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Dynamical System Tools to Navigate in the Earth-Sun System

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Abstract— In this paper we focus on the motion of a solar sail in the Earth-Sun system, where we will show how to use the information on the natural dynamics of the system to navigate around it in a controlled way. We use the Restricted Three Body Problem (RTBP) including the Solar Radiation Pressure (SRP) as a model for the motion of the solar sail. This model has a family of "artificial" equilibrium points parametrised by the orientation of the sail. Most of theme are unstable, we will use their stable and unstable manifolds to drive the trajectory along this family of points.

1 Introduction

Solar Sails are a form of spacecraft propulsion that takes advantage of the Solar Radiation Pressure (SRP) to propel a satellite. By providing a satellite with a large and highly reflecting ultra-thin mirrors, the impact and further reflection of the photos emitted by the Sun on the mirrors will accelerate the probe. This acceleration will be small but unlimited, allowing new and challenging mission concepts [1, 2] as we will see in this paper.

In this paper we focus on the motion of a Solar sail in the vicinity of the Earth. As a model for its motion we use the classical circular Restricted Three Body Problem (RTBP) taking the Earth and the Sun as primaries and adding the effect of the Solar Radiation Pressure (SRP) due to the sail. Where the acceleration given by the solar sail will depend on three parameters: β the sail lightness number which measures the effectiveness of the sail; and two angles $\alpha,\ \delta$ which define the orientation of the sail.

It is well know that the RTBP (when the SRP is not included) has five equilibrium points, $L_{1,\dots,5}$, all of them laying on the ecliptic plane. When we add the SRP these equilibrium points come closer to the Sun, and by changing the sail orientation we can artificially displace their position. These new equilibrium locations result into possible docks for new and challenging mission applications [1, 2]. Most of these equilibrium points are unstable and station keeping strategies are needed to remain close to them [3]. On the other hand we can use their associated stable and unstable manifolds to go from one equilibria to another in a controlled way [4].

In section 2.1 we will describe the family of equilibrium points and their linear dynamics. In section 3 we will show how changes on the sail orientation affect the motion of a solar sail close to a given equilibrium point, and how to use this information to derive simple strategies for the station keeping (section 3.2) or navigating around the family of artificial equilibria (section 3.1) in a controlled way. Finally in section 4 we will apply these strategies to an example mission scenario.

2 Solar Sails in the Earth-Sun system

To model the motion of a solar sail close to the Earth we use the RTBP taking the Earth and Sun as primaries and including the SRP due to sail. Hence, we assume that the two primaries to be point masses that orbit around their mutual centre of mass in a circular due to their mutual gravitational attraction. The solar sail sail is also a point mass that does not affect the motion of the two primaries but is affected by their gravitational attraction as well as the SRP.

Moreover, we take a rotating reference frame where the origin is at the Earth-Sun centre of mass and such that the x-axis is along the line joining the two primaries, the z-axis is perpendicular to the orbital plane and the y-axis completes an orthogonal positive oriented reference system. We also take normalised units of mass, distance and time such that the total mass of the system is 1, the Earth-Sun distance is 1, and their orbital period is 2π. In this units the univer-

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sal gravitational constant is $G = 1$, the mass of the Earth is $\mu = 3.00348060100486 \times 10^6$ and $1 - \mu$ corresponds to the mass of the Sun.

Hence, the equations of motion are given by:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x-\mu+1}{r_{pe}^3} + a_x, \\ \ddot{y} + 2\dot{y} &= y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) y + a_y, \\ \ddot{z} &= -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) z + a_z, \end{aligned} \tag{1}$$

where $\vec{a} = (a_x, a_y, a_z)$ is the acceleration given by the solar sail and $r_{ps} = \sqrt{(x-\mu)^2 + y^2 + z^2}$, $r_{pe} = \sqrt{(x-\mu+1)^2 + y^2 + z^2}$ are the Sun-sail and Earth-sail distances respectively.

