JET TRANSPORT AND APPLICATIONS TO NEOs

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Abstract

One of the basic problems in dynamics, and specially in predicting close approaches of NEOs relies on the fact that, even if the physical laws of motion are known with a reasonable approximation, the set of initial data at a given epoch are not. Typically they are known to be in a given box in the phase space, maybe with some probability density, either known analytically (e.g., a normal multivariate distribution) or in some experimental way.

The problem is how to transport these data to a future epoch. A typical procedure consists in sampling the domain and transport every initial point independently. This is a costly process, specially to transport probability distributions.

A solution to this problem will be presented. It is based on the so-called Taylor methods based on the use of higher order variational equations or, in mathematical jargon, jet transport. In turn, this transport can be done using Taylor integration methods. It is extremely powerful, flexible, accurate and efficient.

We have applied this methodology to the particular case of (99942) Apophis, a NEO that will experience the most significant close approach with the Earth in the next 20 years. We will discuss how the initial uncertainties evolve along time and if we are able to determine the existence or not of possible future collisions.

INTRODUCTION

The asteroid (99942) Apophis represents an interesting case of Near Earth Object with no negligible probability of collision with the Earth. Among the known objects, it will get to the most significant distance with respect to the Earth in the next 20 years. Several studies have actually warded off the threat of an impact in 2029, but it is not clear what this approach will imply for the future behaviour of the asteroid.

It is well known that as long as the asteroid stays far from the planets, its motion can be approximated by the Keplerian laws, which do not hold at a close approach with the Earth–Moon system. As we will see, to model the motion of Apophis we consider a restricted $N-$body problem. Our purpose is to analyse the motion of Apophis taking into account the uncertainties given by the observations using high order variational equations. We will use a Taylor method for the integration of the ODE and the jet transport methodology to transport a box of data along time.

First, we will focus on the neighbourhood of April 13, 2029 to identify up to which order the variational equations are necessary, the minimum distance reached and the role of the Moon in this passage. Clearly, we will also obtain information about the shape of the covered space and the range of velocity swept. Then we will try to propagate the region found by increasing the order of the jet transport, aiming to characterise the second foreseen approach that will take place between 2036 and 2037.

MODEL

We consider the motion of the asteroid to be sufficiently well described by a restricted $N-$body model. That is, Apophis moves driven by the gravitational force exerted by the Sun, the nine planets and Moon ($N = 11$). The major bodies attract each other, but they are not affected at all by the asteroid. We remark that the Moon is taken into account, because we are interested in close approaches of Apophis with respect to the orbit of the Earth.

Then the equations of motion are:

$$\ddot{X}_i = \sum_{j=1, j\neq i}^{11} \frac{Gm_j(X_j - X_i)}{r_{ji}^3}, \quad \text{for } i = 1, \ldots, 11, \qquad \ddot{X}_a = \sum_{j=1}^{11} \frac{Gm_j(X_j - X_a)}{r_{ja}^3}, \tag{1}$$

where $X_1, \ldots, X_{11} \in \mathbb{R}^3$ are the positions of the 11 bodies and $m_1, \ldots, m_{11}$ are their masses; $X_a \in \mathbb{R}^3$ is the position of the asteroid; r_{ij} denotes the distance between major bodies i and j ($i, j = 1, \ldots, 11$); r_{ja} the one between the major body j ($j = 1, \ldots, 11$) and the asteroid and G is the gravitational constant. We take as units of mass, distance and time: 1 kg, 1 AU and 1 day, respectively. To fix criteria, we consider the position of the bodies to be in ecliptic coordinates taking as origin the Solar System barycentre.

We are aware that in the Solar System there are additional effects that must be considered, for instance the gravitational forces due to the planets natural satellites or the relativistic effects. These terms can be taken into account at least for the major bodies by reading their position and velocities from the JPL Solar System Ephemerides DE405 at each step of the integration. However, it is still necessary to compute the jets of derivatives associated with the planets because they appear in the Apophis' one. For this we would just use the gravitational interaction between them, the effect of the changes in the other forces on Apophis motion being negligible in one step of integration. For more details see [1].

