Dynamics of 4d symplectic maps near a double resonance

DyToComp 2012

Bedlewo, Jun 30.

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Introduction: general framework

Let $F_{\boldsymbol{\delta}}$ be a 2-parameter family of analytic symplectic 4d maps, $\boldsymbol{\delta} = (\delta_1, \delta_2) \in \mathbb{R}^2$ small enough parameter.

Assume:

$F_{\boldsymbol{\delta}}(\mathbf{0}) = \mathbf{0}$ totally elliptic fixed point (for all $\boldsymbol{\delta}$),

Spec$(DF_{\boldsymbol{\delta}})(0) = \{\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $k = 1, 2$.

We will always assume that the eigenvalues are simple, i.e., $\alpha_1 \neq \alpha_2$.

The local dynamics can be described using Birkhoff NF.

Set of resonances

$$\Gamma = \{ (k_1, k_2) : k_1\alpha_1 + k_2\alpha_2 = 0 \pmod 1 \} \subset \mathbb{Z}^2 .$$

→ $\mathbf{r} = (k_1, k_2) \in \Gamma$ is *a resonance of order* $|\mathbf{r}| = |k_1| + |k_2|$.

→ $(k_1, k_2) \in \Gamma \Leftrightarrow \lambda_1^{k_1}\lambda_2^{k_2} = 1$.

→ $(0, 0)$ *trivial (or unavoidable)* resonance.

→ We assume $k_1 \geq 0$ to avoid trivial symmetries in resonances.

Introduction: fixed point types

The totally elliptic fixed point of $F_{\mathbf{0}}$, at the origin, can be:

1. Non-resonant (Γ is a trivial group).

   In this case $\{\,\alpha_1, \alpha_2, 1\,\}$ are rationally independent.

2. Simply resonant (Γ is a one-dimensional lattice).

   In this case there are two possibilities:

   a) $\alpha_1 \in \mathbb{Q}$, $\alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$ (or vice versa).

   b) $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$ but $\{\,\alpha_1, \alpha_2, 1\,\}$ are rationally dependent.

3. Doubly resonant (Γ is a two-dimensional lattice).

   In this case $\alpha_1, \alpha_2 \in \mathbb{Q}$

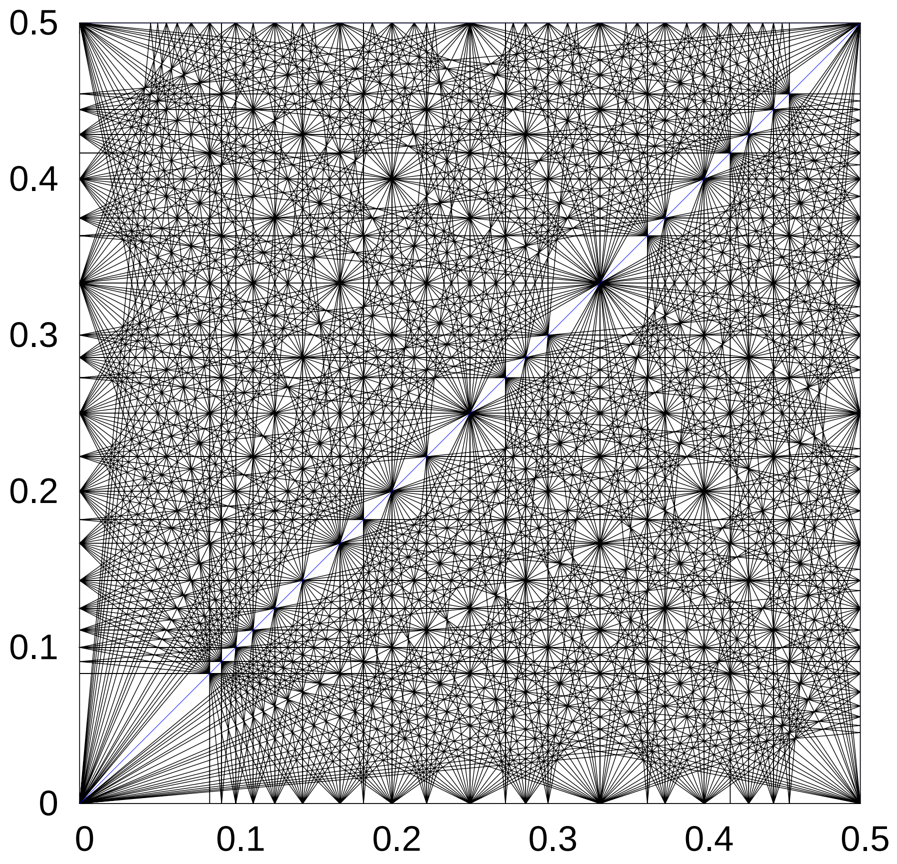
$$\alpha_1 = \frac{p_1}{q_1} \qquad \text{and} \qquad \alpha_2 = \frac{p_2}{q_2}\,, \qquad p_1, p_2, q_1, q_2 \in \mathbb{N}\,.$$

Introduction: frequency space

Each resonant relation ($k_1\alpha_1 + k_2\alpha_2 = k_3$, $k_i \in \mathbb{Z}$) defines a line on the torus

$$\mathcal{T} = \{\, (\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1| = |\lambda_2| = 1 \,\}.$$

Simply and doubly resonant eigenvalues are dense in $\mathcal{T}$.



Resonant lines (of order ≤ 12) on the plane (α_1, α_2).

General idea of this work

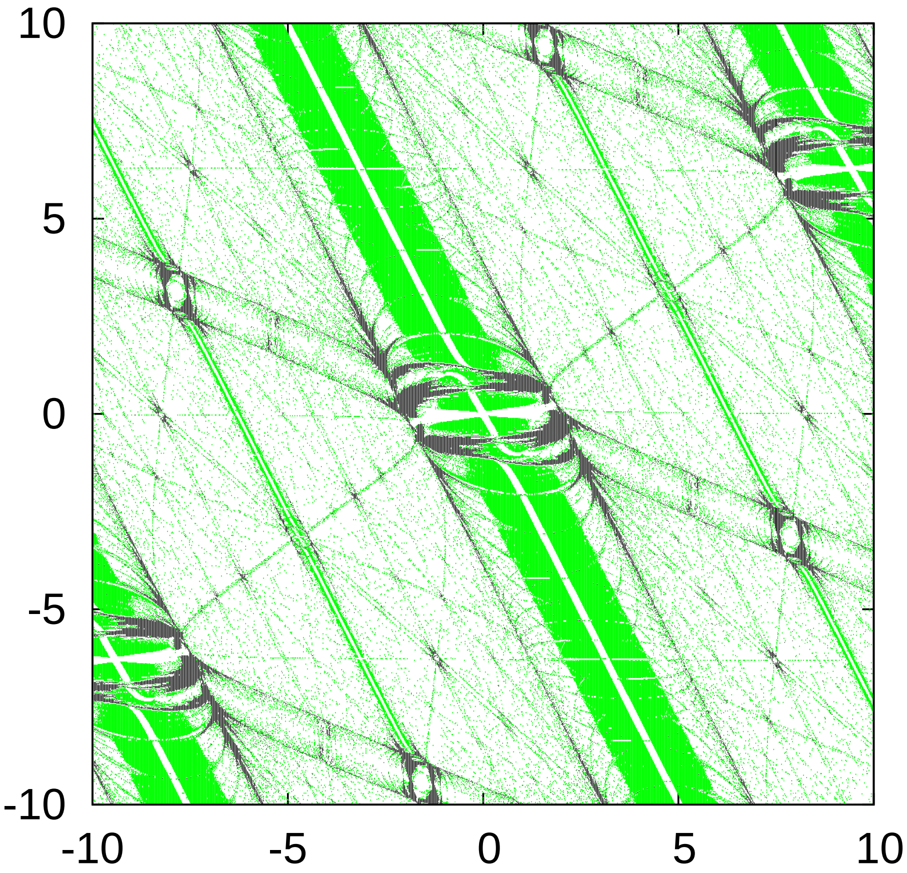
We fix $(\alpha_1, \alpha_2) \in \mathcal{T}$ which we assume close to a double resonant relation ($\alpha_j = p_j/q_j + \delta_j$):

$$\begin{aligned} k_1\alpha_1 + k_2\alpha_2 &= k_3 \\ j_1\alpha_1 + j_2\alpha_2 &= j_3 \end{aligned}$$

and we study the dynamics of $F_{\boldsymbol{\delta}}$ at the double resonance (δ small $\rightarrow$ res. BNF + unfolding).