As a first approach we consider the solar sail to be flat and perfectly reflecting. Then the acceleration due to the Solar Sail will depend its performance and orientation and is given by:

$$\vec{a}_{sail} = \beta \frac{m_s}{r_{ps}} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}. \tag{2}$$

We call **sail lightness number** to the constant β defined as the SRP ratio in terms of the Sun's gravitational attraction, and gives an idea on the performance of the Solar Sail: $\beta = 1.53/\sigma$ where $\sigma =$ mass/area (kg/m^2) of the spacecraft.

The **sail orientation** is given by the normal direction to the surface of sail, $\vec{n}$, which is parametrised by two angles α and δ that measure the vertical and horizontal displacement with respect to the Sun-Sail direction $\vec{r}_s = (x-\mu, y, z)/r_{ps}$.

2.1 Artificial Equilibrium Points

It is well known that the RTBP has 5 equilibrium points, $L_1, \dots, L_5$, when we include the SRP and vary the sail orientation we displace their position. In other words, for small values of β the five equilibrium points are replaced by five families of artificial equilibria parametrised by the angles defining the sail orientation [5, 6].

In particular, if we consider the solar sail to be perpendicular to the Sun-sail line, the points $L_1, \dots, L_5$ are displaced towards the Sun. If we move the sail orientation right or left w.r.t. r_s (i.e. changing α) we will displace the equilibria right/left w.r.t. the Sun-L_i line, and moving up or down the orientation w.r.t. r_s (i.e. changing δ) we displace them up/down w.r.t. the ecliptic plane.

In Figure 1 (top) we have the family of equilibrium points for $\beta = 0.05$ (a sail lightness number similar to the one for the Sunjammer mission `http://www.sunjammermission.com/`). Each point on the surface corresponds to an equilibrium point for a certain fixed sail orientation. At the bottom of Figure 1 we have a zoom of the region close to the Earth. Notice that we have two disconnected regions, the small one corresponds to the equilibria related to L_2, and the other to the equilibria close to L_1.

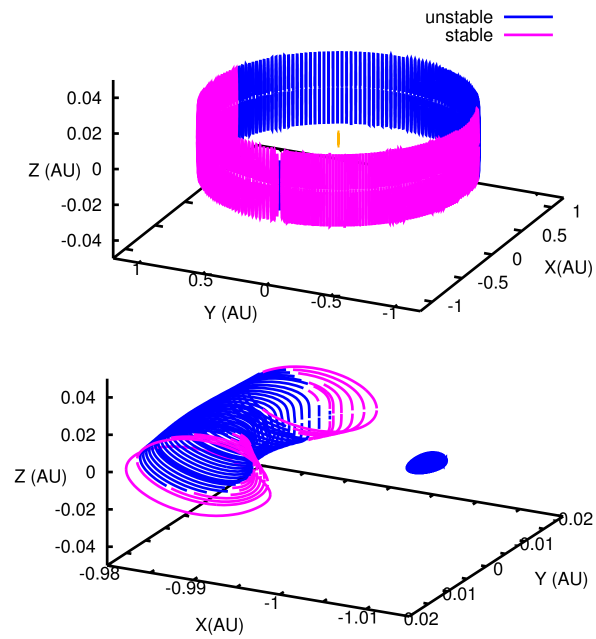

Figure 1: Family of equilibria for $\beta = 0.05$

If we look at the stability of the different equilibrium points, those that are close to $L_{1,2,3}$ are unstable (blue points) and their linear dynamics is of the type saddle×sink×source, and the main instability is given by the saddle as the real component of the complex eigenvalues is very small compared to the positive real eigenvalue. On the other hand, the equilibria close to $L_{4,5}$ are practically stable (purple points) as their linear dynamics is cross products of sinks and source where the real part of the real eigenvalues is small [3].

3 Moving around the Family of Equilibria

Our goal is to derive simple strategies to: (a) remain close to the unstable equilibria and (b) navigate along the family of equilibria in a controlled way. We will use the information on the natural dynamics of the system to derive these strategies. We will focus only on the unstable equilibria (blue points in Figure 1), whose linear dynamics is close to saddle×centre×centre, and take advantage of the unstable manifolds to move along the system.

