Semi-analytical computation of Normal Forms, Centre Manifolds and First Integrals of Hamiltonian systems (II)

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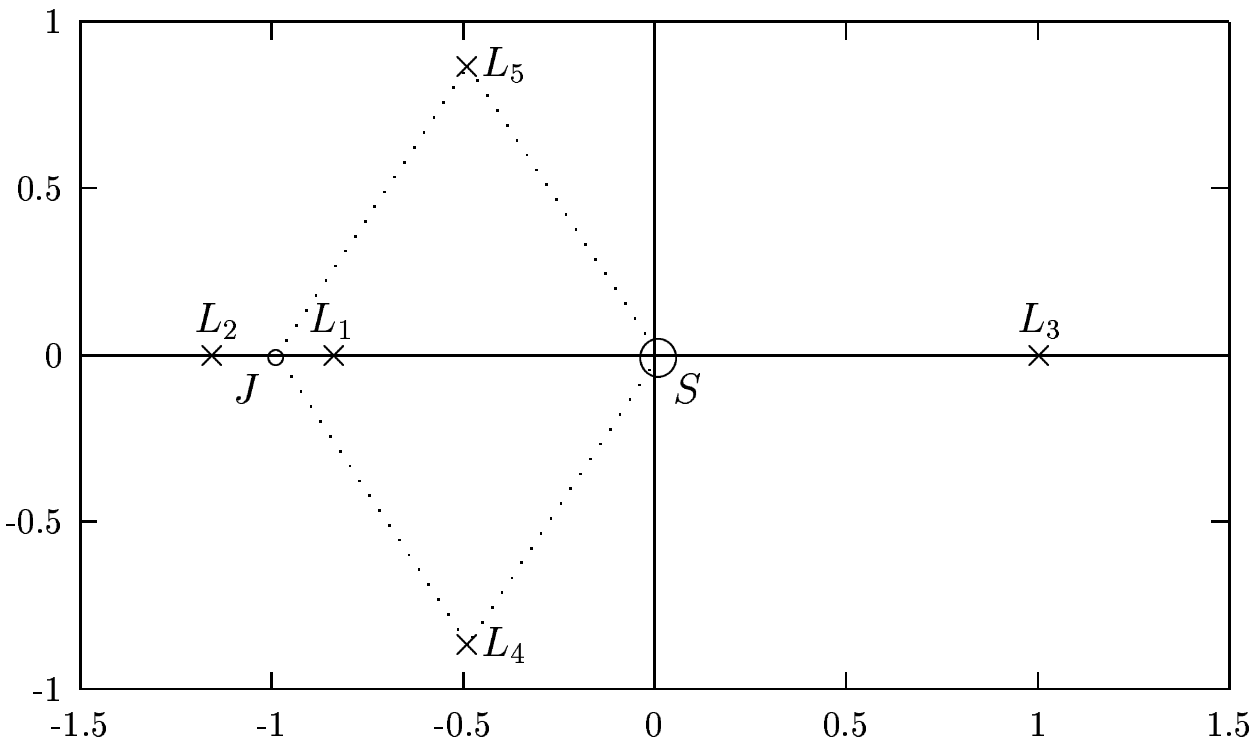
Outline

Centre Manifold of $L_{1,2}$

Let us consider the dynamics near the points $L_{1,2}$ of the RTBP. We recall that the linearization of the vectorfield at these points is of the type centre×centre×saddle.



To give an accurate description of the dynamics close to $L_{1,2}$ one can perform the so-called reduction to the centre manifold.

The idea is the following: assume that the diagonal form of H_2 is

$$H_2 = \lambda q_1 p_1 + \sqrt{-1}\omega_2 q_2 p_2 + \sqrt{-1}\omega_3 q_3 p_3, \qquad \lambda, \omega_2, \omega_3 \in \mathbb{R}.$$

Hence, the hyperbolic direction is given (at first order) by the variables (q_1, p_1).

Let us perform canonical transformations on the Hamiltonian, cancelling monomials such that the exponent of q_1 is different from the exponent of p_1.

After a finite number of transformations, H takes the form

$$H = H^{(0)}(q_1 p_1, q_2, p_2, q_3, p_3) + R(q_1, p_1, q_2, p_2, q_3, p_3),$$

where $H^{(0)}$ is the part that we have arranged and R is the remainder.

As $H^{(0)}$ depends on the product $q_1 p_1$ we can perform the change $I_1 = q_1 p_1$ to produce

$$H = H^{(0)}(I_1, q_2, p_2, q_3, p_3) + R(I_1, \varphi_1, q_2, p_2, q_3, p_3),$$

where φ is the conjugate variable of I_1. If we drop R then I_1 is a first integral of the system and putting $I_1 = 0$ we are skipping the hyperbolic part of the Hamiltonian $H^{(0)}$.

The resulting two degrees of freedom Hamiltonian represents the flow inside the (approximation to the) centre manifold.

So, near the origin, the phase space of the original Hamiltonian must be the phase space of $H^{(0)}(0, q_2, p_2, q_3, p_3)$ times an hyperbolic direction.

To visualize the phase space of $H^{(0)}$ one can fix the value of the Hamiltonian and then use a Poincaré section.

Varying the value of the Hamiltonian we will obtain a collection of 2-D plots representing the dynamics in the phase space.

Let us start by translating the origin of coordinates to the selected point $L_{1,2}$.

It is well known that the distance from L_j to the closest primary, γ_j, is given by the only positive solution of the Euler quintic equation,

$$\gamma_j^5 \mp (3-\mu)\gamma_j^4 + (3-2\mu)\gamma_j^3 - \mu\gamma_j^2 \pm 2\mu\gamma_j - \mu = 0, \quad j = 1, 2,$$

where the upper sign in the first equation is for L_1 and the lower one for L_2.

These equations can be solved numerically by the Newton method, using as starting point $(\mu/3)^{1/3}$.

To have good numerical properties for the coefficients of the Taylor expansion it is very convenient to introduce some scaling.

The translation to the equilibrium point plus the scaling is given by

$$\begin{aligned} X &= \mp\gamma_j x + \mu + a, \\ Y &= \mp\gamma_j y, \\ Z &= \gamma_j z, \end{aligned}$$

where the upper sign corresponds to $L_{1,2}$, the lower one to L_3, $a = -1 + \gamma_1$ for L_1, $a = -1 - \gamma_2$ for L_2 and $a = \gamma$ for L_3.