Introduction: phase space structure

Diffusion along phase space takes place basically along single resonances but multiple resonances play a key role in an explanation of the Arnold diffusion (e.g. Nekhoroshev theory – upper bounds on the rate of diffusion).



$$(\boldsymbol{\psi}, \boldsymbol{J}) \to (\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{J}}) = (\psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2), J_1 + \delta \sin \psi_1, J_2 + \delta \epsilon \sin \psi_2).$$

$\rightarrow \delta = 0.5$, $\epsilon = 0.1$, $a_2 = 0.5$ and $a_3 = 1.25$.

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Takens NF

$F_{\boldsymbol{\delta}}$ symplectic 4d maps ($\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$), $\boldsymbol{\delta} \in \mathbb{R}^2$ small enough,
$F_{\boldsymbol{\delta}}(\mathbf{0}) = \mathbf{0}$, Spec=$\{\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $k = 1, 2$.

$$\alpha_1 \neq \alpha_2 \Longrightarrow DF_{\mathbf{0}}(0) \sim \Lambda_0 = \begin{pmatrix} R_{2\pi\alpha_1} & 0 \\ 0 & R_{2\pi\alpha_2} \end{pmatrix}$$

A canonical change of variables reduces $F_{\boldsymbol{\delta}}$ to BNF $N_{\boldsymbol{\delta}}$:

$$N_{\boldsymbol{\delta}} \circ \Lambda_{\mathbf{0}} = \Lambda_{\mathbf{0}} N_{\boldsymbol{\delta}}.$$

Since $DN_{\mathbf{0}}(\mathbf{0}) = \Lambda_{\mathbf{0}}$ the map $\Lambda_{\mathbf{0}}^{-1} N_{\boldsymbol{\delta}}$ is tangent to the identity
$\Longrightarrow$ it can be formally interpolated (in a compact domain around $\mathbf{0}$) by a (Hamiltonian) vector field:

$$N_{\boldsymbol{\delta}} = \Lambda_{\mathbf{0}} \Phi^1_{H_{\boldsymbol{\delta}}} + \text{ exp. small error}$$

Interpolating Hamiltonian

Moreover $H_{\boldsymbol{\delta}}$ is $\Lambda_{\mathbf{0}}$-invariant ($H_{\boldsymbol{\delta}} = H_{\boldsymbol{\delta}} \circ \Lambda_{\mathbf{0}}$) $\Longrightarrow N_{\boldsymbol{\delta}}^j = \Lambda_{\mathbf{0}}^j \Phi_{H_{\boldsymbol{\delta}}}^j$ for all $j \in \mathbb{N}$

$\Longrightarrow$ study the flow of $H_{\boldsymbol{\delta}}$ instead of iterations of $N_{\boldsymbol{\delta}}$.

To obtain $H_{\boldsymbol{\delta}}$:

→ Complex vbles ($z_k = x_k + iy_k,\ \bar{z}_k = x_k - iy_k$), $\Lambda_{\mathbf{0}} = \mathrm{diag}(\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2)$.

→ $z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$ resonant $\Longleftrightarrow$ $\Lambda_{\mathbf{0}}$-invariant $\Longleftrightarrow (j-k, l-m) \in \Gamma$.

Then $H_{\boldsymbol{\delta}}$ is a sum of res. monomials: $H_{\boldsymbol{\delta}} = \displaystyle\sum_{\substack{(j-k,l-m)\in\Gamma \\ j,k,l,m\geq 0}} h_{jklm}(\boldsymbol{\delta}) z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$

In Poincaré vbles ($I_j = \frac{|z_j|^2}{2},\ \varphi_j = \arg z_j$):

$$H_{\boldsymbol{\delta}} = \sum_{\substack{(k_1,k_2)\in\Gamma \\ p,q\geq 0}} a_{k_1 k_2 pq}(\boldsymbol{\delta}) I_1^{p+k_1/2} I_2^{q+|k_2|/2} \cos(k_1\varphi_1 + k_2\varphi_2 + b_{k_1 k_2 pq})$$

Q: Dominant terms of $H_{\boldsymbol{\delta}}$? Arithmetic properties of Γ depending on (α_1, α_2).

Minimal generators of Γ: primary resonances

Recall: $\Gamma = \{ (k_1, k_2) : k_1\alpha_1 + k_2\alpha_2 = 0 \pmod 1 \} \subset \mathbb{Z}^2$ lattice.

Consider:

→ $\mathbf{r}_0 = (k_1, k_2) \in \Gamma$ a smallest (maybe non-unique) non-trivial element,

→ $\mathbf{r}_1 = (m_1, m_2) \in \Gamma$ any of the smallest elements independent from $\mathbf{r}_0$.

$\Longrightarrow$ $\mathbf{r}_0$ and $\mathbf{r}_1$ generate Γ (provided $\alpha_1 \neq \alpha_2$).

We call $\mathbf{r}_0$, $\mathbf{r}_1$ the **primary resonances** (minimal generators of Γ).

We denote by n_j the order of $\mathbf{r}_j$, $j = 1, 2$ (then, $n_0 \leq n_1$).

Remark: The **primary resonances** are unique *for most* of the frequencies. However, there are two situations of non-uniqueness:

- At the leading order. Consider $\alpha_1 = 1/8$, $\alpha_2 = 3/8$. Then $n_0 = n_1 = 4$ and there are 3 resonances of order 4: $(1, -3)$, $(3, -1)$, $(2, 2)$.
- At order n_1. Consider $\alpha_1 = 1/11$, $\alpha_2 = 4/11$. Then $n_0 = 4$ and the only resonance of order 4 is $(1, -3)$, and $n_1 = 5$ and there are 2 resonances: $(4, -1)$ and $(3, 2)$.

Classification by primary resonances

If $n_0 \geq 5$ the fixed point is called weakly resonant and otherwise it is strongly resonant. There are different situations to study:

- $5 \leq n_0 < n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Moreover, up to order $n_1 - 1$ is also integrable.
- $5 \leq n_0 = n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Adding order n_0-terms becomes generically non-integrable.
- *Simply strong* resonances: $n_0 \leq 4 < n_1$.
- *Doubly strong* resonances: $n_0 = 3$,$n_1 = 4$ and $n_0 = n_1 \leq 4$.

