Com simular de forma precisa el moviment del nostre Sistema Solar?

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Overview of the Talk

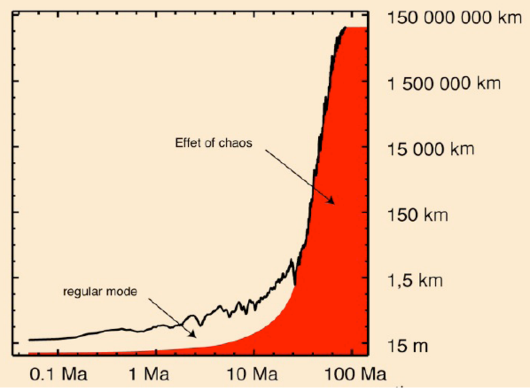
1. *Why do we want long-term integrations of the Solar System ?*
2. *The N-Body Problem (Toy model for the Planetary motion)*
3. *Symplectic Splitting Methods for Hamiltonian Systems*
4. *Conclusions*

GTS_{next}-EU
towards the next generation Geological Time Scale for the last 100 million years

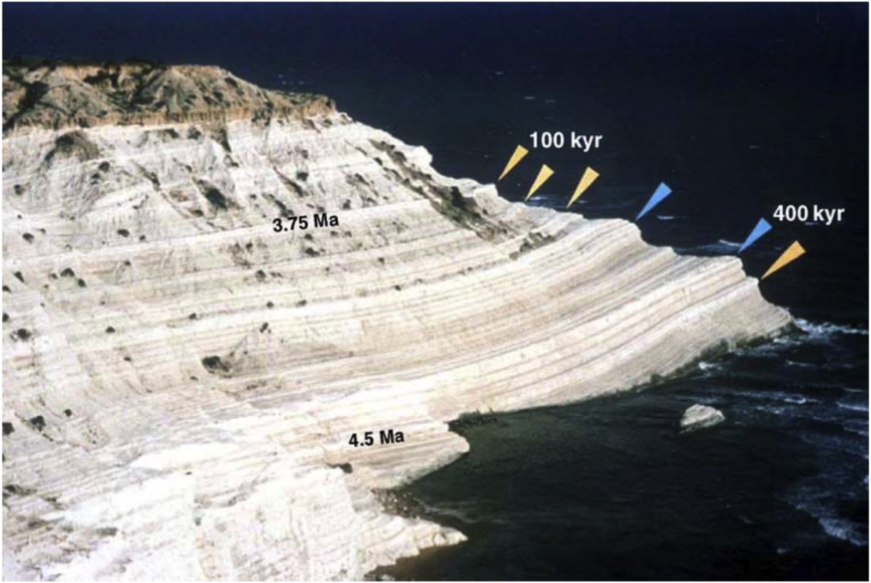
PEOPLE
MARIE CURIE ACTIONS

Chaotic Motion of the Solar System

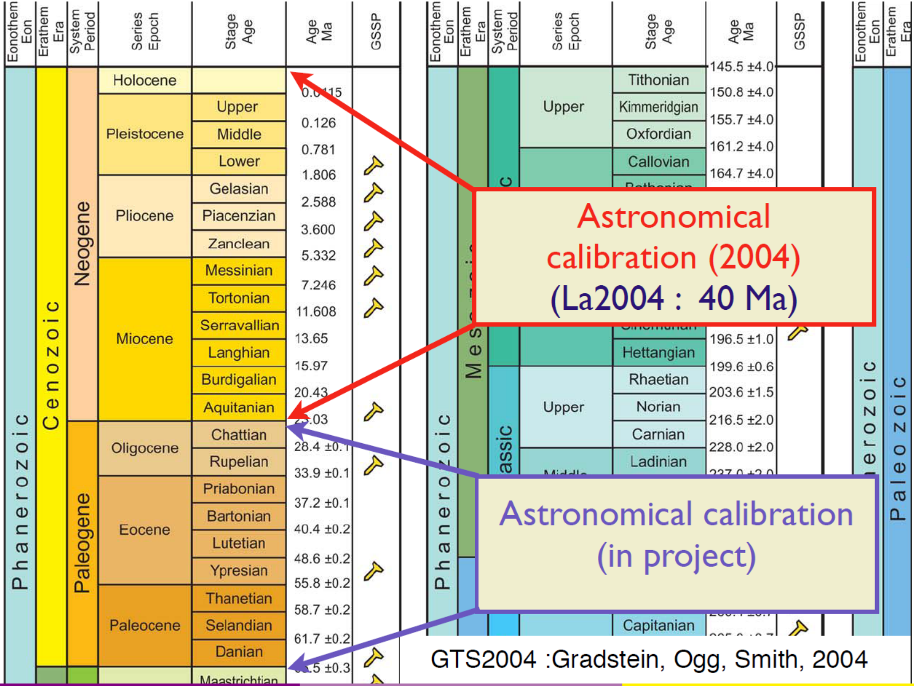
- Secular equations: 200 Ma: Laskar (1989, 1990)
- Direct integration: 100 Ma: Sussman and Wisdom (1992)



$$d(T) \approx d_0 10^{T/10}$$


100 kyr
400 kyr
3.75 Ma
4.5 Ma


Solasaga



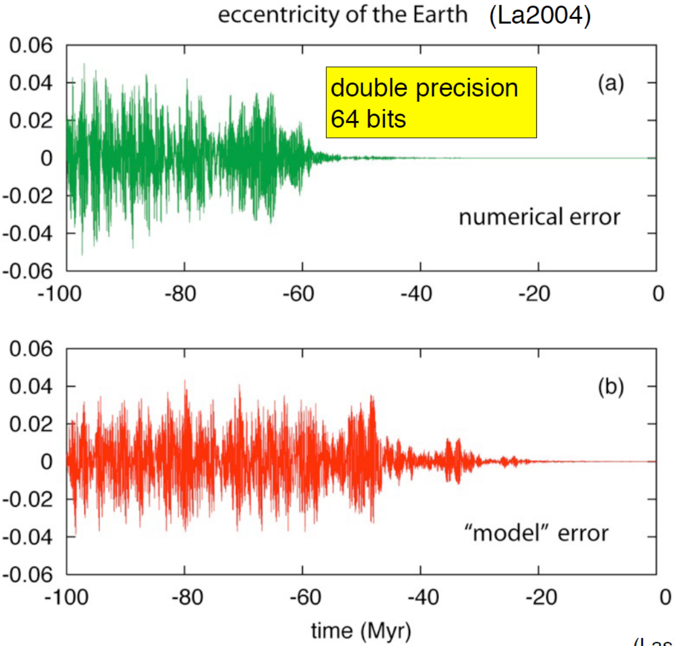
GTS2004 :Gradstein, Ogg, Smith, 2004

Planetary Solution

- La2004 : numerical, simplified,
  tuned to DE406 (6000 yr)

- INPOP : numerical, "complete",
  adjusted to 45000 observations.
  1 Myr : 6 months of CPU.