A detailed description on the JPL Solar System Ephemerides DE405 can be found in [2]. In short, we only recall that the equations of motion used to obtain DE405 include contributions from: (a) point mass interactions among the Moon, planets, and Sun; (b) general relativity (isotropic, parametrised post-Newtonian); (c) Newtonian perturbations of selected asteroids; (d) action of Moon and Sun on Earth's figure; (e) action of Earth and Sun on Moon's figure; (f) physical libration of the Moon, modeled as a solid body with tidal and rotational distortion, including both elastic and dissipation effects, (g) the effect upon the Moon's motion caused by tides raised upon the Earth by Moon and Sun.

Initial Conditions

We set the initial epoch at September 1, 2006 00:00h (JD 2453979.5). The initial condition for Apophis is taken from [3] (and then translated to the Solar System barycentre, see Tab. 1), where we also find information about the uncertainties in position and velocity estimates. Finally, the initial conditions for the main bodies are obtained from the DE405 ephemeris of Caltech's Jet Propulsion Laboratory mentioned above.

The semi-major axis and the mean anomaly of the osculating ellipse covered by Apophis are affected by the errors due to the experimental data (as well as the other orbital parameters). According to [3], these errors follow a Gaussian distribution, with a standard deviation $\sigma_a \approx 9.6 \times 10^{-9}$ AU and $\sigma_M \approx 1.08 \times 10^{-6}$ degrees, respectively. These give rise to uncertainties in position $\approx 9.6 \times 10^{-9}$ AU and $\approx 1.9 \times 10^{-8}$ AU, respectively. They can be translated to uncertainties of 3.5 km along the tangent direction to the orbit and 1.5 km along two directions which are orthogonal to this one.

Table 1: Initial position and velocity for Apophis on September 1 2006 00:00h, given by [3]. Units AU and AU/day.

	x	y	z
position	5.1957863284797057e-01	6.9976953615406001e-01	-2.4547675315725277e-02
velocity	-1.2956402252285342e-02	1.3886168210055569e-02	-1.0476006251579106e-03

GOAL

Let x_0 be an initial data and ξ and initial uncertainty on a given domain $\mathcal{D}$. Our aim is to propagate initial data of the form $x_0 + \xi$ along time and see how they evolve. Typically one would sample $\mathcal{D}$ and transport for many ξ, with a high computational cost as said before. Instead, we will use the jet transport methodology for this purpose. The main idea is to use high order variational equations to express the solution at any desired given time t as a Taylor series in ξ, truncated at a suitable order. Once this is computed, the propagation of any initial small enough sampled domain $\mathcal{D}$ has not significant computational effort.

We want to characterize the first close approach of Apophis with the Earth in 2029 and even the second one between 2036 and 2037. But the whole technique is useful in many different fields of application. In what follows, we explain it in a more general framework.

INTEGRATION OF ODE: THE TAYLOR METHOD

Faced to the problem to integrate a differential equation $\dot{x} = f(t,x),\ x(t_0) = x_0$, there are many methods available. Here we sketch the background for the so-called Taylor methods because they can easily be adapted to the jet transport, to be presented in next section. But, in principle, any integration method can be adapted to jet transport. For simplicity we shall assume f analytic in a neighbourhood of $(t_0, x_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$.

As in many integration methods for IVP one has to provide a way to do one step starting at (t_0, x_0). From $x(t_0+h) = \sum_{j\geq 0} c_j h^j$ and $f(t_0+h, x(t_0+h)) = \sum_{j\geq 0} d_j h^j$ it follows $c_j = d_{j-1}/j$. It remains to compute the d_j.

The key idea is to do this in a *recurrent way*. When c_j are known for $j = 0, \ldots, k$ we know $x(t_0+h)$ up to order k in h. It can be inserted in the expression of f and compute its expansion also up to order k. This gives d_k and from this we obtain c_{k+1}. All the operations are done with formal power series. It is easy to multiply $A(h) = \sum_{j=0}^{k} A_j h^j,\ B(h) = \sum_{j=0}^{k} B_j h^j$, known to order k, to obtain the product $C(h) = A(h)B(h)$ to order k. The well-known convolution formula $C_k = \sum_{j=0}^{k} A_j B_{k-j}$ provides C_k. In a similar way one can compute, recurrently, the coefficients for $A(h)/B(h)$, provided $B_0 \neq 0$, for powers $A(h)^\alpha$, if $A_0 \neq 0$, or for elementary functions, like $\sin, \cos, \exp, \log$, etc.

Note that despite we manipulate (truncated) power series, all operations are done with numbers, i.e., the coefficients of the series. It is enough to store the coefficients of a given variable as a vector with elements of orders from 0 to N.