Doubly strong resonances $\alpha_1 \neq \alpha_2$

num.	α_1	α_2	n_0	n_1	3 resonances	4 resonances
1	$\frac{1}{6}$	$\frac{1}{3}$	3	3	$(2,-1)\ (0,3)$	$(2,2)$
2	$\frac{1}{5}$	$\frac{2}{5}$	3	3	$(2,-1)\ (1,2)$	$(3,1)\ (1,-3)$
3	$\frac{1}{12}$	$\frac{1}{4}$	4	4	—	$(3,-1)\ (0,4)$
4	$\frac{1}{10}$	$\frac{3}{10}$	4	4	—	$(3,-1)\ (1,3)$
5	$\frac{1}{8}$	$\frac{3}{8}$	4	4	—	$(1,-3)\ (3,-1)\ (2,2)$
6	$\frac{1}{4}$	$\frac{5}{12}$	4	4	—	$(1,-3)\ (4,0)$
7	$\frac{1}{4}$	$\frac{1}{3}$	3	4	$(0,3)$	$(4,0)$
8	$\frac{1}{9}$	$\frac{1}{3}$	3	4	$(0,3)$	$(3,-1)$
9	$\frac{2}{9}$	$\frac{1}{3}$	3	4	$(0,3)$	$(3,1)$
10	$\frac{1}{3}$	$\frac{4}{9}$	3	4	$(3,0)$	$(1,-3)$
11	$\frac{1}{8}$	$\frac{1}{4}$	3	4	$(2,-1)$	$(0,4)$
12	$\frac{1}{4}$	$\frac{3}{8}$	3	4	$(1,2)$	$(4,0)$
13	$\frac{1}{7}$	$\frac{2}{7}$	3	4	$(2,-1)$	$(1,3)$
14	$\frac{1}{7}$	$\frac{3}{7}$	3	4	$(1,2)$	$(3,-1)$
15	$\frac{2}{7}$	$\frac{3}{7}$	3	4	$(2,1)$	$(1,-3)$

Weak double resonances: a truncated model

Recall: $F_{\boldsymbol{\delta}}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $\alpha_k = p_k/q_k + \delta_k$ for $k = 1, 2$, $\delta = \|\boldsymbol{\delta}\|$ small.

Takens NF: $H_{\boldsymbol{\delta}} = \sum_{(k_1,k_2)\in\Gamma} a_{k_1 k_2 pq}(\boldsymbol{\delta}) I_1^{p+k_1/2} I_2^{q+|k_2|/2} \cos(k_1\varphi_1 + k_2\varphi_2 + b_{k_1 k_2 pq})$

Assume (most common case!) that

→ α_1, α_2 are close to be doubly resonant,

→ $\mathbf{r_0} = (k_1, k_2)$ and $\mathbf{r_1} = (m_1, m_2)$ are the unique minimal generators of Γ,

→ $5 \le n_0 < n_1$ (*weak* double resonance).

Adapt vbles: $\psi_1 = k_1\varphi_1 + k_2\varphi_2, \psi_2 = m_1\varphi_1 + m_2\varphi_2, I_1 = k_1 J_1 + m_1 J_2, I_2 = k_2 J_1 + m_2 J_2$

$$\boxed{H_{\boldsymbol{\delta}} = H_0(\boldsymbol{J}, \boldsymbol{\delta}) + H_1(\boldsymbol{J}, \psi_1, \boldsymbol{\delta}) + H_2(J_1, J_2, \psi_1, \psi_2, \boldsymbol{\delta}) + \mathcal{O}_{n_1+1}(\boldsymbol{z})}$$

$$\begin{aligned}
H_0 &= A_{00}(J_1, J_2, \boldsymbol{\delta}), \\
H_1 &= \sum_{l_1=1}^{n_1/n_0} I_1^{l_1|k_1|/2} I_2^{l_1|k_2|/2} A_{l_1 0}(J_1, J_2, \boldsymbol{\delta}) \cos\big(l_1\psi_1 + B_{l_1 0}(J_1, J_2, \boldsymbol{\delta})\big), \\
H_2 &= I_1^{|m_1|/2} I_2^{|m_2|/2} A_{01}(0, 0, \boldsymbol{\delta}) \cos\big(\psi_2 + B_{01}(0, 0, \boldsymbol{\delta})\big).
\end{aligned}$$

Localizing around the double resonance

In a neighbourhood of the origin

$$H_0 = c_1\delta J_1 + c_2\delta J_2 + a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta^5)$$

→ inv. $\mathbb{T}^2$ at $J_1 = \delta r_1$, $J_2 = \delta r_2 \Rightarrow$ inv. $\mathbb{T}^2$ for the NF system if $I_1, I_2 > 0$.

Then $J_k = \delta r_k + \delta^{n_0/4}\tilde{J}_k$ and $H = \delta^{n_0/2}\tilde{H}$ gives

$$\begin{aligned}
H_0(J_1, J_2, \boldsymbol{\delta}) &= a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta^{n_0/4}),\\
H_1(J_1, J_2, \psi_1, \boldsymbol{\delta}) &= \sum_{l_1=1}^{n_1/n_0} \delta^{(l_1-1)n_0/2}\tilde{A}_{l_1 0}(J_1, J_2, \boldsymbol{\delta})\cos\big(l_1\psi_1 + \tilde{B}_{l_1 0}(J_1, J_2, \boldsymbol{\delta})\big),\\
H_2(J_1, J_2, \psi_1, \psi_2, \boldsymbol{\delta}) &= \delta^{(n_1-n_0)/2} a_{01}\cos\big(\psi_2 + b_{01}\big).
\end{aligned}$$

Furthermore, if $n_1 < 2n_0$ (different but similar order resonances) then

$$H_1(J_1, J_2, \psi_1, \boldsymbol{\delta}) = \big(a_{10} + \delta^{n_0/4-1}\hat{A}_{10}(J_1, J_2, \boldsymbol{\delta})\big)\cos\psi_1$$

$\longrightarrow$ No other harmonics in H_1 appear!

Analysis of the truncated model

$$H(\psi_1,\psi_2,J_1,J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon\cos(\psi_2)$$

- For the moment $\epsilon \sim \delta^{(n_1-n_0)/2}$ will be considered as a small parameter.
- Change: $\tilde{\psi}_1 = \psi_1,\ \tilde{\psi}_2 = \psi_2 - a_2\psi_1,\ \tilde{J}_1 = J_1 + a_2 J_2,\ \tilde{J}_2 = J_2$

  $$H(\psi_1,\psi_2,J_1,J_2) = J_1^2/2 + dJ_2^2/2 + \cos(\psi_1) + \epsilon\cos(\psi_2 + a_2\psi_1),$$

  where $d = a_3 - a_2^2$. We assume $d \neq 0$.
- 4 fixed points: If $\nu = \epsilon d > 0$ and $|\epsilon|$ small enough

  $p_1 = (0,0,0,0)$ – HH, $\quad p_2 = (0,\pi,0,0)$ – HE

  $p_3 = (\pi,-a_2\pi,0,0)$ – EH, $\quad p_4 = (\pi,(1-a_2)\pi,0,0)$ –EE
- Reversibilities:

$$\begin{aligned} R_0(\psi_1,\psi_2,J_1,J_2) &= (2\pi-\psi_1,-2\pi a_2-\psi_2,J_1,J_2), \\ R_1(\psi_1,\psi_2,J_1,J_2) &= (2\pi-\psi_1,2\pi(1-a_2)-\psi_2,J_1,J_2). \end{aligned}$$

The NHIM

$$H(\psi_1, \psi_2, J_1, J_2) = J_1^2/2 + dJ_2^2/2 + \cos(\psi_1) + \epsilon \cos(\psi_2 + a_2\psi_1)$$

$\epsilon = 0$:

- → ψ_2 is cyclic (J_2 is a first integral) $\Rightarrow$ Pendulum (fast) dynamics in ψ_1, J_1-coord. given by $H_1^0 = \frac{J_1^2}{2} + \cos(\psi_1)$.
- → The cylinder $\Pi_0^0 = \Pi_0^{2\pi} = \{\psi_1 = 0 \pmod{2\pi}, J_1 = 0\}$ is a 2D NHIM.
- → Π_0^0 is foliated by p.o. $C_h^0 = \Pi_0^0 \cap \{H = h\}$.
- → $W^u(\Pi_0^0)$ is given by $J_1 = 2\sin(\psi_1/2)$. $\longleftarrow$ 3D
- → Non-transversal: $W^u(\Pi_0^0) = W^s(\Pi_0^0)$.