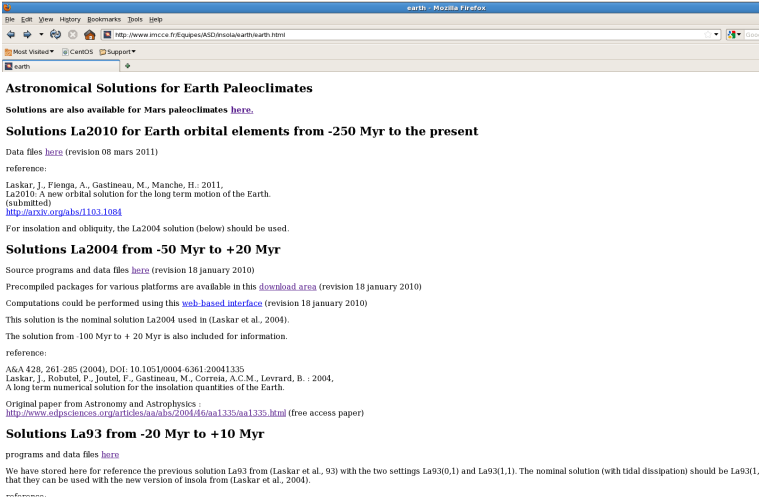
- La2010 : numerical, less simplified,
  tuned to INPOP (1 Myr).
  250Myr : 18 months of CPU.



(Laskar et al, 2004)

For further information

http://www.imcce.fr/Equipes/ASD/insola/earth/earth.html



Astronomical Solutions for Earth Paleoclimates

Solutions are also available for Mars paleoclimates here.

Solutions La2010 for Earth orbital elements from -250 Myr to the present

Data files here (revision 08 mars 2011)

reference:

Laskar, J., Fienga, A., Gastineau, M., Manche, H.: 2011,
La2010: A new orbital solution for the long term motion of the Earth.
(submitted)
http://arxiv.org/abs/1103.1084

For insolation and obliquity, the La2004 solution (below) should be used.

Solutions La2004 from -50 Myr to +20 Myr

Source programs and data files here (revision 18 january 2010)

Precompiled packages for various platforms are available in this download area (revision 18 january 2010)

Computations could be performed using this web-based interface (revision 18 january 2010)

This solution is the nominal solution La2004 used in (Laskar et al., 2004).

The solution from -100 Myr to + 20 Myr is also included for information.

reference:

A&A 428, 261-285 (2004), DOI: 10.1051/0004-6361:20041335
Laskar, J., Robutel, P., Joutel, F., Gastineau, M., Correia, A.C.M., Levrard, B. : 2004,
A long term numerical solution for the insolation quantities of the Earth.

Original paper from Astronomy and Astrophysics :
http://www.edpsciences.org/articles/aa/abs/2004/46/aa1335/aa1335.html (free access paper)

Solutions La93 from -20 Myr to +10 Myr

programs and data files here

We have stored here for reference the previous solution La93 from (Laskar et al., 93) with the two settings La93(0,1) and La93(1,1). The nominal solution (with tidal dissipation) should be La93(1,
that they can be used with the new version of insola from (Laskar et al., 2004).

The Challenge

1. The NUMERICAL PRECISION of the solution. We want to be sure that the precision is not a limiting factor.
2. The SPEED of the algorithm. As La2010a took nearly 18 months to complete 250 Myr.

The N - Body Problem

The N-Body Problem

We consider that we have $n + 1$ particles (n planets + the Sun) interacting between each other due to their mutual gravitational attraction.

We consider:

- $\mathbf{u_0}, \mathbf{u_1}, \dots, \mathbf{u_n}$ and $\dot{\mathbf{u}}_\mathbf{0}, \dot{\mathbf{u}}_\mathbf{1}, \dots, \dot{\mathbf{u}}_\mathbf{n}$ the position and velocities of the $n + 1$ bodies with respect to the centre of mass.
- $\tilde{\mathbf{u}}_\mathbf{i} = m_i \dot{\mathbf{u}}_\mathbf{i}$ the conjugated momenta.

The equations of motion are Hamiltonian:

$$H = \frac{1}{2} \sum_{i=0}^{n} \frac{||\tilde{\mathbf{u}}_\mathbf{i}||^2}{m_i} - G \sum_{0 \leq i < j \leq n} \frac{m_i m_j}{||\mathbf{u_i} - \mathbf{u_j}||}. \quad (1)$$

Notice that the Hamiltonian is naturally split as $H = T(p) + U(q)$.