After computing the coefficients $c_j, j = 0, \ldots, N$ up to a suitable order N, there are different strategies which allow for a reasonable selection of the time step h. It remains then to evaluate a polynomial truncation of the expansion of $x(t_0+h)$, which can be done by Horner's rule. Taylor methods are suitable for *analytic, non-stiff* differential equations.

Some interesting properties are:

- Under simple conditions, the optimal step size h, to minimize the computing time for a fixed truncation error, is almost independent of the number of digits used in the computations.
- The optimal order N is approximately linear in the number of digits. As a rule of thumb, if we want local truncation errors of the order of 10^{-d} then $N \approx 1.2d$. This depends mildly on the vector field f.
- The computational cost to integrate for a fixed time-interval is approximately quadratic in the number of digits. Of course, the elementary operations will be more expensive if one works with more digits.
- It is elementary to produce dense output and to compute Poincaré sections, for instance.
- If f contains terms of different order of magnitude $f_1 + f_2 + f_3 \ldots$ it is easy to implement that they could be expanded up to different orders $N_1, N_2, N_3, \ldots$, with good savings in computing time.
- One of the most important features is that a suitable choice of the local truncation errors allows to reduce the global error to the propagation of the round off errors.
- The method is extremely fast due to its optimality properties.

Further discussion, details, a large variety of examples and implementations freely accessible can be found in [4].

JET TRANSPORT

A step forward concerning the integration of ODE is to know, up to any desired order, how the solution depends on the initial conditions, not only to be able to integrate from given initial conditions x_0 at an initial time t_0. More concretely, assume we denote as $\varphi(t; t_0, x_0)$ the solution of $\dot{x} = f(t,x)$ passing through x_0 at time t_0. Our goal is to obtain $\varphi(t; t_0, x_0 + \xi)$ for small variations ξ of the initial conditions.

It is well-known that the variational equations at first, second, ... order give the linear, quadratic, ... terms in ξ in $\varphi(t; t_0, x_0 + \xi)$. For instance, computations of Lyapunov exponents, stability of periodic orbits, solution of boundary value problems by shooting, etc, require the use of first order variational equations. For some persons this can be familiar under the name of *sensitivity to initial conditions*. To write down and program these equations requires a moderate effort. But to do this for higher order variationals and check and debug from mistakes can be a cumbersome task. The rationale of jet transport is to do this task *automatically*.

The basic idea is to integrate the ODE using any method (e.g., Taylor methods) but replacing operations with numbers (e.g. the coefficients of the Taylor expansion in h of $x(t+h)$) by *operations with polynomials in ξ truncated at the desired order*.

The following considerations hold.

- Typical expansions are full, not sparse. Hence, one has to use efficient methods to multiply multivariate polynomials, to store and retrieve coefficients, etc.

- The computing time is roughly the same that would be required by the integration of the variational equations to the desired order, taking care of the symmetries and avoiding unnecessary computations. But programming and debugging is extremely easier and compact using jet transport.

- At a fixed time t the solution is represented as a truncated Taylor series in $\xi = (\xi_1, ..., \xi_k)$, $k \leq n$. Note that for each ξ_i different truncation order can be considered. The coefficients of this representation evolve in time and its evolution is represented as a Taylor series in h. Taylor methods in general, and the adapted ones to jet transport in particular, allow to choose a suitable truncation orders of the series in h which can depend on the component of x.

- The jet transport methods allow to adapt coordinates according to the shape of the initial uncertainty box: it is equivalent to consider suitable initial conditions in the jet of derivatives with respect to ξ. For example, if the adapted coordinates are a linear combination of the original ones, then the initial condition of the first variational equations is given by the matrix of the change and the adapted coordinates are new coordinates in the tangent space.

- A further advantage is that one can include, with the same methodology, the effect of parameters. So, one can consider equations like $\dot{x} = f(t, x, \lambda)$ containing one or several λ parameters. If one is interested in values of $\lambda = \lambda_0 + \delta$, it is possible to carry out the expansions in (ξ, δ). This allows, for instance, to do bifurcation analysis with respect to parameters of all kinds of objects which are only available by numerical means.

- In the example considered, we assume the uncertainty to be uniformly distributed in the initial box. However, the jet transport can be modified to take into account different distributions for the uncertainty, such as the Gaussian one, to be propagated along time.