The perturbed NHIM

$$H(\psi_1, \psi_2, J_1, J_2) = J_1^2/2 + dJ_2^2/2 + \cos(\psi_1) + \epsilon\cos(\psi_2 + a_2\psi_1)$$

$\epsilon \neq 0$: Normal hyperbolicity theory (Fenichel)

→ $\exists\, \Pi_\epsilon^0$ (resp. $\Pi_\epsilon^{2\pi}$) $\mathcal{O}(\epsilon)$-close to Π_0^0 (resp. $\Pi_0^{2\pi}$).

(the perturbed system is not ψ_1-periodic: $R_k(\Pi_\epsilon^0) \neq \Pi_\epsilon^0$)

→ $\exists\, W^{u/s}(\Pi_\epsilon^0)$ $\mathcal{O}(\epsilon)$-close to $W^{u/s}(\Pi_0^0)$

→ $W^u(\Pi_\epsilon^0)$ given by a graph $J_1 = 2\sin\frac{\psi_1}{2} + \epsilon f(\psi_1, \psi_2, J_2; \epsilon)$,

with $f_0 = \frac{1}{2\sin\frac{\psi_1}{2}} \int_0^{\psi_1} g(s, \psi_2 + dJ_2 \log[\tan(\frac{s}{4})/\tan(\frac{\psi_1}{4})])\, ds$

and $g(\psi_1, \psi_2) = a_2 \sin(\psi_2 + a_2\psi_1)$.

→ Transversality? The distance between $W^{u/s}(\Pi_\epsilon^{0/2\pi})$ on $\psi_1 = \pi$ is

$$J_1^s - J_1^u = \epsilon\Big(f_0(\pi, -2\pi a_2 - \psi_2, J_2) - f_0(\pi, \psi_2, J_2)\Big) + \mathcal{O}(\epsilon^2) =$$
$$= \epsilon A(J_2)\sin(\psi_2 + \pi a_2) + \mathcal{O}(\epsilon^2),$$

where

$$A(J_2) = a_2 \int_0^\pi \cos\big(a_2(s-\pi) + dJ_2 \log[\tan(\tfrac{s}{4})]\big)\, ds.$$

Transversality properties

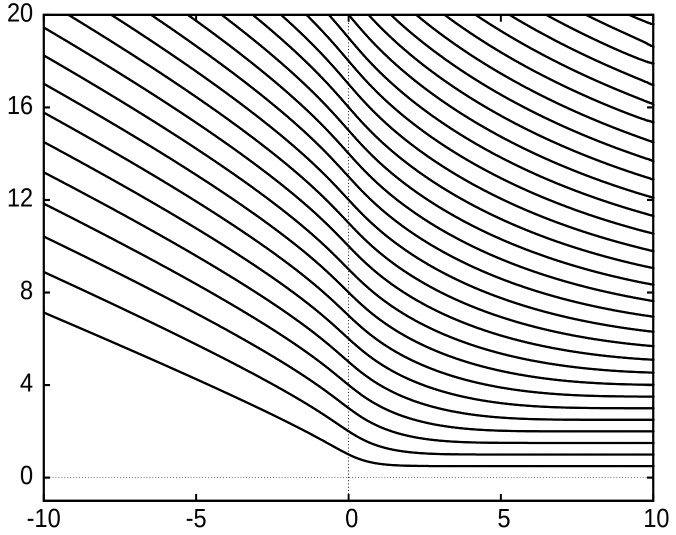
$J_1^s - J_1^u$ vanishes for $\psi_2 = -\pi a_2 \pmod{\pi} \Rightarrow$ 2 lines of homoclinics

$$\begin{aligned}\ell_0 &= \{\,(\psi_1,\psi_2,J_1,J_2) : J_1 = 2+\epsilon f(\psi_1,\psi_2,J_2;\epsilon),\ \psi_1=\pi,\ \psi_2=-\pi a_2\,\},\\ \ell_1 &= \{\,(\psi_1,\psi_2,J_1,J_2) : J_1 = 2+\epsilon f(\psi_1,\psi_2,J_2;\epsilon),\ \psi_1=\pi,\ \psi_2=\pi(1-a_2)\,\}\,.\end{aligned}$$

$W^u(\Pi_\epsilon^0)$ intersects $W^s(\Pi_\epsilon^{2\pi})$ transversally provided $A(J_2) \neq 0$.

→ For $J_2 = 0$, $A(J_2) = \sin(\pi a_2) \Longrightarrow$ transversality if $a_2 \notin \mathbb{Z}$.

→ For J_2 arbitrary, $A(J_2)$ vanishes on the lines



x-axis: dJ_2, y-axis: a_2

Fix a_2 and d, then:

⋆ For $J_2 \to \infty$, the lines accumulate to $k/2$, $k \in \mathbb{Z}$.

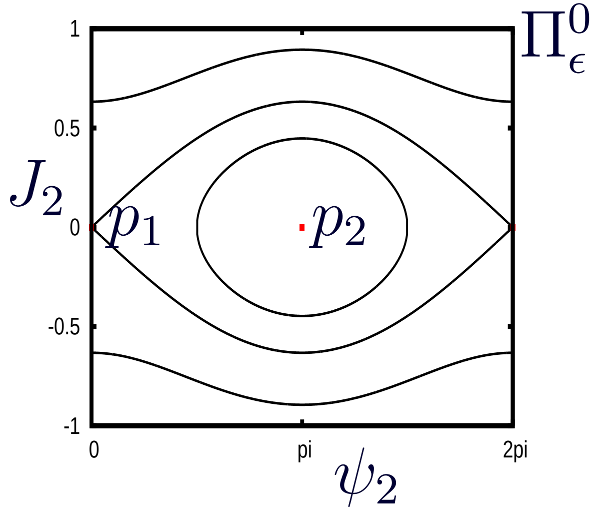
⋆ Finite number of zeros of $A(J_2) \Rightarrow$ Given a_2 except for some concrete values of J_2 the intersections are transversal.

Dynamics on the NHIM and transversality

The dynamics on Π_ϵ^0 is given by

$$H_{2,\epsilon}^0(\psi_2, J_2) = d\frac{J_2^2}{2} + \epsilon\cos(\psi_2) + 1 + \mathcal{O}(\epsilon^2),$$

where (ψ_2, J_2) are used as coordinates on the cylinder.



Π_ϵ^0 contains the fixed points p_1 (HH) and p_2 (HE).

Q: Assume that $W^u(\Pi_\epsilon^0)$ intersects $W^s(\Pi_\epsilon^{2\pi})$ transversally. Does it imply that the separatrices of C_h^ϵ intersect transversally inside $\{ H = h \}$? NO.