Heliocentric Coordinates

We consider relative position of each planet (P_i) with respect to the Sun (P_0).

$$\left.\begin{array}{rcl} \mathbf{r_0} & = & \mathbf{u_0} \\ \mathbf{r_i} & = & \mathbf{u_i} - \mathbf{u_0} \end{array}\right\}, \qquad \left.\begin{array}{rcl} \tilde{\mathbf{r}}_\mathbf{0} & = & \tilde{\mathbf{u}}_\mathbf{0} + \cdots + \tilde{\mathbf{u}}_\mathbf{n} \\ \tilde{\mathbf{r}}_\mathbf{i} & = & \tilde{\mathbf{u}}_\mathbf{i} \end{array}\right\},$$

In this set of coordinates the Hamiltonian is naturally split into two part: $H_H = H_{Kep} + H_{pert}$:

$$H_H = \sum_{i=1}^{n} \left(\frac{1}{2} ||\tilde{\mathbf{r}}_\mathbf{i}||^2 \left[\frac{m_0 + m_i}{m_0 m_i} \right] - G \frac{m_0 m_i}{\mathbf{r_i}} \right) + \sum_{0<i<j\leq n} \left(\frac{\tilde{\mathbf{r}}_\mathbf{i} \cdot \tilde{\mathbf{r}}_\mathbf{j}}{m_0} - G \frac{m_i m_j}{\Delta_{ij}} \right),$$

where $\Delta_{i,j} = ||\mathbf{r_i} - \mathbf{r_j}||$.

Jacobi Coordinates

We consider the position of each planet (P_i) w.r.t. the centre of mass of the previous planets ($P_0, \ldots, P_{i-1}$).

$$\left.\begin{array}{rcl} \mathbf{v_0} & = & (m_0\mathbf{u_0} + \cdots + m_n\mathbf{u_n})/\eta_n \\ \mathbf{v_i} & = & \mathbf{u_i} - (\sum_{j=0}^{i-1} m_j\mathbf{u_j})/\eta_{i-1} \end{array}\right\}, \qquad \left.\begin{array}{rcl} \tilde{\mathbf{v}}_\mathbf{0} & = & \tilde{\mathbf{u}}_\mathbf{0} + \cdots + \tilde{\mathbf{u}}_\mathbf{n} \\ \tilde{\mathbf{v}}_\mathbf{i} & = & (\eta_{i-1}\tilde{\mathbf{u}}_\mathbf{i} - m_i(\sum_{j=0}^{i-1} \mathbf{u_j}))/\eta_i \end{array}\right\}.$$

where $\eta_i = \sum_{j=0}^{i} m_j$.

In this set of coordinates the Hamiltonian is naturally split into two part: $H_J = H_{Kep} + H_{pert}$:

$$H_J = \sum_{i=1}^{n}\left(\frac{1}{2}\frac{\eta_i}{\eta_{i-1}}\frac{||\tilde{\mathbf{v}}_\mathbf{i}||^2}{m_i} - G\frac{m_i\eta_{i-1}}{\mathbf{v_i}}\right) + G\left[\sum_{i=2}^{n} m_i\left(\frac{\eta_{i-1}}{||\mathbf{v_i}||} - \frac{m_0}{||\mathbf{r_i}||}\right) - \sum_{0<i<j\leq n}\frac{m_i m_j}{\Delta_{ij}}\right],$$

where $\Delta_{i,j} = ||\mathbf{u_i} - \mathbf{u_j}||$.

Jacobi Vs Heliocentric

In both cases we have $H = H_{Kep} + H_{pert}$. But:

- $H_H = H_A(p,q) + \varepsilon(H_B(q) + H_C(p))$,
- $H_J = H_A(p,q) + \varepsilon H_B(q)$,

where H_A, H_B and H_C are integrable on their own.

Remarks:

- the size of the perturbation in Jacobi coordinates is smaller that the size of the perturbation in Heliocentric coordinates, giving a better approximation of the real dynamics.
- the expressions in Heliocentric coordinates are easier to handle, and do not require a specific order on the planets.

TEST EXAMPLES

- We know that the most massive planet (i.e. the one that makes the size of the perturbation grow) is Jupiter. So simple models (2 - 4 planets) including Jupiter should be considered.
- We also have a problem with the orbital speed of Mercury. Although it is one of the less massive planets it is by far the fastest one. It has a period of 87.9 days and from our results this reduces enormously the optimal step-size.

With this in mind, from now on we will consider the following test examples:

- **4 planets**: Jupiter, Saturn, Uranus and Neptune & Mercury, Venus, Earth and Mars
- **8 planets**: Mercury to Neptune & Venus to Pluto

Jacobi Vs Heliocentric (size of perturbation)

np,case	Heliocentric Pert.	Jacobi Pert.
2, MV	5.264837243090217E-011	2.507597928893501E-011
2, JS	2.336559877558003E-006	8.255625324341979E-007
4, MM	9.165205211655520E-010	6.334248585000000E-010
4, JN	2.718444355584028E-006	8.716288751176844E-007
8, MN	2.804289442433957E-006	8.715850310304487E-007
8, VP	2.802584202262463E-006	8.715856645507914E-007
9, All	2.804292431703275E-006	8.715852470196316E-007

Table: Size of the perturbation in Heliocentric Vs Jacobi coordinates for different type of planetary configurations. 2planets (Merc. and Venus); 2planets (Jup. and Sat.); 4planets (Merc.-Venus-Earth-Mars); 4planets (Jup.-Sat.-Ura.-Nept.); 8planets (Merc. to Nept.); 8planets (Ven. to Plu.); 9planets (Merc. to Plu.)

Symplectic Splitting Methods for Hamiltonian Systems

Splitting Methods for Hamiltonian Systems

Let $H(q, p)$ be a Hamiltonian, where (q, p) are a set of canonical coordinates.

$$\frac{dz}{dt} = \{H, z\} = L_H z, \tag{2}$$

where $z = (q, p)$ and $\{\ ,\ \}$ is the Poisson Bracket ($\{F, G\} = F_q G_p - F_p G_q$).

The formal solution of Eq. (2) at time $t = \tau$ that starts at time $t = \tau_0$ is given by,

$$z(\tau) = \exp(\tau L_H) z(\tau_0). \tag{3}$$

We want to build approximations for $\exp(\tau L_H)$ that preserve the symplectic character.

Splitting Methods for Hamiltonian Systems

The formal solution of Eq. (2) at time $t = \tau$ that starts at time $t = \tau_0$ is given by,

$$z(\tau) = \exp(\tau L_H) z(\tau_0) = \exp[\tau(A + B)] z(\tau_0). \tag{4}$$

where $A \equiv L_{H_A}, B \equiv L_{H_B}$.

We recall that H_A and H_B are integrable, hence we can compute $\exp(\tau A)$ and $\exp(\tau B)$ explicitly.