- By replacing the usual arithmetic by interval arithmetic and rigorous estimate of the errors, the computations can be made absolutely rigorous, so that the final result becomes, in fact, a mathematical theorem. This will give rigorous realistic estimates, provided the mathematical model is physically correct.

Indeed, the ideas of this methodology can be found in [5] where the jet transport is used to validate numerical results. In our approach we will forget the validation part, not making our results less applicable.

RESULTS

As we have already said, for Apophis we consider that the initial uncertainties are given in a 3D box, which is 3.5 km long on the tangent direction to the orbit and 1.5 km long on two other orthogonal directions. This allows the six polynomials representing the coordinates to be function of 3 coordinates (ξ_1, ξ_2, ξ_3) which lay in these directions. Note that in the case of a 6D initial box, a Taylor series in 6 coordinates is still feasible from a computational point of view.

In Fig. 1, we can see several shots that illustrate the evolution of the initial set of uncertainties along time using first, second and third order variational approximations. As we can appreciate, as we get close to the first approach with the Earth, we need to consider higher order variational equations to have an accurate approximation of the set of uncertainties.

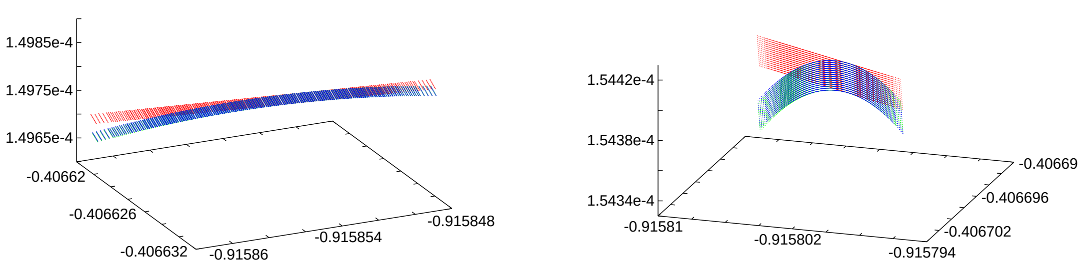

Figure 1: Evolution of the initial uncertainty along time just a few minutes before the first approach and on the first approach. In red: boxes using only the first order variational; in green: boxes using the second order variational approximation; in blue: boxes using the third order variational approximation. Note that the blue boxes essentially overlap the green ones. Left: $T = 8260.95$, right: $T = 8260.955$ days counted from September 1, 2006 00:00h.

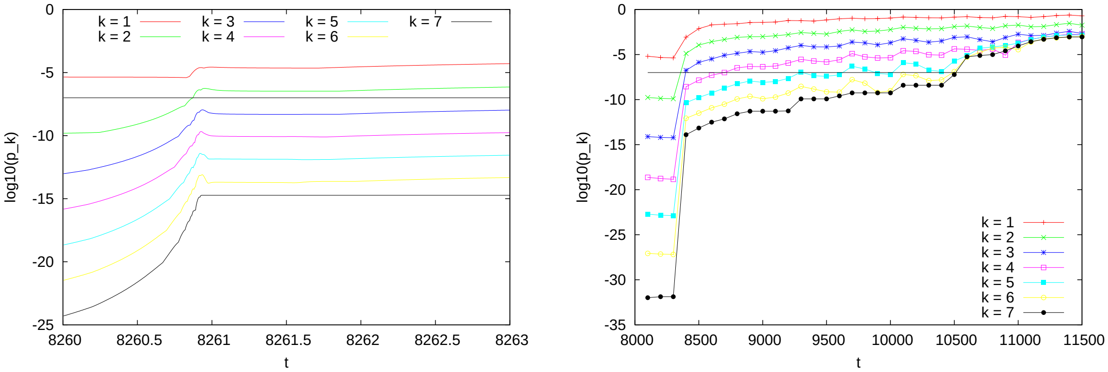

Figure 2: Maximum contribution p_k of the variational terms of order $k = 1, \ldots, 7$ on the estimate of the region covered by the initial uncertainties for Apophis as a function of time. On the x-axis days counted from September 1, 2006 00:00h are shown. On the y-axis the contribution of the variationals in logarithmic scale (units AU and AU/ day). Left: time span close to the first approach with Earth; Right: time span considering the first and second close approaches with the Earth.