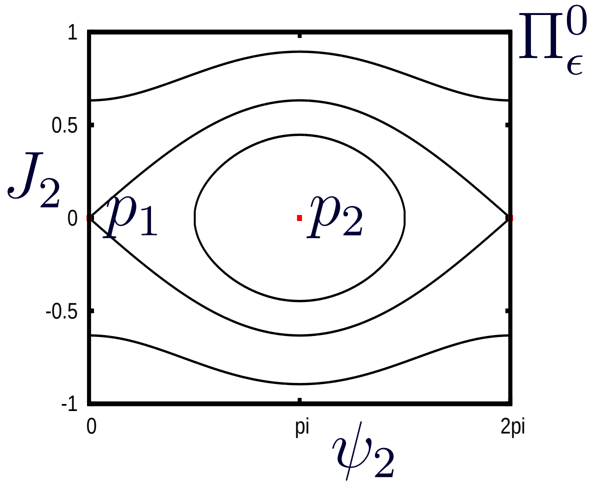
Any $\mathbf{p} \in \ell_k$ is homoclinic to C_h^ϵ with $h = H(\mathbf{p})$. Then,

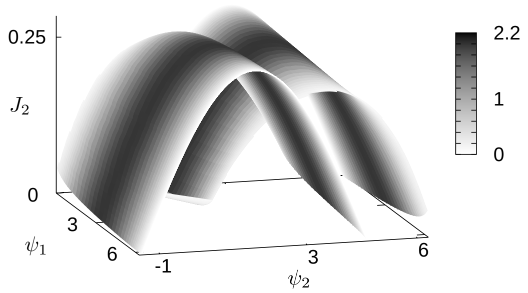
$$\text{transversality inside } \{ H = h \} \Leftrightarrow T_{\mathbf{p}}\ell_k \notin T_{\mathbf{p}}\{ H = h \}. \Leftrightarrow$$

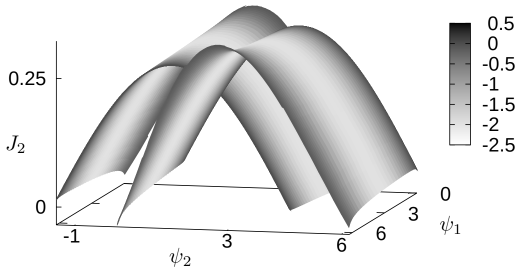
$$h \neq 1 + \epsilon(-1)^k \cos \pi a_2 + \mathcal{O}(\epsilon^2),\ k = 0, 1.$$

Remark. If $a_2 \notin \mathbb{Z} \Rightarrow H(p_2) < E_k < H(p_1)$, $k = 1, 2$ (inside the pendulum within Π_ϵ^0).

Dynamics on the NHIM and transversality (II)







$\epsilon = 0.1$, $a_2 = 0.25$, $d = 0.5$ and $h = 1 + \epsilon$

Assume $a_2 \notin Z$:

- Consider $h = H(p_1) = 1 + \epsilon$ then C_h^ϵ are the separatrices.
  - $C_0^h = \{J_2 = 0\}$ (line of fixed points) $\Rightarrow W^u(C_h^\epsilon)$ and $W^s(R_k(C_h^\epsilon))$ intersect transversally because $A(0) \neq 0$ and $H(p_1) > E_k$.
  - Melnikov (Kovacic) or slow-fast analysis (Haller) $\Rightarrow$ angle is $\mathcal{O}(\sqrt{\epsilon})$.
- Consider $h < H(p_1)$: initially $\ell_k \cap W^u(C_h^\epsilon)$ consists of 2 homoclinic points which collide when $h = E_k$ and disappear $\Rightarrow$ there are no primary homoclinic orbits to p_2 (EH) because $E_k > H(p_2)$.

Non-analyticity of Π_ϵ: a numerical experiment

Normal hyperbolicity theory $\Rightarrow \Pi_\epsilon$ is (at most) $\mathcal{C}^r$ with $r = \tilde{\lambda}_1/\tilde{\lambda}_2$ the quotient of the normal and tangent maximal Lyapunov exponents.

Q: For concrete parameters, can we easily detect the lack of analyticity?

Notations:

- Consider p_1 (HH fixed point). We define

$$W^{u,\Pi_\epsilon^0}_{\text{slow}}(p_1) = W^u(p_1) \cap \Pi_\epsilon^0$$

  (the separatrices of the pendulum inside Π_ϵ^0)

- We say that an 1D invariant submanifold $W^u_{\text{slow}}(p_1) \subset W^u(p_1)$ is a slow unstable invariant manifold if it is tangent at p_1 to the eigenvector related to the slow eigenvalue $\tilde{\lambda}_2$.

  There are infinitely many slow unstable invariant manifolds (in particular, $W^{u,\Pi_\epsilon^0}_{\text{slow}}(p_1)$).

Basic idea

Denote by X_H the vector field related to

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2).$$

$\rightarrow$ In these coord. Π_0^0 is given by $\psi_1 = 0,\, J_1 + a_2 J_2 = 0$.

$\rightarrow$ Denote by $\tilde{\Lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ the unstable eigenvalues of $DX_H(p_1)$.

Basic observations:

1. *If Π_ϵ^0 is an analytic manifold then $W^{u,\Pi_\epsilon^0}_{\text{slow}}(p_1)$ is analytic.*
2. *If $\tilde{\Lambda}$ is non-resonant, then there is a unique analytic slow unstable invariant submanifold $W^{u}_{\text{slow}}(p_1)$ which will be denoted by $W^{u,\omega}_{\text{slow}}(p_1)$. On the other hand, if $\tilde{\Lambda}$ is resonant then generically all slow unstable submanifolds are non-analytic.*

Proof of the basic observations

1. Assume Π_ϵ^0 analytic $\Longrightarrow H|_{\Pi_\epsilon^0}$ is an analytic Hamiltonian which has a non-degenerated saddle fixed point with eigenvalues $\pm\tilde{\lambda}_2$ $\Longrightarrow W_{\text{slow}}^{u,\Pi_\epsilon^0}(p_1)$ is a 1D analytic inv. manifold.
2. The restriction X of the vector field (v.f.) to $W^u(p_1)$ is a 2D analytic v.f. with $(0,0)$ as a repulsive fixed point with eigenvalues $\tilde{\Lambda}$.
   - → If $\tilde{\Lambda}$ non-resonant X is conjugated to $\dot{s}_1 = \tilde{\lambda}_1 s_1$, $\dot{s}_2 = \tilde{\lambda}_2 s_2$.
     Solutions: $s_1 = C s_2^{\tilde{\lambda}_1/\tilde{\lambda}_2}$ (or $s_2 = \hat{C} s_1^{\tilde{\lambda}_2/\tilde{\lambda}_1}$) non-analytic except $\boxed{s_1 = 0}$ and $s_2 = 0$.
   - → If $\tilde{\Lambda}$ resonant (i.e. $\tilde{\lambda}_1 = k\tilde{\lambda}_2$, $k \in \mathbb{N}$) X is conjugated to

$$\begin{aligned} \dot{s}_1 &= \tilde{\lambda}_1 s_1 + \nu s_2^k, \\ \dot{s}_2 &= \tilde{\lambda}_2 s_2. \end{aligned}$$

     Solutions: $s_1 = \nu s_2^k(\log(s_2) + C)/\tilde{\lambda}_2$ non-analytic except $s_2 = 0$.

How to proceed

Conclusion:

- If $\tilde{\Lambda}$ is resonant then Π_ϵ^0 is generically non-analytic.
- If $\tilde{\Lambda}$ is non-resonant then the analyticity of Π_ϵ^0 implies $W^{u,\Pi_\epsilon^0}_{\text{slow}}(p_1) = W^{u,\omega}_{\text{slow}}(p_1)$ by uniqueness of the analytic invariant unstable manifold.