We will construct symplectic integrators, $S_n(\tau)$, that approximate $\exp[\tau(A + B)]$ by an appropriate composition of $\exp(\tau A)$ and $\exp(\tau B)$:

$$\mathcal{S}_n(\tau) = \prod_{i=1}^{n} \exp(a_i \tau A) \exp(b_i \tau B)$$

Splitting Methods for Hamiltonian Systems

Using the Baker-Campbell-Hausdorff (BHC) formula for the product of two exponential of non-commuting operators X and Y:

$$\exp X \exp Y = \exp Z,$$

with

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X,[X,Y]] - [Y,[Y,X]]) + \frac{1}{24}[X,[Y,[Y,X]]] + \dots,$$

and $[X, Y] := XY - YX$.

This ensures us that is we have an nth order integrating scheme:

$$\prod_{i=1}^{k} \exp(a_i \tau A) \exp(b_i \tau B) = \exp(\tau D_{\tilde{H}}).$$

Then $\tilde{H} = H + \tau^n H_n + o(\tau^n)$ and the error in energy is of order τ^n.

Two simple examples

- $S_1(\tau) = \exp(\tau A)\exp(\tau B)$,

$$K = A + B + \frac{\tau}{2}[A,B] + \frac{\tau^2}{12}\left([A,[A,B]] + [B,[B,A]]\right) + \dots$$

- $S_2(\tau) = \exp(\tau/2A)\exp(\tau B)\exp(\tau/2A)$ (Leap-Frog),

$$K = A + B + \frac{\tau^2}{6}\left([A,[A,B]] + [B,[B,A]]\right) + \dots$$

Many Authors like Ruth(1983), Neri (1987) and Yoshida(1990) among others have found appropriate set coefficientscients a_i, b_i in order to have a High Order symplectic integrator (4th, 6th, 8th, ...).

From now on we will focus on the special case $H = H_A + \varepsilon H_B$, where H_A and H_B are integrable on its own. This is the case of the N-body planetary system, where the system can be expressed as a Keplerian motion plus a small perturbation due to their mutual interaction.

Splitting Methods for Hamiltonian Systems

Let us call $\mathcal{S}_n(\tau) = \exp(\tau K)$. Where,

$$\mathcal{S}_n(\tau) = \prod_{i=1}^{n} \exp(a_i \tau A) \exp(b_i \tau \varepsilon B) = \exp(\tau K), \qquad (5)$$

The BCH theorem ensures us that $K \in L(\{A, B\})$, the Lie algebra generated by A and B, and it can be expanded as a double asymptotic series in τ and ε:

$$\begin{aligned} \tau K \;&=\; \tau p_{1,0} A \;+\; \varepsilon \tau p_{1,1} B \;+\; \varepsilon \tau^2 p_{2,1} [A, B] \\ &+ \varepsilon \tau^3 p_{3,1} [A, [A, B]] \;+\; \varepsilon^2 \tau^3 p_{3,2} [B, [B, A]] \\ &+ \varepsilon \tau^4 p_{4,1} [A, [A, [A, B]]] \;+\; \varepsilon^2 \tau^4 p_{4,2} [A, [B, [B, A]]] \;+\; \varepsilon^3 \tau^4 p_{4,3} [B, [B, [B, A]] + \dots, \end{aligned}$$

where $p_{i,j}$ are polynomials in a_i and b_i.

Splitting Methods for Hamiltonian Systems

If is a splitting method $\mathcal{S}_n(\tau)$ such that $K = A + \varepsilon B + o(\tau^p)$. Then, the coefficients a_i, b_i must satisfy:

$$p_{1,0} = 1, \qquad p_{1,1} = 1, \qquad p_{i,j} = 0, \text{ for } i = 2, \dots, p.$$

Remark:

- It is easy to check that,

$$p_{0,1} = a_1 + a_2 + \cdots + a_n = 1,$$
$$p_{1,1} = b_1 + b_2 + \cdots + b_n = 1.$$

- If $S_n(\tau) = S_n(-\tau)$ then all the terms of order τ^{2k+1} are cancelled out.

Splitting Methods for Hamiltonian Systems

$$\mathcal{S}_n(\tau) = \prod_{i=1}^{n} \exp(a_i \tau A) \exp(b_i \varepsilon \tau B) = \exp(\tau K),$$

In general $\varepsilon \ll \tau$ (or at least $\varepsilon \approx \tau$), so we are more interested in killing the error terms with small powers of ε. We will find the coefficient a_i, b_i such that:

$$|\tau K - \tau(A + \varepsilon B)| = \mathcal{O}(\varepsilon \tau^{s_1+1} + \varepsilon^2 \tau^{s_2+1} + \varepsilon^3 \tau^{s_3+1} + \cdots + \varepsilon^m \tau^{s_m+1}). \qquad (6)$$

Definition

We will say that the method $\mathcal{S}_n(\tau)$ has n stages if it requires n evaluations of $\exp(\tau A)$ and $\exp(\tau B)$ per step-size.

Definition

We will say that the method $\mathcal{S}_n(\tau)$ has order $(s_1, s_2, s_3, \ldots)$ if it satisfies Eq. (6).

Remarks

- The spliting schemes integrate in an exact way (up to machine accuracy) the approximated Hamiltonian $\tilde{H}$.
- We will always integrate using a constant step-size τ such that $H = \tilde{H} + \mathcal{O}(\tau^n)$.
- To compare the different methods we will check the variation of energy $|H(t_0) - H(t)|$ for different step-sizes τ.
- We will use as test examples the N-body problem for different planetary configurations.

$\mathcal{SABA}_n$ or McLachlan (2n,2) methods

McLachlan, 1995; Laskar & Robutel, 2001, considered symmetric schemes that only killed the terms of order $\tau^k \varepsilon$ for $k = 1, ..., 2n$.

$$\mathcal{S}_m(\tau) = \exp(a_1 \tau A) \exp(b_1 \tau B) \dots \exp(b_1 \tau B) \exp(a_1 \tau A).$$

The main advantages are that:

- We only need n stages to have a method of order $(2n, 2)$.
- We can guarantee that for all n the coefficients a_i, b_i will always be positive.