Hence, if a solar sail is close to one of these equilibrium points, p_0, with a fixed sail orientation α_0, δ_0, the trajectory will escape along the unstable manifold while rotating in the other two centre directions. If we change the sail orientation, the position of the equilibrium point is displaced as well as its stable and unstable directions. Now the trajectory will escape along the new unstable manifold. If we choose an appropriate new sail orientation we can make the sail come back to the original equilibrium point, p_0, or to surf towards a new equilibria p_1.

We do not have an explicit expression, $p(\alpha, \delta)$, for the position of the equilibrium points as a function of the sail orientation. But for a given point $p_0 = p(\alpha_0, \delta_0)$, if $|\alpha - \alpha_0|$ and $|\delta - \delta_0|$ are small, the position of the equilibrium points is well approximated by:

$$(3) \quad p(\alpha, \delta) = p(\alpha_0, \delta_0) + \frac{\partial p}{\partial \alpha}(\alpha - \alpha_0) + \frac{\partial p}{\partial \delta}(\delta - \delta_0),$$

where $\dfrac{\partial p}{\partial \alpha}$ and $\dfrac{\partial p}{\partial \delta}$ can be easily computed numerically.

In order to decide which new sail orientation, (α_1, δ_1), places the equilibrium point, $p_1 = p(\alpha_1, \delta_1)$, such that its unstable manifold drives the trajectory where we want to go, we need an appropriate reference system. This reference system must contain information on the relative position of the trajectory with respect to the stable and unstable manifolds. For this purpose we use a reference frame centered at the equilibrium point p_0 and defined by its eigenvectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6\}$. Hence, the position of the sail at time t, $\phi(t)$, is written as:

$$(4) \qquad \phi(t) = p_0 + \sum_{i=0}^{6} s_1(t)\vec{v}_i,$$

where $\vec{v}_1, \vec{v}_2$ are the unstable and stable directions, $\vec{v}_3, \vec{v}_4$ define the first centre projection and $\vec{v}_5, \vec{v}_6$ the second centre projection.

We use equation (3) in the reference frame defined by equation (4) to find the appropriate sail orientation for our purpose. Where $p(\alpha, \delta)$ is where we want to have the equilibrium point in coordinates $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4, \bar{s}_5, \bar{s}_6)$. Notice that there are more equations than unknowns, which corresponds to the fact that we do not have equilibrium points in arbitrary places. We will use the least-squares method to solve the system and have a fixed point close to the desired positions. We might have to add some restrictions when we solve the system in order to guarantee that the trajectory behaves as expected.

In Figure 2 we try to sketch the effect on the sail trajectory of two new equilibrium point (p_1, p_2), that can appear when we change the sail orientation, in the saddle×centre×centre reference frame defined in equation (4). In blue we have the trajectory of the sail close to p_0 for $\alpha = \alpha_0$, $\delta = \delta_0$, and the dashed line represents the variation of the equilibria with respect to one of the sail angles. The lines in red and green represent the effect on the sail trajectory for two given changes on the sail orientation. The red curve corresponds to the effect of having p_1 as the new equilibria, notice how here the trajectory in the saddle component comes back to the stable manifold of p_0 and how the trajectory in the centre components tries to come close to p_0. On the other hand, the green curve corresponds to the effects of p_2, here the trajectory on the saddle direction continues to escape from p_0 as well as in the two centre projections. Situations like the one generated by p_1 are interesting for station keeping strategies while situations like the one given by p_2 are interesting for surfing strategies.

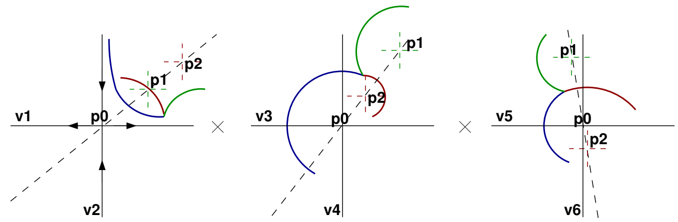

Figure 2: Sketch for possible effects on the sail trajectory for changes on the sail orientation.

3.1 Surfing strategies

Let us briefly describe the main ideas behind the surfing scheme, where the goal is to go from one point p_{ini} to another p_{end}.