To check non-analyticity of Π_ϵ^0:

1. Consider the generic non-resonant case.
2. Numerically check that the analytic $W^{u,\omega}_{\text{slow}}(p_1)$ leaves the cylinder Π_ϵ^0.
3. It follows that $W^{u,\Pi_\epsilon^0}_{\text{slow}}(p_1) \neq W^{u,\omega}_{\text{slow}}(p_1)$ and, consequently, that Π_ϵ^0 is non-analytic.

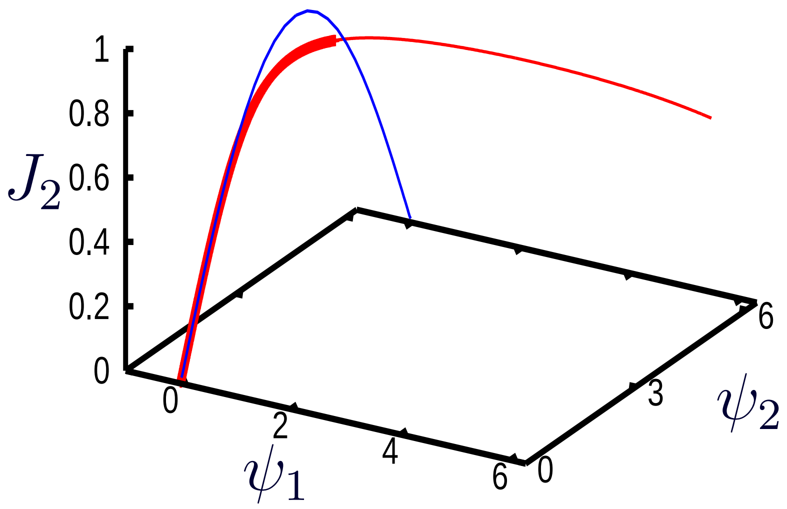
Remark. The tangency order between the analytic solution $s_1 = 0$ and the other solutions of $\dot{s}_1 = \tilde{\lambda}_1 s_1$, $\dot{s}_2 = \tilde{\lambda}_2 s_2$ is $r_* = \tilde{\lambda}_1/\tilde{\lambda}_2 \Rightarrow$ necessary to approximate $W^{u,\omega}_{\text{slow}}(p_1)$ up to order $k > r_*$.

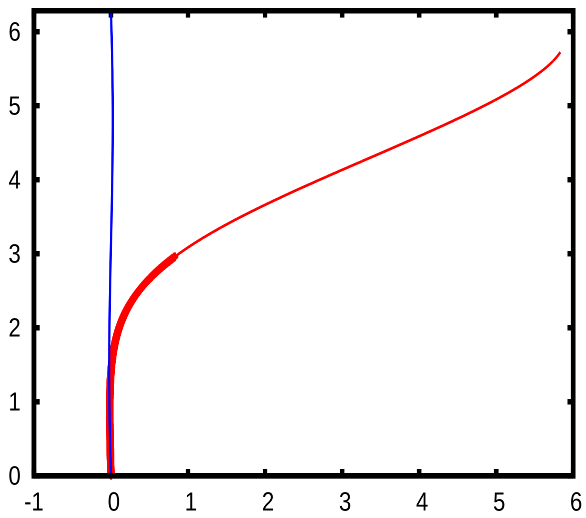
Example

- Parameters: $a_2 = 0.25$, $a_3 = 0.5625$ and $\epsilon = 0.1$.
- $\tilde{\Lambda}$ non-resonant ($\tilde{\lambda}_1 \approx 1.003282954$ and $\tilde{\lambda}_2 \approx 0.222875109$, then $r_* \approx 4.501547788$).
- Compute the parametric representation $\psi_i(s)$, $J_i(s)$ of $W^u_{\text{slow}}(p_1)$.

$$\psi_i(s) = \sum_{k\geq 1} \psi_k^{(i)} s^k, \; J_i(s) = \sum_{k \geq 1} J_k^{(i)} s^k, \; i = 1, 2$$

- Truncating at order $k = 120$ the approximation of $W^u_{\text{slow}}(p_1)$ has an error below 10^{-15} for $|s| < s_* \approx 0.3635$.





A posteriori check: Π^0_ϵ is C^4 but not C^5

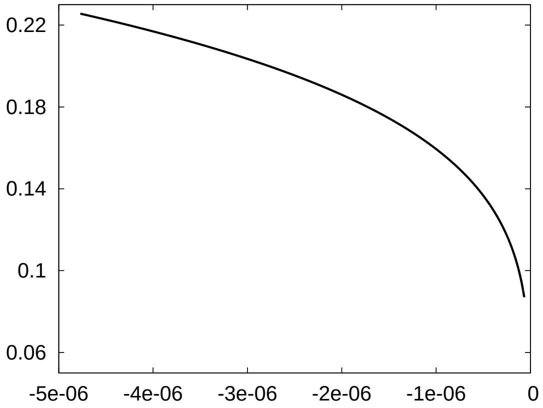
We compute $W^u(p_1)$. Parametrization:

$g(s_1, s_2) = (\psi_1(s_1, s_2), \psi_2(s_1, s_2), J_1(s_1, s_2), J_2(s_1, s_2))$ where

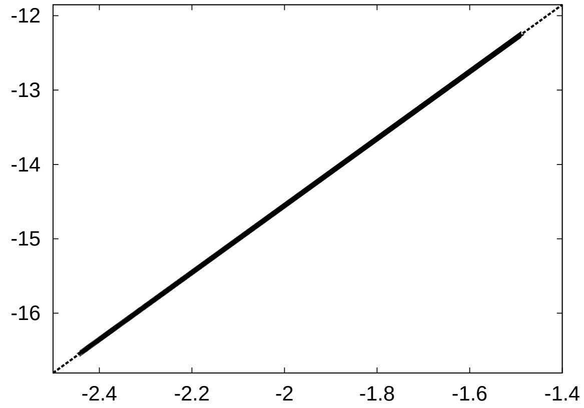
$\psi_i(s_1, s_2) = \sum_{k,l \geq 1} \psi^{(i)}_{k,l} s_1^k s_2^l$, $J_i(s_1, s_2) = \sum_{k,l \geq 1} J^{(i)}_{k,l} s_1^k s_2^l$, $i = 1, 2.$

At order 20: error below 10^{-15} for $s = (s_1, s_2)$ s.t. $\|s\| < s_* \approx 0.282.$

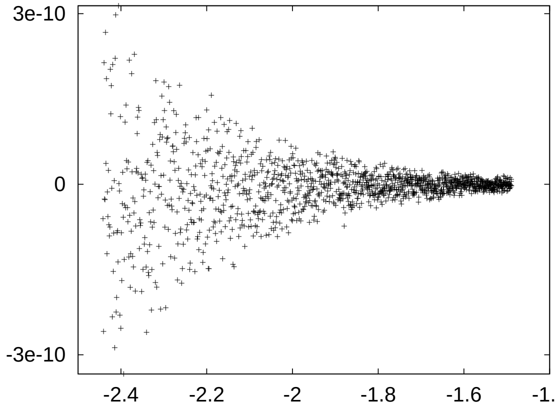
- $s_1 = 0$ corresponds to $W^{u,\omega}_{\text{slow}}(p_1)$ (analytic)
- We compute the non-analytic $1D$ submanifold of $W^u(p_1)$ within Π^0_ϵ by imposing that the homoclinic has $\psi_1 = 0$.



s_1, s_2



$\log s_2, \log s_1$



$\log s_2, |\log s_1 - f(\log s_2)|$

$f(x) = r_* x - 5.55007880852$

Beyond NF: a 4D map

Truncated NF: $H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2)$

Generating function: $S(\psi_1, \psi_2, J_1, J_2) = \psi_1 \bar{J}_1 + \psi_2 \bar{J}_2 + \delta \mathcal{H}(\psi_1, \psi_2, J_1, J_2)$.