- McLachlan, 1995: "*Composition methods in the presence of small parameters*", BIT 35(2), pp. 258-268.

- Laskar & Robutel, 2001: "*High order symplectic integrators for perturbed Hamiltonian systems*", Celestial Mechanics and Dynamical Astronomy 80(1), 39-62.

$\mathcal{SABA}_n$ or McLachlan (2n,2) methods

McLachlan, 1995; Laskar & Robutel, 2001

id	order	stages	a_i	b_i
SABA1 or ABA22	$(2,2)$	1	$a_1 = 1/2$	$b_1 = 1$
SABA2 or ABA42	$(4,2)$	2	$a_1 = 1/2 - \sqrt{3}/6$ $a_2 = \sqrt{3/3}$	$b_1 = 1/2$
SABA3 or ABA62	$(6,2)$	3	$a_1 = 1/2 - \sqrt{15}/10$ $a_2 = \sqrt{15}/10$	$b_1 = 5/18$ $b_2 = 4/9$
SABA4 or ABA82	$(8,2)$	4	$a_1 = 1/2 - \sqrt{525 + 70\sqrt{30}}/70$ $a_2 = \left(\sqrt{525 + 70\sqrt{30}} - \sqrt{525 - 70\sqrt{30}}\right)/70$ $a_3 = \sqrt{525 - 70\sqrt{30}}/35$	$b_1 = 1/4 - \sqrt{30}/72$ $b_2 = 1/4 + \sqrt{30}/72$

Table: Table of coefficients for the ABA, BAB methods of order $(2s, 2)$ for $s = 1, \ldots, 4$.

$\mathcal{SABA}_n$ or McLachlan (2n,2) methods

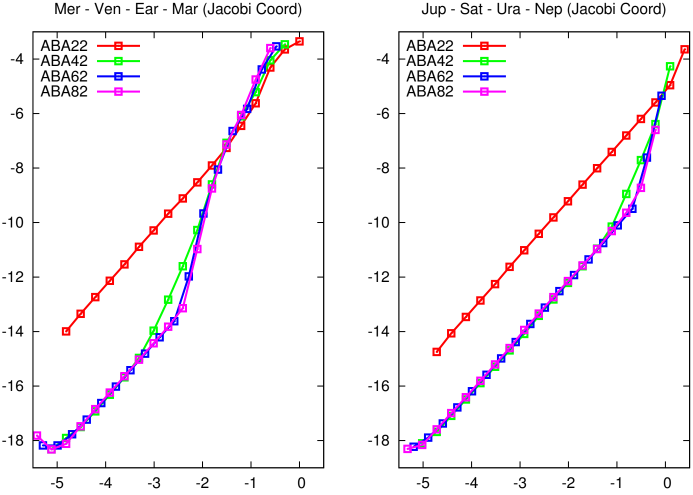


Figure: Comparison of the performance of the $SABA_n$ schemes for Jacobi Coordinates. Using log scale maximum error energy Vs. cost (τ/n).

$\mathcal{SABA}_n$ or McLachlan (2n,2) methods

- As we have seen in the figures above, the main limiting factor of these methods are the terms of order $\tau\varepsilon^2$, which become relevant when τ is small.

- We recall that in the methods described above we have:

$$K = (A + \varepsilon B) + \varepsilon\tau^{2n} p_{2n,1}[A,[A,[A,B]]] + \varepsilon^2\tau^2 p_{3,2}[B,[B,A]] + \dots,$$

- There are in the literature several options to kill the terms of order $\tau^2\varepsilon^2\{\{A,B\},B\}$.

Symplectic Integrator (killing the terms of higher order)

Let $S_0(\tau)$ be any of the given symmetric symplectic schemes previously described:

$$\mathcal{S}_0(\tau) = \exp(a_1\tau A)\exp(b_1\tau B)\ldots\exp(b_1\tau B)\exp(a_1\tau A) = \exp(\tau K),$$

where $K = (A + \varepsilon B) + \varepsilon\tau^{2n}p_{2n,1}[A,[A,[A,B]]] + \varepsilon^2\tau^2 p_{3,2}[B,[B,A]] + \ldots$.

In order to kill the terms of order $\varepsilon^2\tau^2$ we can:

1. Add a corrector term: $\exp(-\tau^3\varepsilon^2 c/2L_C)S_0(\tau)\exp(-\tau^3\varepsilon^2 c/2L_C)$.
2. Composition method: $S_0^m(\tau)S_0(c\tau)S_0^m(\tau)$, where $c = -(2m)^{-1/3}$.
3. Add extra stages: $S(\tau) = \prod_{i=1}^m \exp(a_i\tau A)\exp(b_i\tau B)$, with $m > n$.

Hence, the reminder will be $\tau^{2n}\varepsilon + \tau^4\varepsilon^2$, having methods of order $(2n, 4)$.

The corrector term L_C

This option was proposed by *Laskar & Robutel, 2001*.

$$K = (A + \varepsilon B) + \varepsilon^2\tau^2 p_{3,2}[B,[B,A]] + \varepsilon\tau^{2n} p_{2n,1}[A,[A,[A,B]]] + \dots,$$

Notice that if A is quadratic in p and B depends only of q then $[B,[B,A]]$ is integrable.

We will consider $SC_n(\tau) = \exp(-\tau^3\varepsilon^2 b/2L_C)S_n(\tau)\exp(-\tau^3\varepsilon^2 b/2L_C)$, with $C = \{\{A,B\},B\}$.

order	c_{ABA_n}	c_{BAB_n}
1	$1/12$	$1/24$
2	$(2-\sqrt{3})/24$	$1/72$
3	$(54-13\sqrt{15})/648$	$(13-5\sqrt{5})/288$
4	$0.003396775048208601331532157783492144$	$(3861-791\sqrt{21})/64800$

REMARK: This procedure only works in Jacobi coordinates.

Composition method

The idea behind this option was first discussed by *Yoshida (1990)*. generalise

- He showed that if $S(\tau)$ is a symplectic methods of order $2k$, then it is possible to find a new method of order $2k+2$ by taking

$$S(\tau)S(c\tau)S(\tau),$$

where c must satisfy, $c^{2k+1}+2=0$.