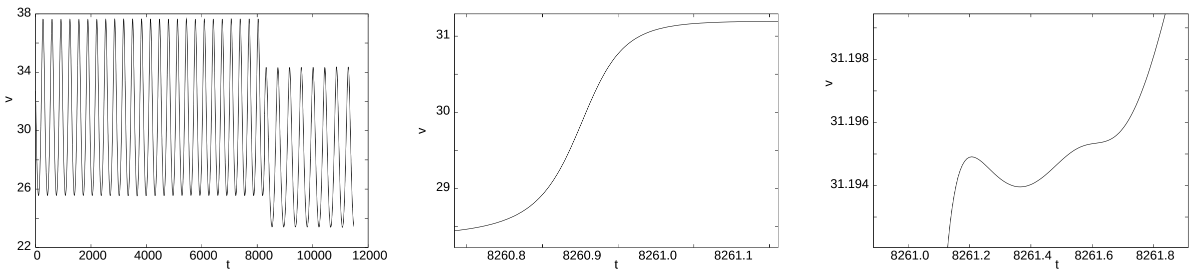

Figure 3: Speed of Apophis as function of time. Left: from September 1, 2006 00:00h to February 15, 2038 00:00h. Middle (resp. right): in a neighbourhood of the first close approach with the Earth (resp. the Moon). On the x-axis days counted from September 1, 2006 00:00h. Units day and km/s.

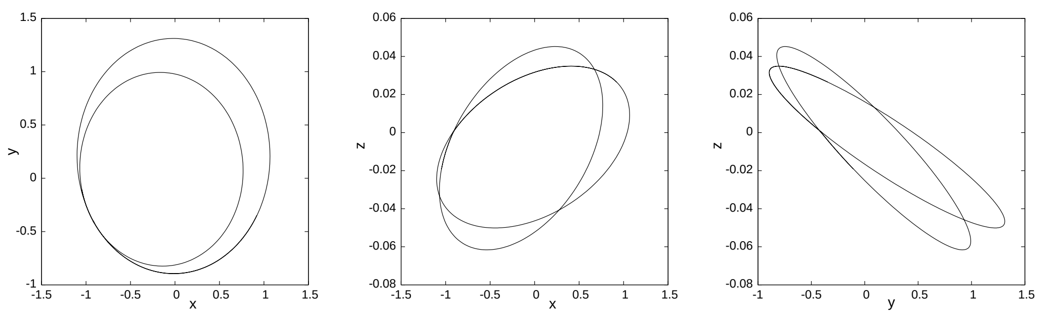

Figure 4: Projections of the orbit of Apophis from May 3, 2028 00:00h to October 4, 2030 12:00 h. From the left, $x - y$, $x - z$ and $y - z$. Unit AU.

The first question that we must face is to determine the minimum order of variational equations that we need to take in the jet of derivatives to have an accurate approximation of the set of uncertainties at a given time T^*. To this end, we estimate the maximum contribution of each variational order for a set of points in the uncertainty region. As we can see in the left-hand side of Fig. 2, for the first close approach with the Earth in 2029 the fourth order variationals contribution is less than 10^{-10} AU and AU/day. So we can use a third order variational approximation to have an accurate description of the shape of the final uncertainty set.

Note, also, that a plot like the one displayed in Fig. 2 carries out much more information that just, say, the maximal Lyapunov exponent. This indicator gives an idea of the average stretching along the full period, while the plot shows the epochs at which stretching is created. The modulus of the dominant eigenvalue of the matrix of the first variational equations changes from a few tens just before the first approach to $\approx 10^5$ a few days after that approach, and then to

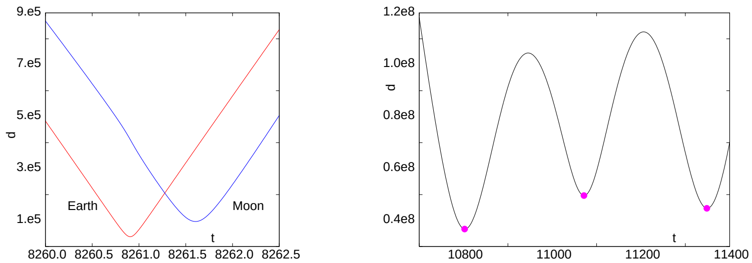

Figure 5: On the left, distance of Apophis w.r.t. the centre of the Earth and of the Moon as function of time in the neighbourhood of the first close approach. On the right, minima of the distance w.r.t. the centre of the Earth reached by the asteroid between 2036 and 2037. The x-axis refers to the number of days counted from September 1, 2006 00:00h. Units day and km.