Note that this change redefines the unit of distance as the distance from the equilibrium point to the closest primary. As scalings are not canonical transformations, they have to be applied on the equations of motion.

To expand the nonlinear terms, we will use that

$$\frac{1}{\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}} = \\ = \frac{1}{D}\sum_{n=0}^{\infty}\left(\frac{\rho}{D}\right)^n P_n\left(\frac{Ax+By+Cz}{D\rho}\right),$$

where $D^2 = A^2 + B^2 + C^2$, $\rho^2 = x^2 + y^2 + z^2$ and P_n is the polynomial of Legendre of degree n.

After some calculations, one obtains that the Hamiltonian can be expressed as

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \sum_{n\geq 2} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right),$$

where $\rho^2 = x^2 + y^2 + z^2$ and the coefficients $c_n(\mu)$ are given by

$$c_n(\mu) = \frac{1}{\gamma_j^3}\left((\pm 1)^n \mu + (-1)^n \frac{(1-\mu)\gamma_j^{n+1}}{(1 \mp \gamma_j)^{n+1}}\right), \quad \text{for } L_j,\ j = 1, 2$$

As usual, the upper sign is for L_1 and the lower one for L_2.

P_n denotes the Legendre polynomial of degree n.

For instance, if we define

$$T_n(x, y, z) = \rho^n P_n\left(\frac{x}{\rho}\right),$$

then, it is not difficult to check that T_n is a homogeneous polynomial of degree n that satisfies the recurrence

$$T_n = \frac{2n-1}{n} x T_{n-1} - \frac{n-1}{n}(x^2 + y^2 + z^2) T_{n-2},$$

starting with $T_0 = 1$ and $T_1 = x$.

The linearization around the equilibrium point is given by the second order terms (linear terms must vanish) of the Hamiltonian that, after some rearranging, takes the form,

$$H_2 = \frac{1}{2}\left(p_x^2 + p_y^2\right) + yp_x - xp_y - c_2x^2 + \frac{c_2}{2}y^2 + \frac{1}{2}p_z^2 + \frac{c_2}{2}z^2.$$

As $c_2 > 0$ (for the three collinear points), the vertical direction is an harmonic oscillator with frequency $\omega_2 = \sqrt{c_2}$.

As the vertical direction is already uncoupled from the planar ones, in what follows we will focus on the planar directions, i.e.,

$$H_2 = \frac{1}{2}\left(p_x^2 + p_y^2\right) + yp_x - xp_y - c_2x^2 + \frac{c_2}{2}y^2,$$

where, for simplicity, we keep the name H_2 for the Hamiltonian.

Next step will be to compute a symplectic change of variable such that Hamiltonian takes a simpler (diagonal) form.
This change is given by the symplectic matrix

$$\begin{pmatrix} \frac{2\lambda_1}{s_1} & 0 & 0 & \frac{-2\lambda_1}{s_1} & \frac{2\omega_1}{s_2} & 0 \\ \frac{\lambda_1^2-2c_2-1}{s_1} & \frac{-\omega_1^2-2c_2-1}{s_2} & 0 & \frac{\lambda_1^2-2c_2-1}{s_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_2}} & 0 & 0 & 0 \\ \frac{\lambda_1^2+2c_2+1}{s_1} & \frac{-\omega_1^2+2c_2+1}{s_2} & 0 & \frac{\lambda_1^2+2c_2+1}{s_1} & 0 & 0 \\ \frac{\lambda_1^3+(1-2c_2)\lambda_1}{s_1} & 0 & 0 & \frac{-\lambda_1^3-(1-2c_2)\lambda_1}{s_1} & \frac{-\omega_1^3+(1-2c_2)\omega_1}{s_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\omega_2} \end{pmatrix},$$

and casts the second order Hamiltonian into its real normal form,

$$H_2 = \lambda_1 x p_x + \frac{\omega_1}{2}(y^2 + p_y^2) + \frac{\omega_2}{2}(z^2 + p_z^2),$$

where, for simplicity, we have kept the same name for the variables.

To simplify the computations, we have used a complex normal form for H_2 because this allows to solve very easily the homological equations that determine the generating functions used during the computations of the center manifold. This complexification is given by

$$\begin{aligned} x &= q_1, \quad y = \frac{q_2 + \sqrt{-1}p_2}{\sqrt{2}}, \quad z = \frac{q_3 + \sqrt{-1}p_3}{\sqrt{2}}, \\ p_x &= p_1, \quad p_y = \frac{\sqrt{-1}q_2 + p_2}{\sqrt{2}}, \quad p_z = \frac{\sqrt{-1}q_3 + p_3}{\sqrt{2}}, \end{aligned}$$

that puts the 2nd order Hamiltonian into its complex normal form,

$$H_2 = \lambda_1 q_1 p_1 + \sqrt{-1}\omega_1 q_2 p_2 + \sqrt{-1}\omega_2 q_3 p_3,$$

being λ_1, ω_1 and ω_2 real (and positive) numbers.

Summarizing:

We have a real Hamiltonian

$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

with an equilibrium point.

Summarizing:

We have a real Hamiltonian

$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

with an equilibrium point.

We want to expand it around that point, composing the expansion with a linear change,

$$U = CV + d,$$

Now, the Hamiltonian takes the form

$$H(q,p) = H_2(q,p) + \sum_{n\geq 3} H_n(q,p),$$

where $H_2 = \lambda_1 q_1 p_1 + \sqrt{-1}\omega_1 q_2 p_2 + \sqrt{-1}\omega_2 q_3 p_3$ and H_n denotes an homogeneous polynomial of degree n.

The changes of variables are implemented by means of the Lie series method: if $G(q,p)$ is a Hamiltonian system, then the function $\hat{H}$ defined by

$$\hat{H} \equiv H + \{H,G\} + \frac{1}{2!}\{\{H,G\},G\} + \frac{1}{3!}\{\{\{H,G\},G\},G\} + \cdots,$$

is the result of applying a canonical change to H. This change is the time one flow corresponding to the Hamiltonian G. G is usually called the generating function of the transformation.