Before we start we need to know the position of the final point p_{end} in the reference frame centred at p_{ini} and draw an imaginary line that goes from one point to the other. As it might be not feasible to get from p_{ini} to p_{end} with one change of the sail orientation, we will need to get a sequence of points, q_i (i.e. a sequence of changes in the sail orientation (α_i, δ_i)) to arrive to p_{end}. In order not to loose information each time we change the sail orientation we will recompute the reference frame to be centre around q_i and draw the imaginary line between q_i and p_{end}

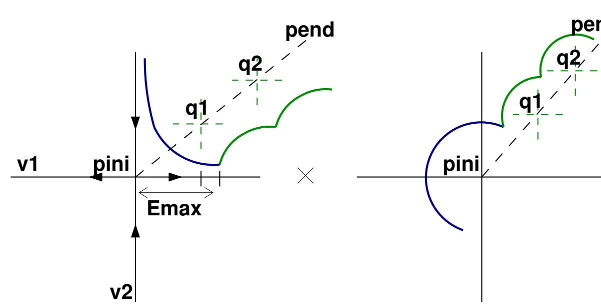

Figure 3: Sketch of the surfing strategy on the saddle and centre projections of the trajectory.

Hence, imagine we are close to a certain equilibrium point q_i. Then the trajectory will escape along the unstable direction, and rotate around the centre projections. When the trajectory is far away from the q_i, that is to say that $|s_1(t^*)| > \varepsilon_{max}$ we will change the sail orientation. We will choose a new orientation such that the new point q_{i+1} satisfies $|\bar{s}_1| < d \cdot \varepsilon_{max}$ with $d < 1$, and $||(\bar{s}_3, \bar{s}_4)||_2 > ||(s_3(t^*), s_4(t^*))||_2$ and $||(\bar{s}_5, \bar{s}_6)||_2 > ||(s_5(t^*), s_6(t^*))||_2$. In other words, the new unstable manifold takes the trajectory towards p_{end}, and the trajectory in the centre projection must also moves towards p_{end}. In Figure 3 we can see a sketch of these phenomena.

We will repeat this until we get close to the final point. We can play with the parameters ε_{max} and d in order to remain close to the surface of equilibria or to control the surfing speed.

3.2 Station Keeping strategies

Let us briefly describe the ideas behind the station keeping scheme, where the goal is to remain close to the equilibrium point p_{ini} for a long time.

We will always start with a reference frame centred around the equilibrium point we want to remain close (p_{ini}). As before the trajectory will escape along the unstable direction and rotate in the centre projections. When the trajectory is far from p_{ini} (i.e $|s_1(t^*)| > \varepsilon_{max}$) we will change the sail orientation. Now we will chose a new orientation such that the new equilibrium point, q_i, satisfies $|\bar{s}_1| > d \cdot \varepsilon_{max}$ with $d > 1$, and $||(\bar{s}_3, \bar{s}_4)||_2 < ||(s_3(t^*), s_4(t^*))||_2$ and $||(\bar{s}_5, \bar{s}_6)||_2 < ||(s_5(t^*), s_6(t^*))||_2$. In other words, the new unstable manifold must take the trajectory towards the stable manifold of p_{ini} and the centre projections must remain bounded. Once the trajectory comes close to p_{ini} (i.e. $|s_1(t^*)| < \varepsilon_m in$) we will restore the original sail orientation. In Figure 4 we have a sketch of this phenomena.

We will repeat this as long as we want to remain close to p_{ini}. By playing with the parameters ε_{max} and d we can modify how far away we can get from the orbit and the time it takes the sail to come back.

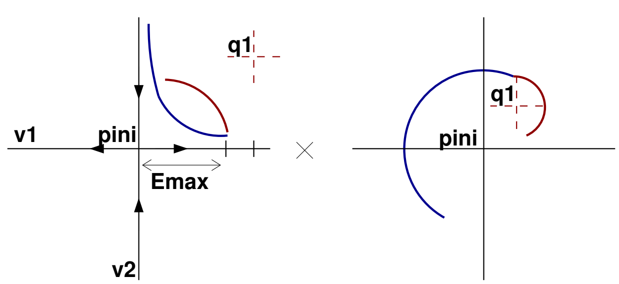

Figure 4: Sketch of the station keeping strategy on the saddle and centre projections of the trajectory.