- We can generalise these as:

$$S^m(\tau)S(c\tau)S^m(\tau),$$

where now, $c=-(2m)^{1/(2k+1)}$.

- With this simple composition methods we can transform any of the $(2s,2)$ methods described above to $(2s,4)$ method.

REMARK: This procedure works for both set of coordinates.

Adding an extra stage (McLachlan (2s,4))

McLachlan discussed the possibility of adding an extra stage to methods of order $(2s, 2)$ in order to get rid of the $\varepsilon^2\tau^2$ terms:

$$S(\tau) = \prod_{i=1}^{n+1} \exp(a_i \tau A) \exp(b_i \tau B)$$

.

id	order	stages	a_i	b_i
ABA64	$(6,4)$	4	$- - -$	$- - -$
BAB64	$(6,4)$	4	$a_1 = -0.0437514219173$ $a_2 = 0.5437514219173$	$b_1 = 0.53163862458135$ $b_2 = -0.3086019704406$ $b_3 = 0.55392669171851$
ABA84	$(8,4)$	5	$a_1 = 0.07534696026989$ $a_2 = 0.51791685468825$ $a_3 = -0.0932638149581$	$b_1 = 0.19022593937367$ $b_2 = 0.84652407044352$ $b_3 = -1.0735000196344$
BAB84	$(8,4)$	5	$a_1 = -0.00758691311877$ $a_2 = 0.31721827797316$ $a_3 = 0.38073727029120$	$b_1 = 0.81186273854451$ $b_2 = -0.6774803995321$ $b_3 = 0.36561766098765$

Notice that we no longer have positive values for the coefficients a_i, b_i.

Jacobi Coordinates (first results)

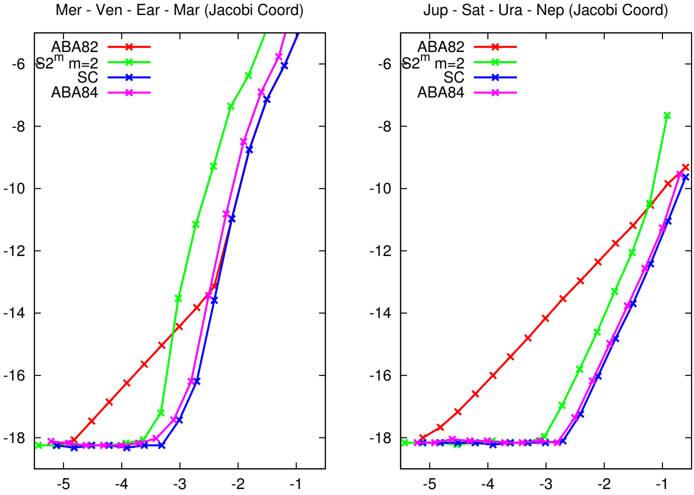


Figure: Simulations for the inner planets: 4BP case = Merc-Ven-Earth-Mars (left) and Jup-Sat-Ura-Nept (right). Plotting cost vs precision for different integrating schemes.

Remark 1: Splitting Methods in Heliocentric Coordinates

We recall that in Heliocentric coordinates:

$$H(p, q) = H_A(p, q) + \varepsilon(H_B(q) + H_C(p)).$$

- We can use the same integrating schemes introduced above:

$$S(\tau) = \prod_{i=1}^{n} \exp(a_i \tau A) \exp(b_i \tau (B + C)),$$

- We can use the approximation:

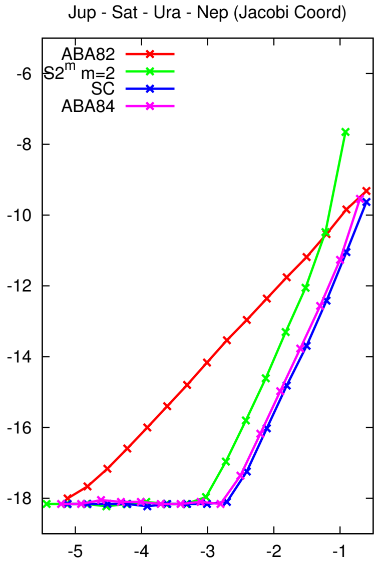
$$\exp(\tau(B + C)) = \exp(\tau/2C) \exp(\tau B) \exp(\tau/2C).$$

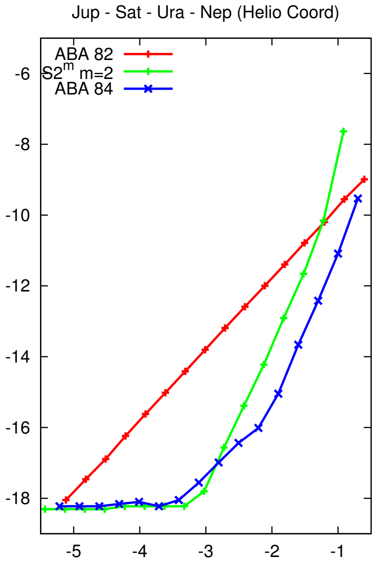
Example (Leap-Frog method):

$$S_1(\tau) = \exp(\tau/2A) \exp(\tau/2C) \exp(\tau B) \exp(\tau/2C) \exp(\tau/2A).$$

REMARK: this introduces an extra error term in the approximation of order $\varepsilon^3 \tau^3$.

Jacobi vs Heliocentric (first results)





Heliocentric Coordinates (Improving McLachlan)

As we have already discussed, in Heliocentric coordinates, we use $\exp(\tau/2C)\exp(\tau B)\exp(\tau/2C)$ to integrate the perturbation part.