It is easy to check that, if P and Q are two homogeneous polynomials of degree r and s respectively, then $\{P, Q\}$ is a homogeneous polynomial of degree $r + s - 2$.

This property is very useful to implement in a computer a transformation given by a generating transformation G.

For instance, let us assume that we want to eliminate the monomials of degree 3, as it is usually done in a normal form scheme.

Let us select as a generating function a homogeneous polynomial of degree 3, G_3. Then, it is immediate to check that the terms of $\hat{H}$ satisfy

- degree 2: $\hat{H}_2 = H_2$,
- degree 3: $\hat{H}_3 = H_3 + \{H_2, G_3\}$,
- degree 4: $\hat{H}_4 = H_4 + \{H_3, G_3\} + \frac{1}{2!}\{\{H_2, G_3\}, G_3\}$,
- $\vdots$

Hence, to kill the monomials of degree 3 one has to look for a G_3 such that $\{H_2, G_3\} = -H_3$.

Let us denote

$$H_3(q,p) = \sum_{|k_q|+|k_p|=3} h_{k_q,k_p} q^{k_q} p^{k_p},$$
$$G_3(q,p) = \sum_{|k_q|+|k_p|=3} g_{k_q,k_p} q^{k_q} p^{k_p},$$

where $\eta_1 = \lambda_1$, $\eta_2 = \sqrt{-1}\omega_1$ and $\eta_3 = \sqrt{-1}\omega_2$. As

$$\{H_2, G_3\} = \sum_{|k_q|+|k_p|=3} \langle k_p - k_q, \eta \rangle \, g_{k_q,k_p} q^{k_q} p^{k_p}, \quad \eta = (\eta_1, \eta_2, \eta_3),$$

it is immediate to obtain

$$G_3(q,p) = \sum_{|k_q|+|k_p|=3} \frac{-h_{k_q,k_p}}{\langle k_p - k_q, \eta \rangle} q^{k_q} p^{k_p}.$$

Observe that $|k_q| + |k_p| = 3$ implies $\langle k_p - k_q, \eta \rangle \neq 0$. Note that G_3 is so easily obtained because of the "diagonal" form of H_2.

We are not interested in a complete normal form, but only in uncoupling the central directions from the hyperbolic one.

Hence, it is not necessary to cancel all the monomials in H_3 but only some of them. Moreover, as we want the radius of convergence of the transformed Hamiltonian to be as big as possible, we will try to choose the change of variables as close to the identity as possible. This means that we will kill the least possible number of monomials in the Hamiltonian.

To produce an approximate first integral having the center manifold as a level surface (see below), it is enough to kill the monomials $q^{k_q} p^{k_p}$ such that the first component of k_q is different from the first component of k_p

This implies that the generating function G_3 is

$$G_3(q,p) = \sum_{(k_q,k_p)\in\mathcal{S}_3} \frac{-h_{k_q,k_p}}{\langle k_p - k_q, \eta\rangle} q^{k_q} p^{k_p},$$

where $\mathcal{S}_n$, $n \geq 3$, is the set of indices (k_q, k_p) such that $|k_q| + |k_p| = n$ and the first component of k_q is different from the first component of k_p.

Then, the transformed Hamiltonian $\hat{H}$ takes the form

$$\hat{H}(q,p) = H_2(q,p) + \hat{H}_3(q,p) + \hat{H}_4(q,p) + \cdots,$$

where $\hat{H}_3(q,p) \equiv \hat{H}_3(q_1p_1, q_2, p_2, q_3, p_3)$ (note that $\hat{H}_3$ depends on the product q_1p_1, not on each variable separately).

This process can be carried out up to a finite order N, to obtain a Hamiltonian of the form

$$\bar{H}(q,p) = \bar{H}_N(q,p) + R_N(q,p),$$

where $H_N(q,p) \equiv H_N(q_1 p_1, q_2, p_2, q_3, p_3)$ is a polynomial of degree N and R_N is a remainder of order $N+1$ (note that H_N depends on the product $q_1 p_1$ while the remainder depends on the two variables q_1 and p_1 separately).

Neglecting the remainder and applying the canonical change given by $I_1 = q_1 p_1$, we obtain the Hamiltonian $\bar{H}_N(I_1, q_2, p_2, q_3, p_3)$ that has I_1 as a first integral.

Setting $I_1 = 0$ we obtain a 2DOF Hamiltonian, $\bar{H}_N(0, \bar{q}, \bar{p})$, $\bar{q} = (q_2, q_3)$, $\bar{p} = (p_2, p_3)$, that represents (up to some finite order N) the dynamics inside the center manifold.

Note the absence of small divisors during this process.

The denominators that appear in the generating functions, $\langle k_p - k_q, \eta \rangle$, can be bounded from below when $(k_q, k_p) \in \mathcal{S}_N$: using that η_1 is real and that $\eta_{2,3}$ are purely imaginary, we have

$$|\langle k_p - k_q, \eta \rangle| \geq |\lambda_1|, \qquad \text{for all } (k_q, k_p) \in \mathcal{S}_N, \quad N \geq 3.$$

For this reason, the divergence of this process is very mild.

An explicit expression for the change of variables that goes from the coordinates of the center manifold to the initial coordinates can be obtained in the following way: once the generating function G_3 has been obtained, we can compute

$$\begin{aligned}\tilde{q}_j &= q_j + \{q_j, G_3\} + \frac{1}{2!}\{\{q_j, G_3\}, G_3\} + \frac{1}{3!}\{\{\{q_j, G_3\}, G_3\}, G_3\} + \cdots \\ \tilde{p}_j &= p_j + \{p_j, G_3\} + \frac{1}{2!}\{\{p_j, G_3\}, G_3\} + \frac{1}{3!}\{\{\{p_j, G_3\}, G_3\}, G_3\} + \cdots\end{aligned}$$

that produces the transformation that sends the old coordinates, given by the variables $(\tilde{q}, \tilde{p})$ to the new coordinates represented by the variables (q, p).

In the next step, the generating function G_4 is used to obtain the change corresponding to fourth order, and so on.

Substituting $q_1 = p_1 = 0$ one obtains six power expansions (corresponding to the six initial variables), each one depending on the four variables of the center manifold.