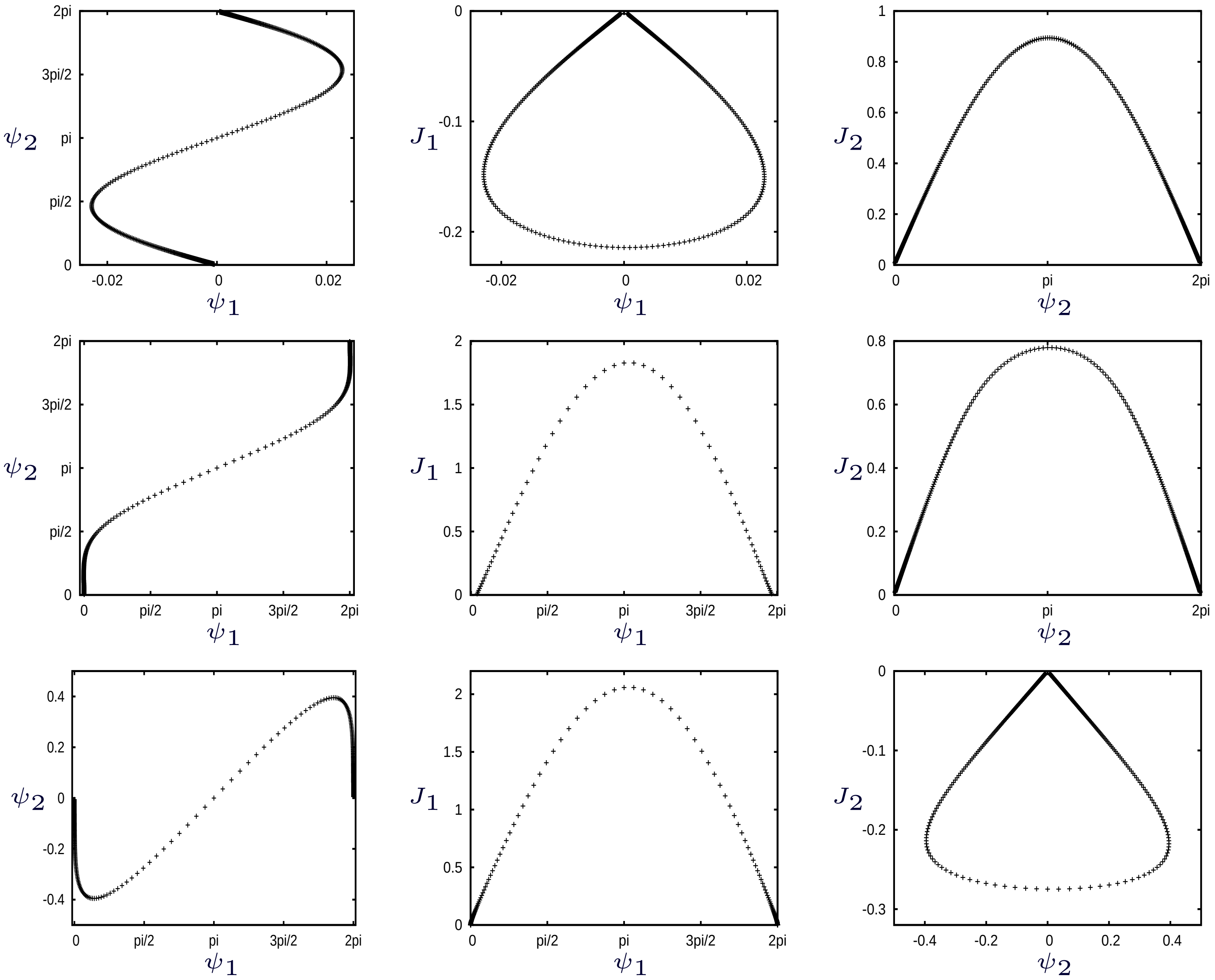
$$T_\delta : \begin{pmatrix} \psi_1 \\ \psi_2 \\ J_1 \\ J_2 \end{pmatrix} \to \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{J}_1 \\ \bar{J}_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2) \\ \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2) \\ J_1 + \delta \sin(\psi_1) \\ J_2 + \delta \epsilon \sin(\psi_2) \end{pmatrix}$$

Phase space structure similar to H (but the homoclinic trajectories split!):

- 4 fixed points: p_1 HH, p_2 HE, p_3 EH, p_4 EE.
- NHIM $\Pi^0_{\epsilon,\delta}$.
- Reversible: $R_1 = (-\psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2)$,
  $R_2 = (2\pi - \psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2)$ and $R_3 = (2\pi - \psi_1, -\psi_2, \bar{J}_1, \bar{J}_2)$.

The homoclinic trajectories

Reversibilities $\Rightarrow T$ has 6 primary homoclinic trajectories.



"Pendulum" separatrix in $\Pi^0_{\epsilon,\delta}$

$$\delta = 0.1$$

$$\epsilon = 0.1$$

$$a_2 = \frac{1}{4}$$

$$a_3 = \frac{9}{16}$$

Splitting of the invariant manifolds

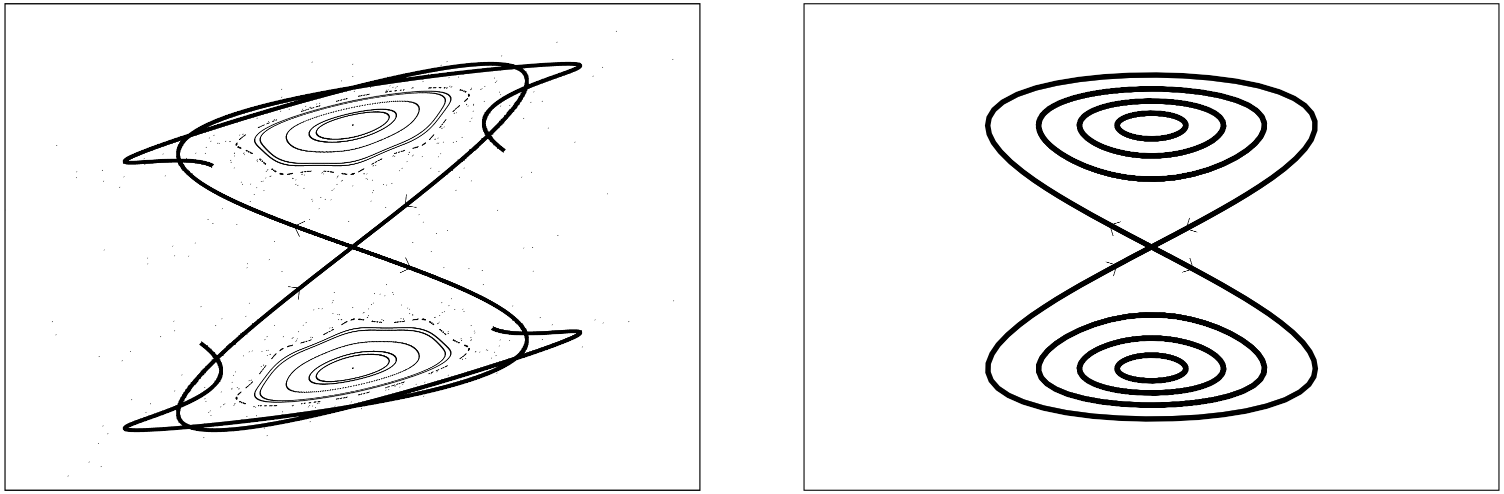
To measure the splitting of the manifolds $W^{u,s}(p_1)$, at the homoclinic point p_h on Σ_{R_k}, $k = 1, 2, 3$, we compute the volume V defined as follows:

- Let
  $(\psi_1, \psi_2, J_1, J_2) = (G_1(s_1, s_2), G_2(s_1, s_2), G_3(s_1, s_2), G_4(s_1, s_2))$,
  $s_1, s_2 \in \mathbb{R}$, be the parametrisation of the $2D$ local $W^u(p_1)$.
- Consider (s_1^h, s_2^h) such that the point with coordinates
  $(G_1(s_1^h, s_2^h), G_2(s_1^h, s_2^h), G_3(s_1^h, s_2^h), G_4(s_1^h, s_2^h))$ belongs to the homoclinic trajectory defined by p_h.
- V is the volume defined by the unitary tangent vectors $v_j = \tilde{v}_j / \|\tilde{v}_j\|$
  where
  $\tilde{v}_1(s_1^h, s_2^h) = (\partial G_i / \partial s_1)(s_1^h, s_2^h)$, $\tilde{v}_2(s_1^h, s_2^h) = (\partial G_i / \partial s_2)(s_1^h, s_2^h)$,
  $\tilde{v}_3(s_1^h, s_2^h) = R_1(\tilde{v}_1(s_1^h, s_2^h))$ and $\tilde{v}_4(s_1^h, s_2^h) = R_1(\tilde{v}_2(s_1^h, s_2^h))$.

Background from 2D maps

Consider $F:\mathbb{R}^2 \to \mathbb{R}^2$ analytic APM. Assume:

- ⋆ $F(0)=0$ hyperbolic fixed point, Spec=$\{\lambda, 1/\lambda\}$, $\lambda \approx 1+\epsilon$, $\epsilon << 1$.
- ⋆ $F \sim \varphi^H_{t=\log\lambda}$, H is the so-called limit Hamiltonian.



Let $\Gamma(t)$ the separatrix of $H \Longrightarrow \Gamma(t)$ has singularities for $t \in \mathbb{C}\setminus\mathbb{R} \Longrightarrow$

Let τ the closest singularity to the real axis.

Then (generically):

$$\sigma \sim A(\log\lambda)^B e^{-2\pi \,\mathrm{Im}\,\tau/\log\lambda}$$

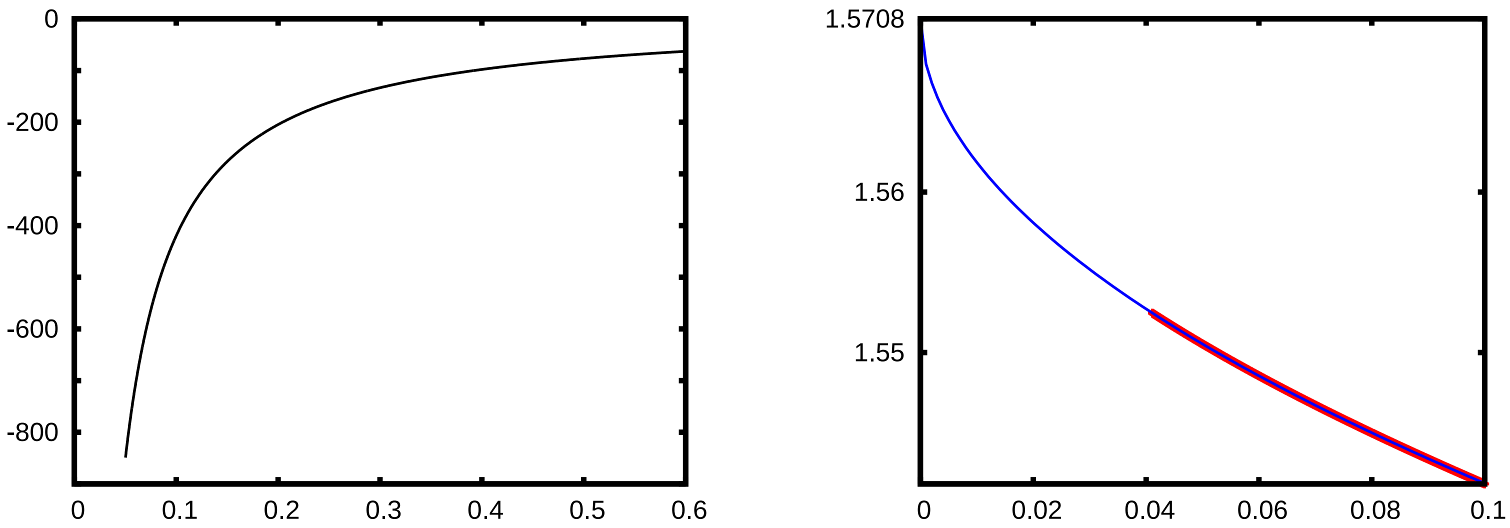
Asymptotic behaviour of V *in* Σ_{R_1}

For a fixed ϵ, a_2 and a_3 parameters we study the behaviour as $\delta \to 0$.

At p_h in Σ_{R_1} (homoclinic trajectory on $\Pi^0_{\epsilon,\delta}$):

$$V \sim A\mu_2^B \mathrm{e}^{-2\pi \operatorname{Im} \hat{\tau}_2/\mu_2}$$

where $\mu_2 = \log \tilde{\lambda}_2$, $A, B \in \mathbb{R}$ and $\hat{\tau}_2 = i\pi/2 + \mathcal{O}(\sqrt{\epsilon})$ is related to the "closest" singularity τ_2 of the homoclinic trajectory of the limit vector field.



$\epsilon = 0.1$, $a_2 = 0.25$, $a_3 = 0.5625$. Left: $\log V$ vs. δ. Right: $\operatorname{Im} \hat{\tau}_2$ vs. ϵ.

Different values of ϵ, a_2 and a_3 have been considered.

Asymptotic behaviour of V in Σ_{R_2} and Σ_{R_3}

In both cases, the volume V behaves like

$$V(\delta) \sim A\mu_1^B \mathrm{e}^{-2\pi \mathrm{Im}\hat{\tau}_1/\mu_1},$$

with $\mu_1 = \log \tilde{\lambda}_1$ and where $\hat{\tau}_1$ is related to the closest singularity τ_1 of the homoclinic trajectory of the limit Hamiltonian flow which has a homoclinic point on the plane $(\psi_1, \psi_2) = (\pi, \pi)$ (or $(\psi_1, \psi_2) = (\pi, 0)$).

→ Note that the limit homoclinic trajectory is not explicitly known.

→ There is no developed theory available (4D!) supporting the previously given asymptotic formulas.

Remark. Our numerics support the fact that $B = -3$ in all the cases and independently of ϵ, a_2 and a_3. Further investigations are needed.

Future directions

- Consider $|\epsilon|$ in a non-perturbative regime (e.g. two resonances of equal order).

  In particular, for $|\epsilon|$ large the EE fixed point can suffer a Hamiltonian-Hopf bifurcation (complex instability).

- Clarify the situations where a different truncated model is obtained, and study the strong doubly resonant cases.

- Analyse the diffusion properties (obtain quantitative data from massive numerical simulations, and relate it with the geometry at the simple/double resonances).

$\longrightarrow$ Work in progress with E.Fontich, V.Gelfreich and C.Simó.

Thanks for your attention!!