- This introduces in our approximation error terms of order $\varepsilon^3\tau^2$ that can become important for small step-sizes. For instance, the McLachlan methods of order $(8,4)$ becomes a method of order $(8,4,2)$
- In order to improve the performance of these scheme, we can add an extra stage to get rid of these term.

$$\prod_{i=1}^{m+1} \exp(a_i\tau A)\exp(b_i\varepsilon\tau B)$$

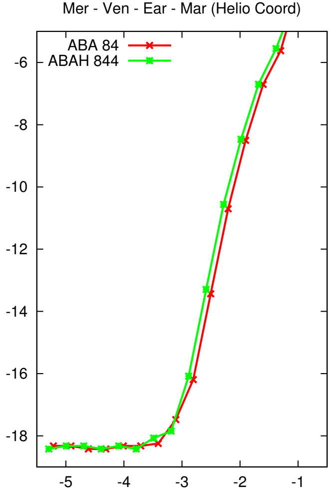
- We must add the extra condition:

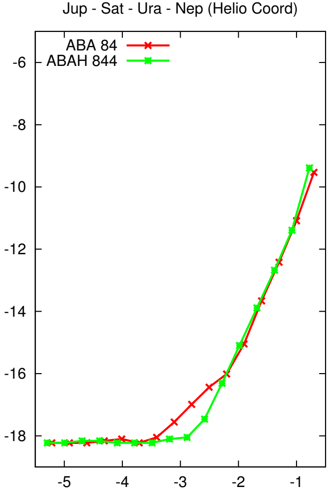
$$b_1^3 + b_2^3 + \cdots + b_m^3 = 0$$

Heliocentric Coordinates (Improving McLachlan)

id	order	n	a_i	b_i
ABAH84	$(8,4)$	5	$a_1 = 0.07534696026989288842$ $a_2 = 0.51791685468825678230$ $a_3 = -0.09326381495814967072$	$b_1 = 0.19022593937367661925$ $b_2 = 0.84652407044352625706$ $b_3 = -1.07350001963440575260$
ABAH844	$(8,4,4)$	6	$a_1 = 0.2741402689434018762$ $a_2 = -0.1075684384401642306$ $a_3 = -0.0480185025906016926$ $a_4 = 0.7628933441747280943$	$b_1 = 0.6408857951625127178$ $b_2 = -0.8585754489567828567$ $b_3 = 0.7176896537942701389$

Heliocentric Coordinates (Improving McLachlan)





New Schemes for a generalised stepsize $(s_1, s_2, \ldots)$

In this philosophy, we can always add extra stages in order to kill the desired terms in the error approximation.

$$S_m(\tau) = \prod_{i=1}^{m} \exp(a_i \tau A) \exp(b_i \varepsilon \tau B)$$

We need:

- First to decide which are the most relevant terms that might be limiting our splitting scheme.
- Find the minimal set of coefficients that fulfil our requirements (not trivial).

Possible drawbacks:

- Sometimes to much stages are required and no actual gain in the performance of the scheme is observed.
- The coefficients a_i, b_i will no longer be positive. We have no control on their size and this can sometimes produce big rounding error propagation for long term-integration.

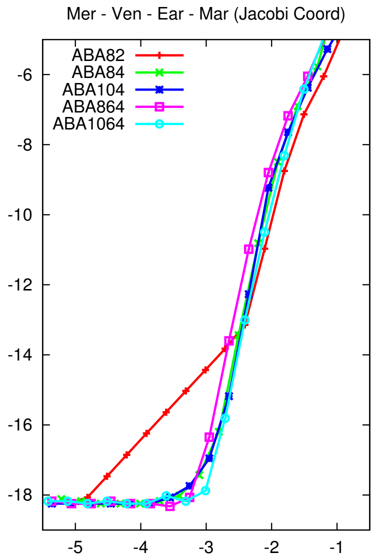
New Schemes for Jacobi Coordinates

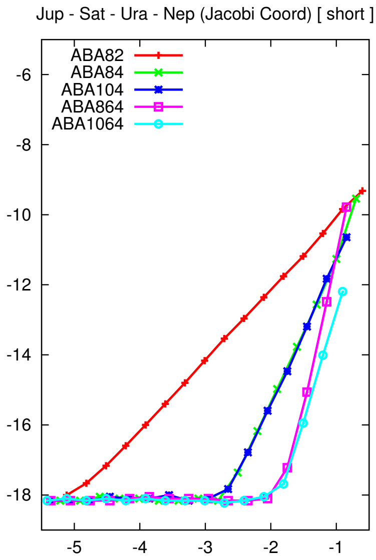
id	order	n	a_i	b_i
ABA82	$(8,2)$	4	$a_1 = 0.06943184420297371$ $a_2 = 0.26057763400459815$ $a_3 = 0.33998104358485626$	$b_1 = 0.17392742256872692$ $b_2 = 0.32607257743127307$
ABA84	$(8,4)$	5	$a_1 = 0.07534696026989288$ $a_2 = 0.51791685468825678$ $a_3 = -0.09326381495814967$	$b_1 = 0.19022593937367661$ $b_2 = 0.84652407044352625$ $b_3 = -1.07350001963440575$
ABA104	$(10,4)$	7	$a_1 = 0.04706710064597250$ $a_2 = 0.18475693541708810$ $a_3 = 0.28270600567983620$ $a_4 = -0.01453004174289681$	$b_1 = 0.11888191736819701$ $b_2 = 0.24105046055150156$ $b_3 = -0.27328666670532380$ $b_4 = 0.82670857757125044$
ABA864	$(8,6,4)$	7	$a_1 = 0.07113342649822311 7$ $a_2 = 0.24115342795664009 8$ $a_3 = 0.52141176177281478 9$ $a_4 = -0.33369861622767800$	$b_1 = 0.18308368747219722 1$ $b_2 = 0.31078285989857486 9$ $b_3 = -0.02656461851195880$ $b_4 = 0.06539614228237341 8$
ABA1064	$(10,6,4)$	8	$a_1 = 0.03809449742241219$ $a_2 = 0.14529871611691374$ $a_3 = 0.20762769572554125$ $a_4 = 0.43590970365152615$ $a_5 = -0.65386122583278670$	$b_1 = 0.09585888083707521$ $b_2 = 0.20444615314299878$ $b_3 = 0.21707034797899110$ $b_4 = -0.01737538195906509$