Finally, these expansions are put into real form in the same way as the Hamiltonian.

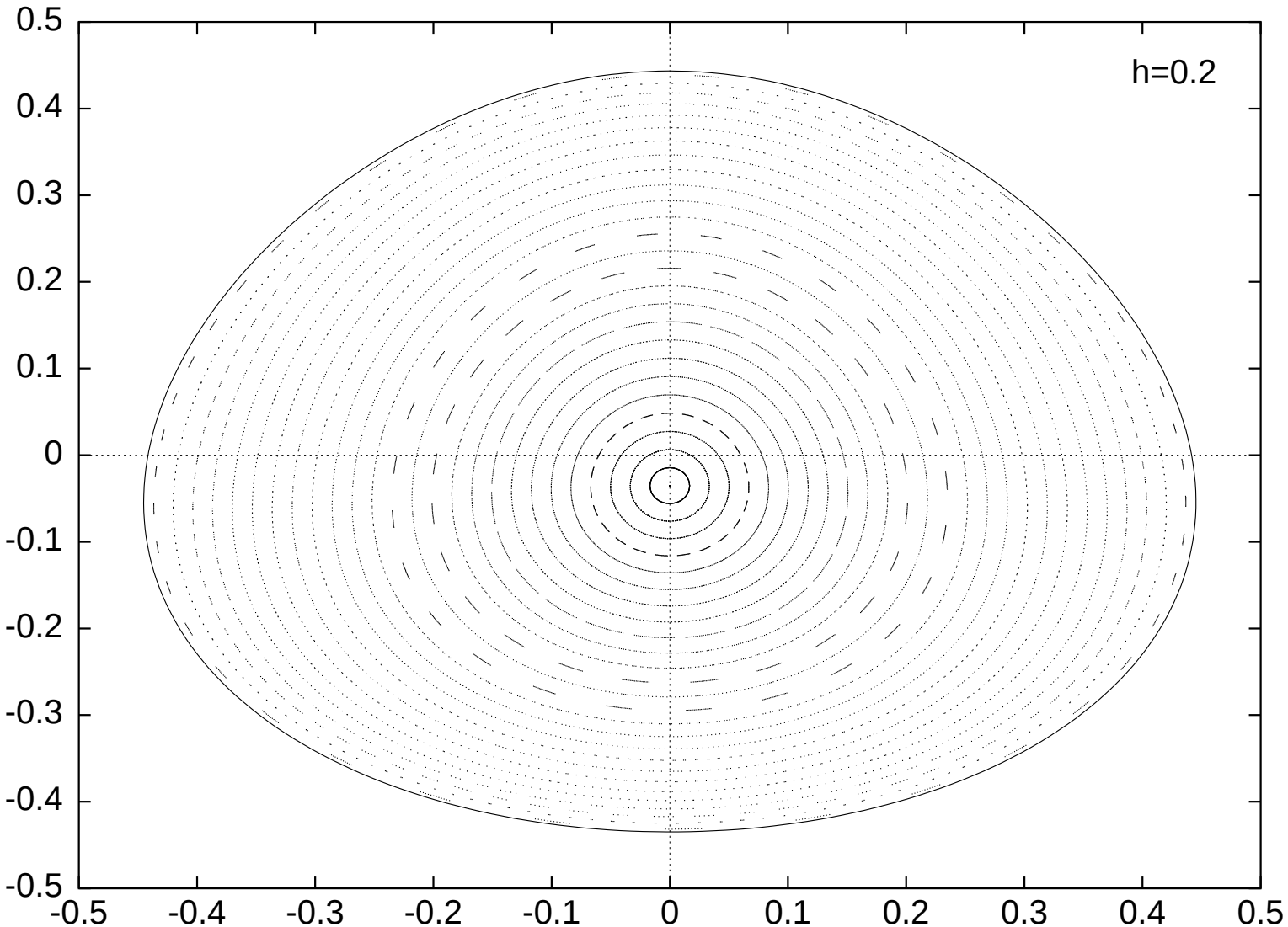
As a first example we focus on the L_1 point corresponding to the mass parameter $\mu = 3.040423398444176 1 \times 10^{-6}$.

This is an approximate value for the Earth-Sun case.