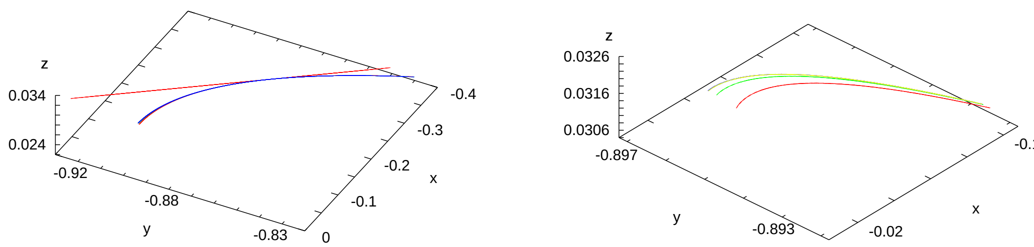

Figure 6: Uncertainty box at $T = 10000$ days from September 1, 2006 00:00h. Different colors refer to different orders of the variational approximation. Unit AU.

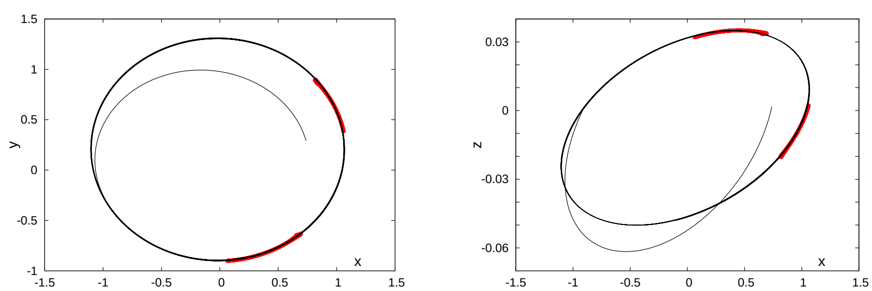

Figure 7: Relation between the uncertainty boxes at $T = 11300$ (left) and $T = 11400$ (right) days from September 1, 2006 00:00h and the nominal orbit of Apophis. Left: x, y projection of the orbit, Right: x, z projection. Unit AU.

$\approx 10^7$ on year 2038.

The plot also reflects another feature. Assume we start an integration at time t_0 and consider how initial variations ξ propagate at time t. The solution, for fixed t and as a function of ξ has a radius of convergence $R(t_0, t)$. It is clear the sudden decrease in $R(t_0, t)$ at the first approach to the Earth. This decrease is related to the collision singularity with the Earth. In the real path this collision does not occur, but certainly it is present for some complex value of time. One should remember that the radius of convergence takes also into account the complex part of the phase space.

Analyzing the effects caused by the first passage, after 2029 the asteroid slows down (on the average and in the long range) by about 3 km/s. The only role played by the Moon is a tiny change of the modulus of the velocity at the time of the lunar closest approach (see Fig. 3). Moreover, we note that Apophis' orbit suffers a change mainly in the value of inclination w.r.t. the ecliptic plane and in the value of semi-major axis (see Fig. 4) and thus in the value of the period, which changes from ≈ 320 days to ≈ 425 days. Concerning the eccentricity, no significant variations can be appreciated.

Following the idea explained earlier, we try to see if we can get to the second close approach with a reasonable approximation of the set of uncertainties. As we can see in the right-hand side of Fig. 2, after approximately 10600 days from the initial epoch, not even the seventh order variationals yield something reliable. At this time we are approaching a minimum of the distance to the Earth (see Fig. 5 on the right), but it takes a value which cannot justify such behaviour,

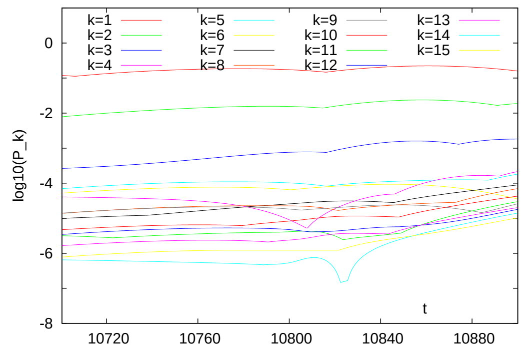

Figure 8: Maximum contribution p_k of the variational terms of order $k = 1, 2, \ldots, 15$ on the estimate of the region covered by the initial uncertainties for Apophis as a function of time. On the x-axis days counted from September 1, 2006 00:00h are shown. On the y-axis the contribution of the variationals in logarithmic scale (units AU and AU/ day).

and neither can do the distance of Apophis w.r.t. another major body.