New Schemes for Heliocentric Coordinates

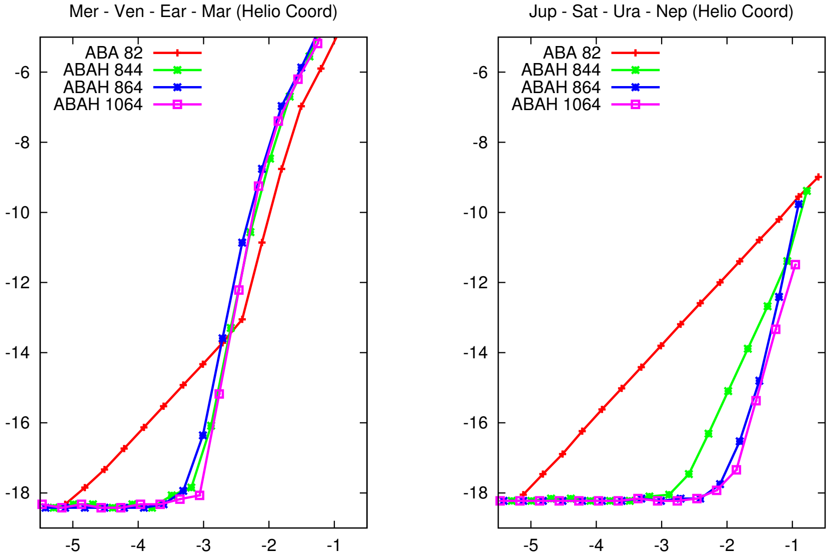
id	order	n	a_i	b_i
ABAH82	$(8,2)$	4	$a_1 = 0.0694318442029737123$ $a_2 = 0.2605776340045981552$ $a_3 = 0.3399810435848562648$	$b_1 = 0.1739274225687269286$ $b_2 = 0.3260725774312730713$
ABAH84	$(8,4)$	5	$a_1 = 0.07534696026989288842$ $a_2 = 0.51791685468825678230$ $a_3 = -0.09326381495814967072$	$b_1 = 0.19022593937367661925$ $b_2 = 0.84652407044352625706$ $b_3 = -1.07350001963440575260$
ABAH844	$(8,4,4)$	6	$a_1 = 0.2741402689434018762$ $a_2 = -0.10756843844016423066$ $a_3 = -0.048018502590601692667$ $a_4 = 0.7628933441747280943$	$b_1 = 0.6408857951625127178$ $b_2 = -0.8585754489567828567$ $b_3 = 0.7176896537942701389$
ABAH864	$(8,6,4)$	8	$a_1 = 0.068102356516583720847$ $a_2 = 0.251136038722103323307$ $a_3 = -0.07507264957216562516$ $a_4 = -0.00954471970174500781$ $a_5 = 0.530757948070447177634$	$b_1 = 0.168443259361895453431$ $b_2 = 0.424317717374267722430$ $b_3 = -0.58581096946817568123$ $b_4 = 0.493049992732012505369$
ABAH1064	$(10,6,4)$	9	$a_1 = 0.04731908697653382270$ $a_2 = 0.26511052357487851595$ $a_3 = -0.00997652288381124084$ $a_4 = -0.05992919973494155126$ $a_5 = 0.25747611206734045344$	$b_1 = 0.11968846245853220353$ $b_2 = 0.37529558553793742504$ $b_3 = -0.46845934183259937836$ $b_4 = 0.33513973427558970103$ $b_5 = 0.27667111912108009750$

Results for Jacobi

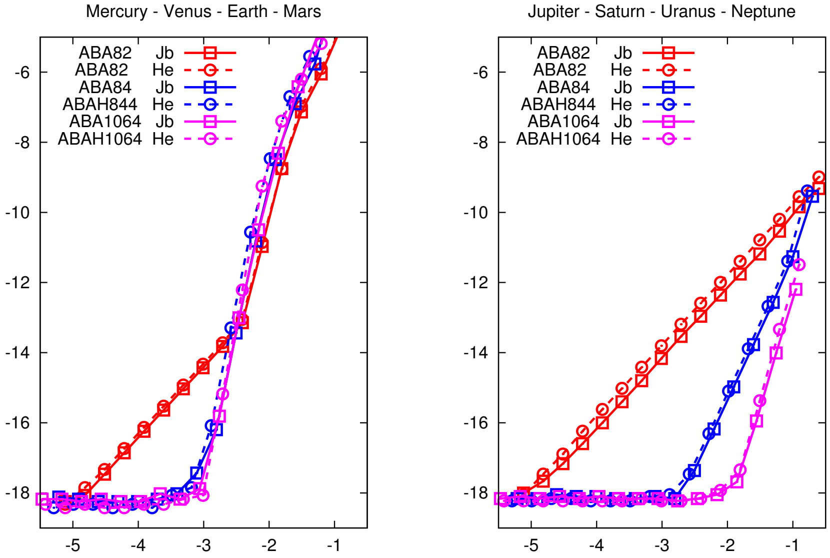




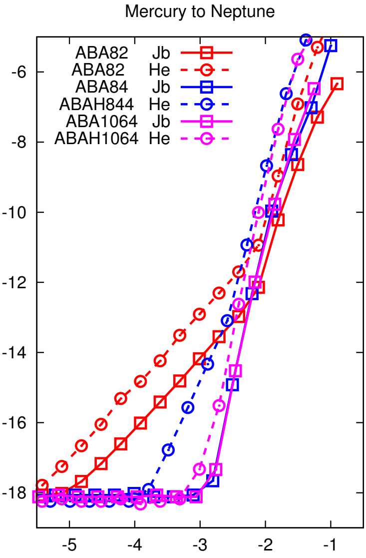
Results for Heliocentric



Results Jacobi Vs Heliocentric (I)



Results Jacobi Vs Heliocentric (II)



Final Comments

- Jacobi coordinates offer better results than Heliocentric coordinates.
- Adding extra stages in order to improve the error approximation (i.e. methods of order $(8,4,4)$, $(8,6,4)$, $(10,6,4)$, ...) in most of the cases improves the results.
- The high angular momenta of Mercury is the main limiting factor on the optimal step-size.

Thank You for Your Attention