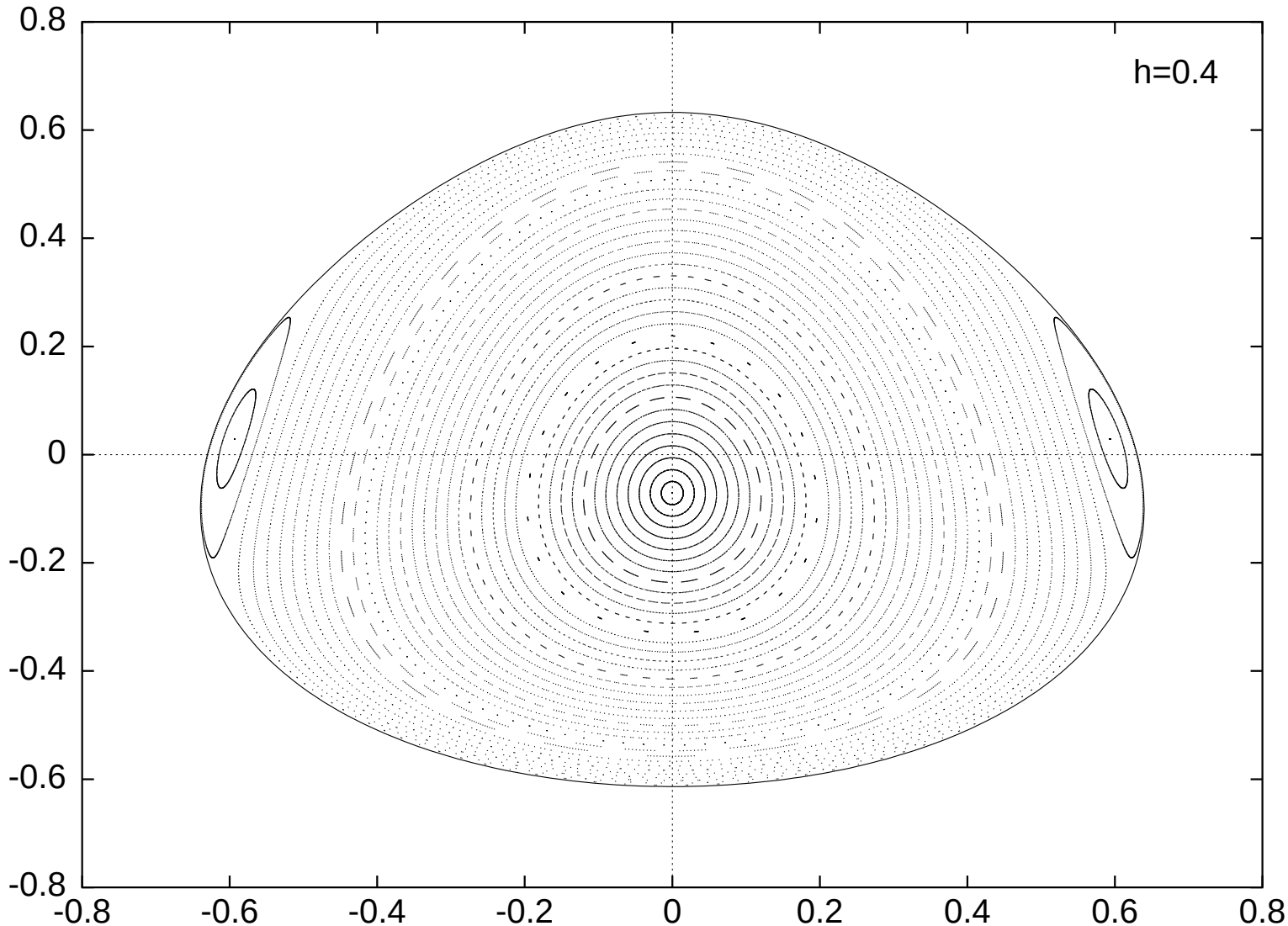
All the expansions have been performed up to degree $N = 32$.

Let us see a run of the program.