4 Test Mission

As a test mission we propose a round tour visiting 4 equilibrium points on the surface of equilibria combining the two strategies proposed in the previous section. The 4 points we want to visit are displaced 5° from the Earth-Sun line, two of them above and below the ecliptic plane (p_1, p_3) and the other two left and right from the Earth-Sun (p_0, p_2), forming a rhombus with the Sun in the middle if you look at them from the Earth.

We have considered a solar sail performance $\beta = 0.051689$, which corresponds to the sail lightness number for the Sunjammer mission (a sail with $\approx$ 32kg of payload mass and an area of $38 \times 38\text{m}^2$). In Table 1 we have the position of the 4 equilibrium points we want to visit and their corresponding sail orientation.

	x	y	z
p_1	-9.79998e-01	1.81889e-03	0.00000e+00
p_2	-9.80036e-01	0.00000e+00	1.73948e-03
p_3	-9.79998e-01	-1.81889e-03	0.00000e+00
p_4	-9.80036e-01	0.00000e+00	-1.73948e-03

	α (deg)	δ (deg)
p_1	-0.74	0.00
p_2	0.00	2.61
p_3	0.74	0.00
p_4	0.00	-2.61

Table 1: Coordinates of the equilibrium points to visit in the example mission and their corresponding sail orientation for equilibria.

We have divided the mission into four stages, where each of them has two different parts. In stage 1 we start close to p_0 and first we surf from p_0 to p_1, once we are close enough to p_1 we use the station keeping algorithm to remain there for two years. Stages 2, 3 and 4 follow the same objective, going from p_1 to p_2 (stage 2), from p_2 to p_3 (stage 3) and from p_3 to p_0 (stage 4). The idea of the mission is to orbit around the solar disc while surfing along the equilibrium points.

When we surf from one point to the other we use the surfing strategy described in section 3.1 and when we control the trajectory to remain close to one of the equilibria for 2 years we use the station keeping strategy described in section 3.2

In Figure 5 we have the trajectory the solar sail follows through the mission, where each colour corresponds to the different stages of the mission.

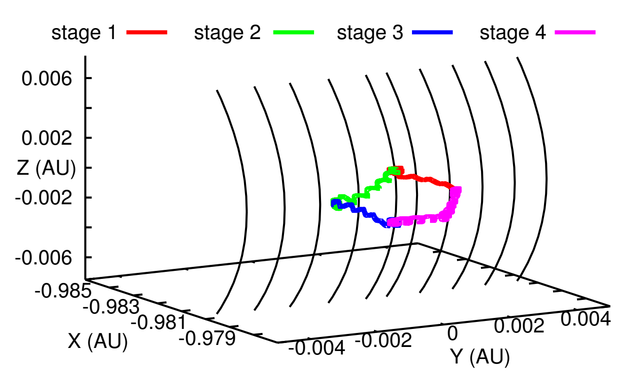

Figure 5: XYZ projection of the solar sail trajectory. Stage 1 in red; Stage 2 in green; Stage 3 in blue; Stage 4 in purple.

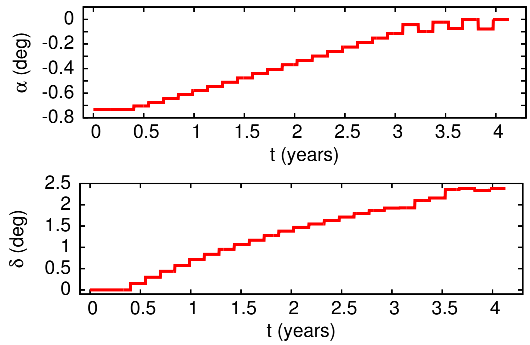

Figure 6: Stage 1, surfing from p_0 to p_1: α, δ variation.

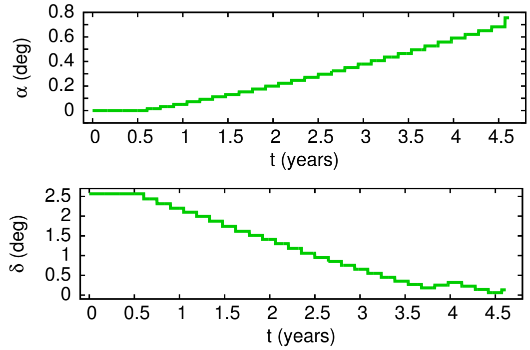

Figure 7: Stage 2, surfing from p_1 to p_2: α, δ variation.

