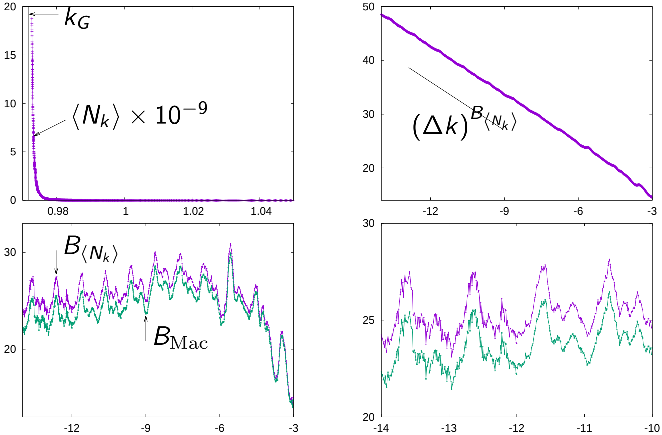
As a consequence, we expect the mean escape rate $\langle N_k \rangle$ to behave as

$$\langle N_k \rangle = A(\Delta k)^B$$

Mather, J.N., A criterion for the nonexistence of invariant circles, IHES Publ. Math 63, 153–204, 1986.

Escape rates: numerical results

To compute $\langle N_k \rangle$ for $\tilde{k} = -12(-0.01) - 5.99$, we consider 10^5 – 10^7 (depending on $\tilde{k}$) initial conditions in $W^u_{\text{loc}}(0.5, 0.5)$ to guarantee they escape.



Top left: $\langle N_k \rangle \times 10^{-9}$ as a function of k. Top right: $\log_\delta \langle N_k \rangle$ as a function of $\tilde{k}$. Bottom left: $\langle N_k \rangle \times (k - k_G)^{-B}$, with $B = B_{\langle N_k \rangle}$ and $B = B_{\mathrm{Mac}}$ as a function of $\tilde{k}$. Bottom right: Detail of bottom left plot. $B_{\mathrm{Mac}} = -\log_\delta(\alpha\beta) \approx -3.0117218914$

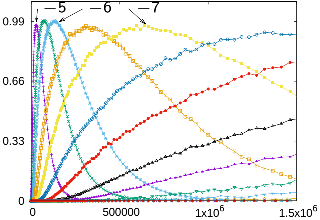
Final comments

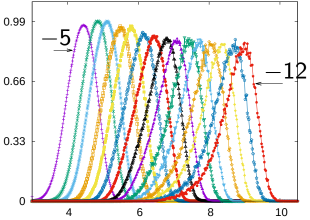
- We approached the breakdown by $\delta^{-14} \approx 0.00109$ and obtain $|B_{\mathrm{Mac}} - B_{\langle N_{\tilde{k}} \rangle}| = \mathcal{O}(10^{-2})$.
  $\rightarrow$ still far from the limit,
  $\rightarrow$ but computations for $\tilde{k} < -14$ are still far from practical.
  Closer to the limit, as $\tilde{k} \rightarrow -\infty$, the oscillations of the escape rates should take place around a horizontal line.

- It needs to be clarified which are the objects responsible for the actual transport probabilities. One expects the oscillations to be strongly related to the area of the heteroclinic lobes of intersecting invariant manifolds of hyperbolic periodic orbits and the turnstile areas of "approximant" orbits. Our computations show that, as k varies, the area in the phase space that is accessible to orbits that can escape changes as the islands of stability do, and not in a monotone way in $\tilde{k}$, but if we conveniently scale the phase space, the area occupied by evolving islands varies in a periodic way.

Open questions/future work

- What can be said about the probability law of escape rates?





  Numerically computed histogram (pdf) of the number of iterates needed to escape, N_k, for different values of $\tilde{k}$ Left: N_k in the original scale of time. Right: Same as left plot but in decimal scale in time.

- Behaviour of the parameterization of the golden RIC as $k \nearrow k_G$?
  Olvera, A. and Petrov, N.P. Regularity Properties of Critical Invariant Circles of Twist Maps, and Their Universality. SIAM J.Appl.Dyn.Syst. 7, 2008.

- Our numerics indicate that the transport properties near the Cantorus might be described by a nearest neighbour Markov process with different states (as many as relevant islands near the Cantorus). How?

Thanks for your attention!