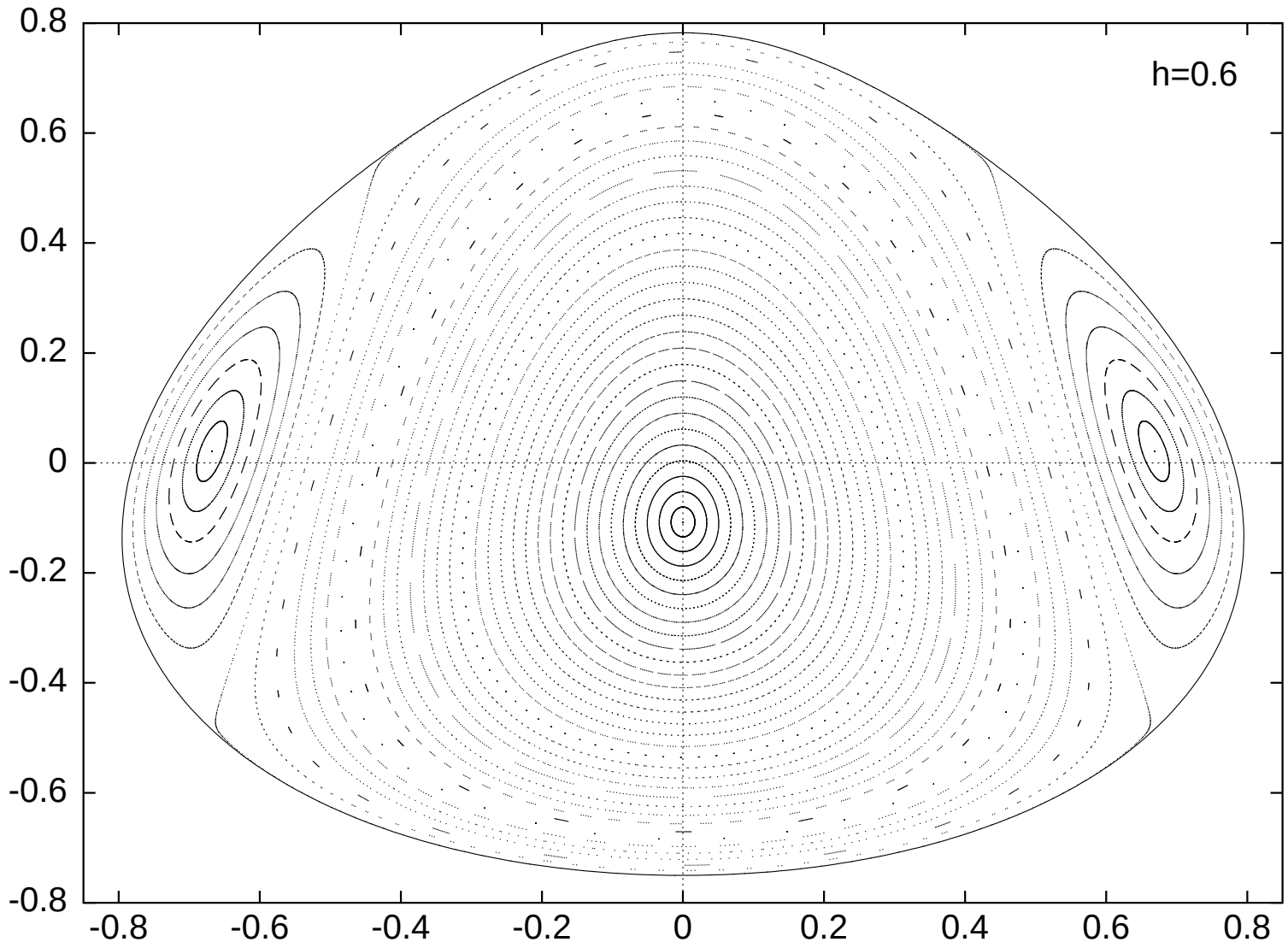
Poincaré sections



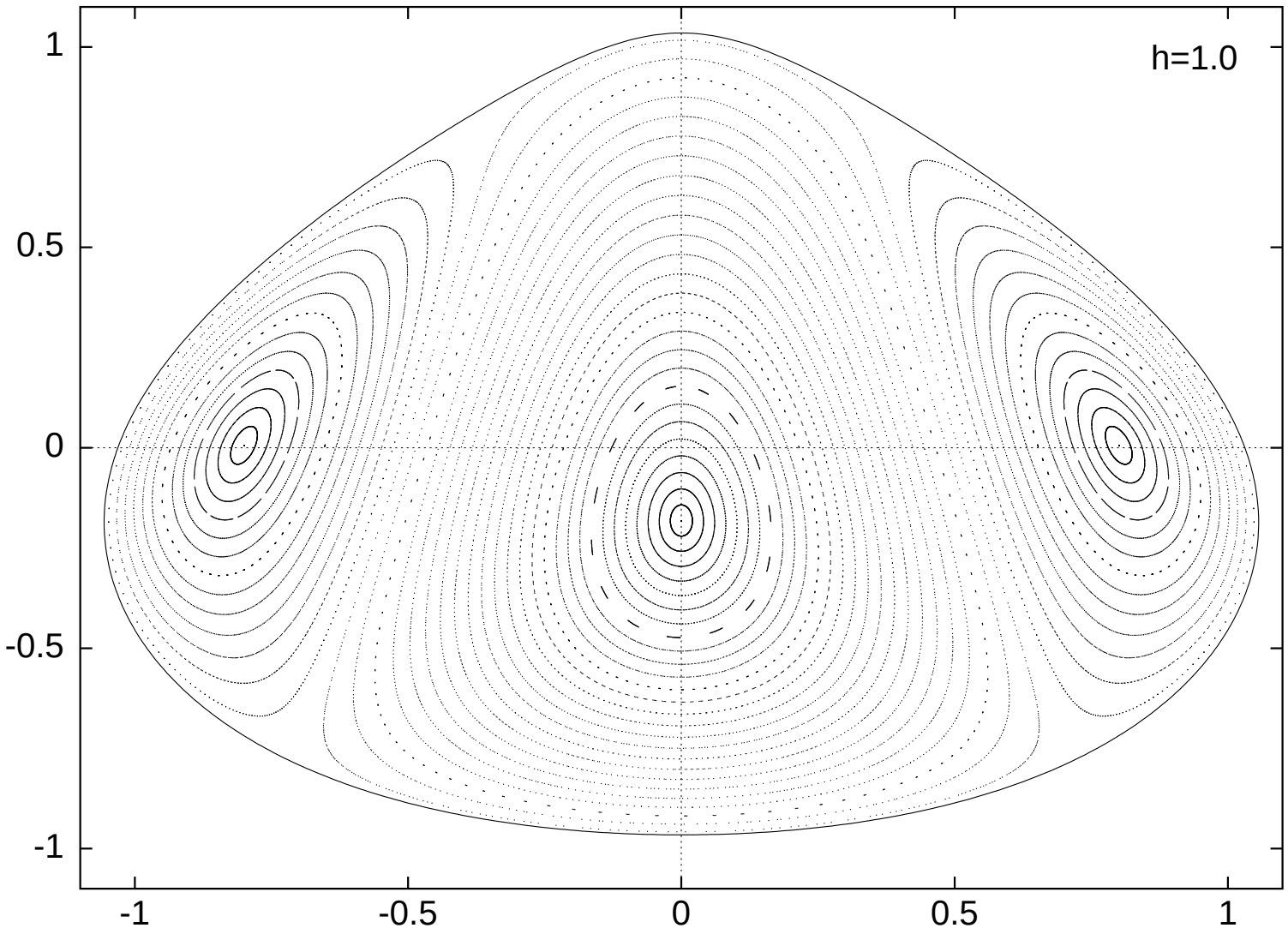
Poincaré sections



Poincaré sections



Poincaré sections



To estimate the radius of convergence of this series, we have computed (numerically) the values

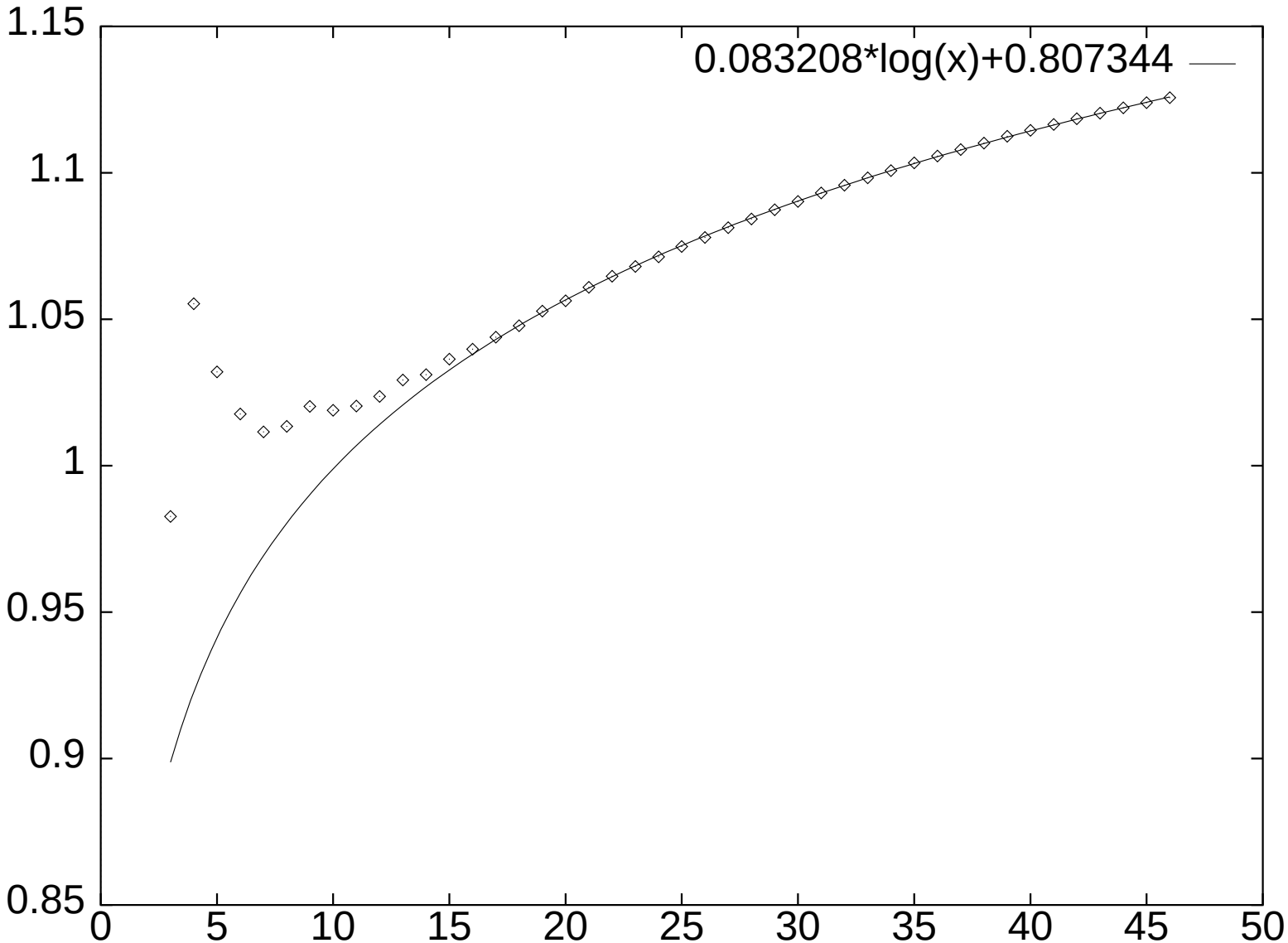
$$r_n^{(1)} = \frac{\|H_n\|_1}{\|H_{n-1}\|_1}, \qquad r_n^{(2)} = \sqrt[n]{\|H_n\|_1}$$