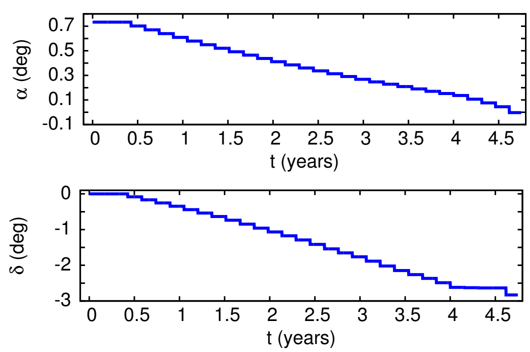

Figure 8: Stage 3, surfing from p_2 to p_3: α, δ variation.

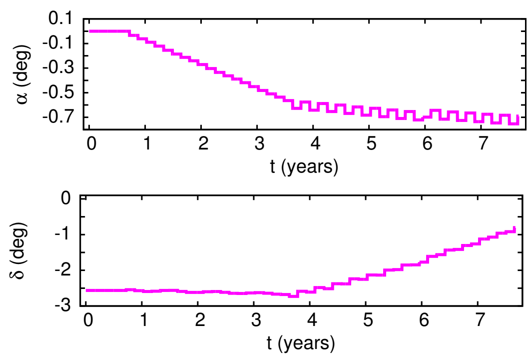

Figure 9: Stage 4, surfing from p_3 to p_0: α, δ variation.

In Figures 6, 7, 8 and 9 we show the variation of the two angles defining the sail orientation during the mission. As we can see the average time to get from one point to the other is around 4 years, except in stage 4 where the sail takes a longer tour as it can be seen in Figure 5. Moreover, notice how at each 'step' the sail orientation gets closer to the orientation required for the target point.

Let us now describe in a little more detail the trajectory for each strategy. For simplicity and due to space limitations we only present results for the two parts of stage 1, as the rest of the stages behave in a very similar way.

4.1 Stage 1A: surfing form p_0 to p_1

On the top-left hand-side of Figure 10 we can see the YZ projection of the first stage of the mission, where we can clearly see how the trajectory gains altitude while moving towards the left until it reaches a vicinity of p_1.

The other three plots in Figure 10 are the different projections of the trajectory in the saddle and centre planes related to p_0, and the black crosses represent the projection of the equilibria, q_i, that play a role in the surfing scheme, these are the equilibrium positions for the different sail orientations that we use. Notice how the trajectory in the saddle projection (top-right) is a succession of saddle arcs, each of them centered around the equilibria q_i. The trajectory on the other two centre directions (bottom) also moves along the family of equilibria q_i and rotates around them. A similar behaviour is observed for the other three stages.

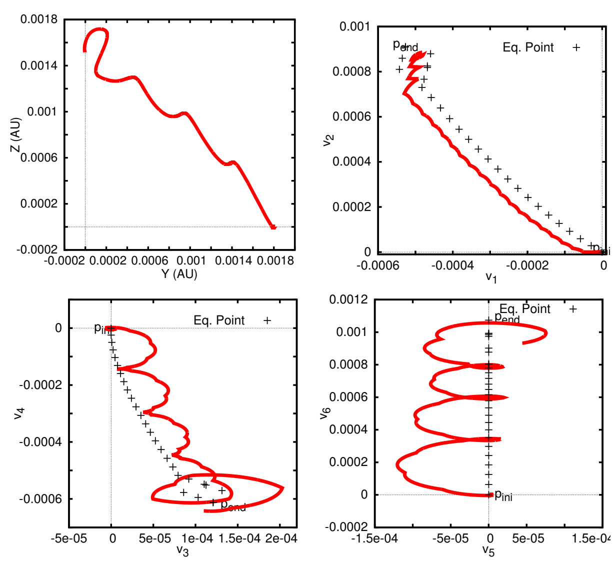

Figure 10: Different projections of the trajectory of the 1st stage surfing form p_0 to p_1.