In Fig. 6, we display the shape of the transported box 10000 days after the initial epoch. One checks that it is completely bent along the orbit and it has a large size. On the right plot a zoom of one of the sides is shown. One can distinguish the different approximations that the second, third, up to seventh order variationals give. However, after order four the contributions of next orders are small.

Finally, Fig. 7 show the relation between two boxes after a long time of integration, in particular $T = 11300$ and $T = 11400$ days from the initial epoch. Although (see Fig. 2) the seventh order variational equations still don't give a good description of the uncertainty region, we do have an idea of how large these uncertainties are going to be.

We must mention that to get close to the second close approach between 2036 and 2037, we need to consider higher order variational equations or to use a different approach (see final Section). In Fig. 8, we display an analysis up to order 15 in the neighbourhood of the first minimum in Fig. 5. We can appreciate that the error decreases as the variational order increases (as it was expected), and it seems that taking order 13 or 14 variationals give a good approach a couple of years before this approach.

To illustrate the flexibility of the method implemented, as the jet transport is independent from the initial box, we look for the initial size in the uncertainties required to obtain an accurate description of the second close approach up to 11500 days from the initial epoch. It can be seen that if we start with an initial set 4 times smaller than the one considered so far, the seventh order variational would be enough for this purpose.

CONCLUSIONS & FUTURE WORK

Here we have used the jet transport to see how uncertainties in position and velocity evolve along time. The technique that we have presented is very flexible and fast, and allow us to have a good understanding of the dynamics of the problem. Moreover, it permits to compute the evolutions of uncertainties in a more efficient way that the classical sampling rule.

We have seen that we can accurately describe the first close approach of Apophis with the Earth by means of third order variational equations. With respect to the second close approach, we have shown that seventh order variational equations are not enough to guarantee an accurate description of the final set. One way to overcome this difficulty is to consider higher order approximation. It seems that order 14th would be sufficient for the first minimum of the distance at the epoch of the second approach.

As a further step, we will consider a 6D uncertainty box, mainly to look if it brings relevant modifications to the conclusions obtained with a 3D one. It should be explained the role of the errors in the velocity directions.

On the other hand, in the near future we will have more precise experimental data, hence we will be allowed to propagate the initial box with a lower order of variational equations and higher level of accuracy. In particular we have checked that errors in the initial data with standard deviations 4 times smaller would be enough to make good estimates for the second approach. Hence, we can expect that good data in a few years will make this prediction reliable.

Alternative ways to propagate big uncertainties via Taylor jet transport will be studied. For instance, one can divide the initial sampled domain into several pieces and consider their evolution independently. This subdivision technique has been widely used in computer assisted proof providing good results (see [1]). Special attention will be devoted to the division's strategy, in order to select an optimal number of boxes.

More concretely, before the vicinity of the first approach the initial set of data is stretched out along the tangent direction to the orbit, having a typical shear behaviour. During the first approach the character of the dynamics is hyperbolic (both in the celestial mechanics and in the dynamical systems senses). Beyond the possibility of using coordinates ξ adapted to the expanding directions and the most suitable orders in each direction, this has a direct effect on the subdivision strategies.

Indeed, the cost of the integration depends on the variational order M at which the expansions are carried out and on which is the dimension D of the ξ variables used for the expansion. The computational cost is proportional, asymptotically and for large M, to M^{2D}. On the other hand, for given radius of convergence $R(t_0, t)$, tolerance ϵ and assuming that the number of expanding directions is E, the required number of boxes is of the order of $\epsilon^{E/M}$. Of course, different expanding directions can require different numbers of divisions. A rough estimate of the optimal value of the order M, the one which requires less computational effort, gives $M \approx E|\log(\epsilon)|/(2D)$. This rule helps to decide the best subdivision strategy that has to be applied at the time it is required, that is, when the size of the propagated box exceeds some size.

Finally, one of the ideas considered in order to prevent a possible future collision consists in introducing a small change in the asteroid's velocity at a certain epoch. Using jet transport, we will analyze the possibilities of collision avoidance, collision and even capture of the asteroid by the Earth-Moon system.

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