where

$$\|H_n\|_1 = \sum_{|k|=n} |h_k|, \qquad 3 \leq n \leq N,$$

being h_k the coefficient of the monomial of exponent k.

The values $r_n^{(2)}$ have been plotted in the next figure.


0.083208*log(x)+0.807344
1.15
1.1
1.05
1
0.95
0.9
0.85
0
5
10
15
20
25
30
35
40
45
50

Numerical (and very realistic) estimates of the radius of convergence are obtained as follows:

- take an initial condition inside the center manifold and, by means of a numerical integration of the reduced Hamiltonian, produce a sequence of points for the corresponding trajectory.
- Use the change of variables to send these points back to the initial RTBP coordinates.
- Use a numerical integration of the RTBP to test if those points belong to the same orbit (note that we can not use a very long time span for those integrations, since the hyperbolic character of the center manifold in the RTBP amplifies the errors exponentially).

This gives an idea of the error we have in the determination of the center manifold.

Note that the normalizing method used here is slightly different than the standard Lie triangle.

The main difference is that this process uses less computer memory, since it does not require to store the complete triangle.

Let us discuss this with more detail.

Assume we are working with an expansion of H up to degree N:

$$H = H_2 + H_3 + \cdots + H_{N-1} + H_N,$$

and, for instance, we want to transform it using a generating function G_3 (of degree 3):

$$H' = H + \{H, G_3\} + \frac{1}{2!}\{\{H, G_3\}, G_3\} + \frac{1}{3!}\{\{\{H, G_3\}, G_3\}, G_3\} + \cdots,$$

To save memory the result is stored in the same space used for H.
To give the idea, let us write explicitly the firsts steps:

step 1.1 $H_N \leftarrow H_N + \{H_{N-1}, G_3\}$
step 2.1 $H_{N-1} \leftarrow H_{N-1} + \{H_{N-2}, G_3\}$
step 2.2 $H_N \leftarrow H_N + \frac{1}{2!}\{\{H_{N-2}, G_3\}, G_3\}$
step 3.1 $H_{N-2} \leftarrow H_{N-2} + \{H_{N-3}, G_3\}$
step 3.2 $H_{N-1} \leftarrow H_{N-1} + \frac{1}{2!}\{\{H_{N-3}, G_3\}, G_3\}$
step 3.3 $H_N \leftarrow H_N + \frac{1}{3!}\{\{\{H_{N-3}, G_3\}, G_3\}, G_3\}$
$\vdots$

step 1.1 $H_N \leftarrow H_N + \{H_{N-1}, G_3\}$

step 2.1 $H_{N-1} \leftarrow H_{N-1} + \{H_{N-2}, G_3\}$

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step 3.1 $H_{N-2} \leftarrow H_{N-2} + \{H_{N-3}, G_3\}$

step 3.2 $H_{N-1} \leftarrow H_{N-1} + \frac{1}{2!}\{\{H_{N-3}, G_3\}, G_3\}$

step 3.3 $H_N \leftarrow H_N + \frac{1}{3!}\{\{\{H_{N-3}, G_3\}, G_3\}, G_3\}$

$\vdots$

Note that the Poisson bracket done in step **2.1** can be re-used to compute step **2.2**, the one in **3.1** can be used in **3.2** and this last one in **3.3**, and so on.

In this way,

- we are minimizing the number of arithmetic operations (each Poisson bracket is done only once),
- we can work on the initial Hamiltonian (the parts of it that are overwritten are not needed in further steps),
- the need of working space is not very big: we need working space for two homogeneous polynomial of degree N in the worst case (one is used to store the Poisson bracket done in **i.j-1** to be used in **i.j**, the other one is to compute the next Poisson bracket).

This has been implemented in routine `traham`.

When one considers the optimality of a given calculation, there are two main things to be taken into account: the algorithm used and its implementation.

We are not going to discuss the efficiency of the algorithm selected, we are only going to focus on their implementation.

We will focus on the memory and speed used.

As the memory is allocated and freed dynamically, we will focus on the "worst moment" of the program, that is, when the maximum amount of memory is needed.

Next table is for normal forms and reduction to centre manifols.

degree	RAM	HD
8	0.058	0.025
12	0.306	0.153
16	1.218	0.609
24	9.595	4.798
32	44.435	22.217

Note: a series of degree 32 has 1,388,577 monomials.

Next, we can use a profiler on the code.

Each sample counts as 0.01 seconds.