4.2 Stage 1B: 2 year station keeping

As we have mentioned before, once we are close to the target equilibrium points p_i we apply the station keeping strategy to remain close to them for 2 years. The main idea is to check how well we are surfing along the surface of equilibria, as well as to know if we are able to stay around any of the points if requiered.

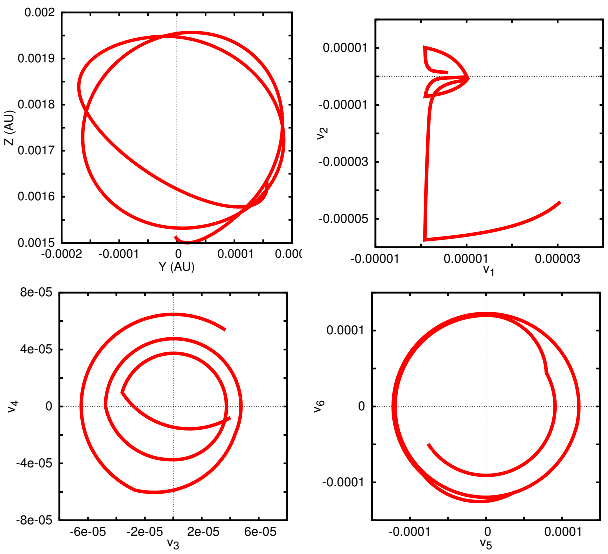

Figure 11: Different projections of the trajectory of the 1st stage control around p_1.

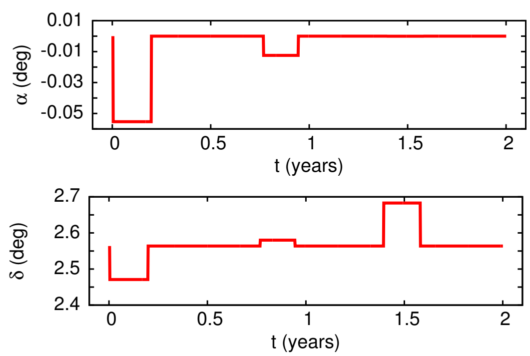

Figure 12: Stage 1, control around p_1: α, δ variation

On the top-left hand side of Figure 11 we have the YZ projection of the trajectory of the solar sail during the control period. As we can see, despite the instability of the region, the trajectory remains close of the equilibrium point for two years.

The other three plots are the projection of the sail trajectory on the saddle and two centre projections around the target point p_1. As we can see, in the saddle projection (top-right) the trajectory is a connection of saddle arcs, where each time the trajectory is escaping along the unstable direction, the sail orientation is changed and the trajectory comes back along the saddle arc of the new equilibria and comes close to the stable direction of p_1. The trajectory in the two centre projections is a connection of rotations around different points, which remains bounded along time.

Finally in Figure 12 we see the variation of sail orientation during the two years.

5 Conclusions

In this paper we have shown how to use the information on the dynamics of a system to navigate around it in a controlled way. The ideas are general enough to be applied to different kind of dynamical systems. We have applied them to the particular case of a satellite propelled by a solar sail in the Earth-Sun system.

We have seen that in the Sun-Earth RTBP with a solar sail we have a surface of equilibrium points, each point corresponding to a certain sail orientation. Some of these equilibrium points are linearly unstable and have a stable and unstable manifolds associated to them. Hence, for a fixed sail orientation, if we are close to equilibria, the trajectory will escape along the unstable manifolds. We also know that when we change the sail orientation the position of equilibria shifts, then the trajectory will escape along the new unstable manifold. If we can compute how these invariant manifolds vary with the sail orientation, we can derive schemes to make the solar sail surf around the system in a controlled way.

We have shown how to derive some of these strategies and applied them to an example mission to surf around the equilibrium points close to L_1. The results show that the strategy is robust and that we are able to surf around the surface of equilibria. Nevertheless, we still need to do a more extensive study in order to minimise the surfing time to go from one point to the other and see how it depends on the parameters in our algorithms.

Acknowledgements

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