% time	cumulative seconds	self seconds	calls	self ms/call	total ms/call	name
40.15	51.96	51.96	269	193.16	347.79	papu6s
26.48	86.23	34.27				mcount
26.46	120.47	34.24	84136095	0.00	0.00	exll6s
6.02	128.26	7.79	55490539	0.00	0.00	llex6s
0.58	129.01	0.75	14	53.57	6737.27	traham
0.24	129.32	0.31	66	4.70	11.02	pph6s
0.04	129.37	0.05	14	3.57	3.95	cage
0.01	129.38	0.01	14	0.71	1.10	put0
0.01	129.39	0.01	1	10.00	747.48	exp_l5
0.01	129.40	0.01	1	10.00	54.09	reste
0.01	129.41	0.01	1	10.00	15.34	rnf6s
0.00	129.41	0.00	76062	0.00	0.00	kill_nf
0.00	129.41	0.00	38044	0.00	0.00	check_rlf
0.00	129.41	0.00	38032	0.00	0.00	prxk6s
0.00	129.41	0.00	1474	0.00	0.00	ntph6s
0.00	129.41	0.00	164	0.00	0.00	exll3
0.00	129.41	0.00	164	0.00	0.00	llex3

We have done some checks on the software, to (try to) be sure that there are no bugs present.

We have done some checks on the software, to (try to) be sure that there are no bugs present.

One of them is the following.

- We select an initial condition at distance h from the origin, say $x_h^{(0)}$.
- We integrate the corresponding orbit for a short time T, to obtain a new point $x_h^{(1)}$.
- We send both points to the initial (RTBP) coordinates, let us call them $y_h^{(0)}$ and $y_h^{(1)}$
- We integrate (in the RTBP) from $y_h^{(0)}$ to obtain a point $\bar{y}_h^{(1)}$.
- We compute the "error" $e_h = \|\bar{y}_h^{(1)} - y_h^{(1)}\|$.

The idea is that, if the numerical integrations are sufficiently accurate, e_h is determined by the truncation of the series.

Let us illustrate this.

h	e_h
0.00001	2.482807824522209 3e-16
0.00002	5.1198523403369423e-15
0.00004	1.3192410121093586e-12
0.00008	3.4023375555581652e-10
0.00016	8.8211434435268124e-08
0.00032	2.3101212284736493e-05

If the software is working properly, e_h is due to the truncation of the power series.

Hence, e_h should behave like ch^n, where n is the last order in the expansions that we have taken into account.

Then, one has that the order of the error can be approximated by

$$n \approx \frac{\ln\left(\frac{e_1}{e_2}\right)}{\ln\left(\frac{h_0^{(1)}}{h_0^{(2)}}\right)}.$$

Applying this to the results in the previous table we obtain:

$h_0^{(1)}$	$h_0^{(2)}$	n
0.00001	0.00002	4.366
0.00002	0.00004	8.009
0.00004	0.00008	8.011
0.00008	0.00016	8.018
0.00016	0.00032	8.033

The code we have presented admit several extensions, the more natural are to change the kind of coefficients for the polynomial expansions.

The use of C++ allows for a quite clean substitution of these types.

Standard options are extended precision or intervalar arithmetic.

Intervalar arithmetic is based on using intervals instead of real numbers.

In this way, an interval $[a, b]$ accounts for the error of the true quantity.

When we add two intervals, the lower bounds are added using rounding to $-\infty$, upper bounds use rounding towards $+\infty$.

Therefore, we can guarantee that the final result is included in the final interval.

			lower bound	upper bound
1	0	0	9.5450087346978552e-01	9.5450087346991741e-01
0	1	0	-2.9820811951634596e-01	-2.9820811951573489e-01
0	0	1	1.0000000000000000e+00	1.0000000000000000e+00
2	0	0	1.1568661303889360e-01	1.1568661401345537e-01
1	1	0	-1.7127952451731403e+00	-1.7127952303486182e+00
0	2	0	3.3855424323176919e-01	3.3855425662676453e-01
1	0	1	8.9130919836368838e-02	8.9130920112820977e-02
0	1	1	2.2531870640182916e-01	2.2531870757604811e-01
0	0	2	-2.2354591590257877e-03	-2.2354591074729147e-03
3	0	0	-2.9479121860441637e-01	-2.9478447589701773e-01
2	1	0	8.1656201621290165e+00	8.1657691558011720e+00
1	2	0	-5.4586913901624575e+02	-5.4586860598896601e+02
0	3	0	-5.1021371160130911e+01	-5.1021185629532283e+01
2	0	1	-4.3799836956028315e-01	-4.3799552187429924e-01
1	1	1	1.4116969490546651e+01	1.4116998940124972e+01
0	2	1	2.0186927381142823e+00	2.0187190572228246e+00
1	0	2	-5.5905224456048508e-02	-5.5904854484518651e-02
0	1	2	-1.7898271680742539e-01	-1.7898147963031263e-01
0	0	3	-5.1334316020434586e-05	-5.1317165340935330e-05

			lower bound	upper bound
4	0	0	1.267768034100299 7e+00	1.28733452418236 99e+00
3	1	0	-3.5434024811722338e+01	-3.4703682770952582e+01
2	2	0	-5.4877274309542030e+04	-5.4872743283411488e+04
1	3	0	3.2220252371445298e+04	3.2226686164319515e+04
0	4	0	3.5177942440398037e+03	3.5192072384618223e+03
3	0	1	2.1707021092443028e+00	2.1811671985342400e+00
2	1	1	1.9986363951466046e+01	2.0216307091992348e+01
1	2	1	1.3647290105217136e+04	1.3647973048501415e+04
0	3	1	1.4506020027436316e+03	1.4508753202967346e+03
2	0	2	2.1927585054381780e+00	2.1948837131021719e+00
1	1	2	-4.9551211330863225e+01	-4.9529208559599283e+01
0	2	2	-1.0188391081203008e+01	-1.0169093839605921e+01
1	0	3	3.5386579632358917e-02	3.5564130055718124e-02
0	1	3	7.0933774363425073e-02	7.1488715816371950e-02
0	0	4	5.1925348264703075e-04	5.2452355219756441e-04

A more interesting option is to use, instead of numeric coefficients, Fourier series.

This allows to deal with autonomous systems affected of a periodic or quasiperiodic perturbation.

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