Quantitative global phase space analysis of APM

Workshop on

Stability and Instability in Mechanical Systems: Applications and Numerical Tools

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Goal

We look for properties of the phase space of an area preserving map (APM) that help in understanding its qualitative structure providing quantitative data.

Part I:

Local and semi-global analysis.

Part II:

Global analysis.

Clearly, these two parts are related.

Object to study

We consider a one-parameter family of maps

$$F_\delta : \mathcal{U} \to \mathbb{R}^2, \quad \mathcal{U} \subset \mathbb{R}^2 \text{ domain,}$$

such that

1. F_δ analytic in the (x, y)-coordinates of $\mathcal{U}$,
2. $\det DF_\delta(x, y) = 1$, for all $(x, y) \in \mathbb{R}^2$ and for all $\delta \in \mathbb{R}$,
3. F_δ has a fixed point E_0 that will be assumed to be at the origin for all $\delta \in \mathbb{R}$,
4. spec $DF(E_0) = \{\lambda, \lambda^{-1}\}$, $\lambda = \exp(2\pi i\alpha)$, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$.

For some local results it will be assumed δ small enough and irrational.

Hénon map

As an example consider the Hénon map

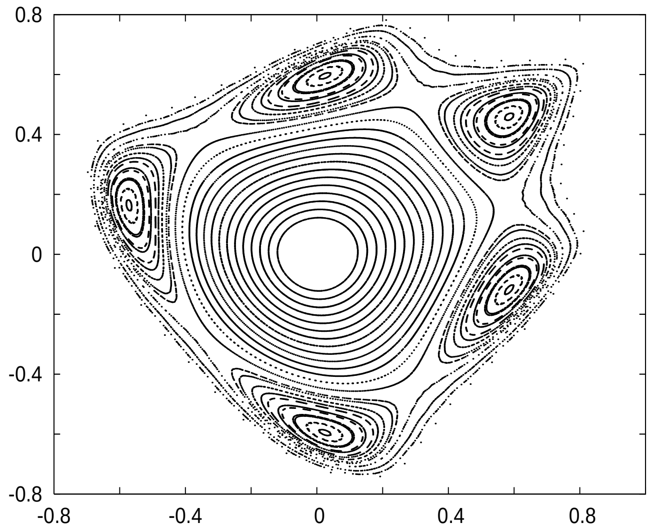
$$H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2), \qquad \alpha \in (0, 1/2)$$

- It has two fixed points:

  the origin is an elliptic fixed point E_0,

  the point $P_h = (2\tan(\pi\alpha), 2\tan^2(\pi\alpha))$ is a hyperbolic fixed point.

- Reversible with respect to $y = x^2/2$ and $y = \tan(\pi\alpha)x$.



Local and semi-global analysis

Normal form of APM. Interpolating flow. Description of resonances.	Well-known
Inner and outer splitting of separatrices. Strong resonances.	"New"

Birkhoff Resonant Normal Form

Given F as before ($\alpha = q/m + \delta$, δ irrational small), the Birkhoff Normal Form to order m around E_0 can be expressed as

$$\text{BNF}_m(F)(z) = R_{2\pi\frac{q}{m}}\left(\underbrace{e^{2\pi i\gamma(r)}z}_{\text{unavoidable res.}} + \underbrace{i\bar{z}^{m-1}}_{m\text{-order res.}}\right) + R_{m+1}(z,\bar{z}),$$

where

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \ldots + b_s r^{2s},$$

being

$z = x + iy,\ \bar{z} = x - iy,\ r = |z|$, (complex variables)

$s = [(m-1)/2]$,

$b_i \in \mathbb{R}$ are the so-called Birkhoff coefficients,

$R_{m+1}(z,\bar{z})$ denotes the remainder which is of $\mathcal{O}(m+1)$.

Remarks

1. Effect of other resonances.

To get BNF expression it is assumed that the m-order resonance cannot be removed but we have removed the others. It can be seen that in a neighbourhood of the m resonance the effect of the others can be ignored (at least if they are of similar order and in a first order approximation).

2. BNF dynamics reduces to near-the-identity map dynamics.

$$\text{BNF}_m(F)(z) = R_{2\pi\frac{q}{m}} \circ K(z, \bar{z}, \delta)$$

with

$$K(z, \bar{z}, \delta) = \exp(2\pi i \gamma(r))z + i\bar{z}^{m-1} + R_{m+1}(z, \bar{z}).$$

The m-jet of K commutes with the rotation $R_{2\pi\frac{q}{m}}$, hence BNF is dynamically equivalent to the near-the identity map K.

Interpolating flow of the BNF

(I,φ)-Poincaré variables ($z=\sqrt{2I}\exp(i\varphi)$).

$$\mathcal{H}_{nr}(I)=\pi\sum_{n=0}^{s}\frac{b_n}{n+1}(2I)^{n+1}\quad\text{and}\quad\mathcal{H}_r(I,\varphi)=\frac{1}{m}(2I)^{\frac{m}{2}}\cos(m\varphi).$$

Let r_* such that $\gamma(r_*)=0$, that is $r_*\approx(-b_0/b_1)^{1/2}$, $b_0=\delta$.

$\longrightarrow$ The flow ϕ generated by the Hamiltonian

$$\mathcal{H}(I,\varphi)=\mathcal{H}_{nr}(I)+\mathcal{H}_r(I,\varphi)$$

interpolates K with an error of order $m+1$ with respect to the $(z,\bar{z})$-coordinates, that is,

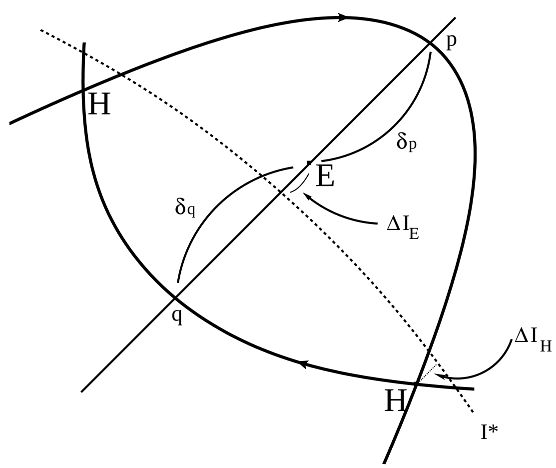
$$K(I,\varphi)=\phi_{t=1}(I,\varphi)+\mathcal{O}\left(I^{\frac{m+1}{2}}\right).$$

If we assume $b_1\neq 0$ this approximation holds in an annulus centred in the resonance radius r_* of width $r_*^{1+\nu}$, for $\nu>0$.

Description of resonances

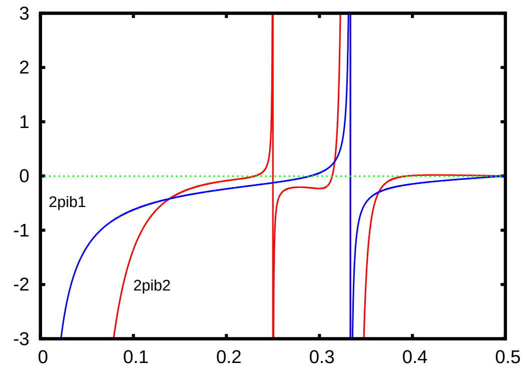
Generic case: $\alpha = q/m + \delta$, $m > 5$, δ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then F **has** a resonant island of order m.
- The resonant zone is determined by **two periodic orbits** of period m located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.

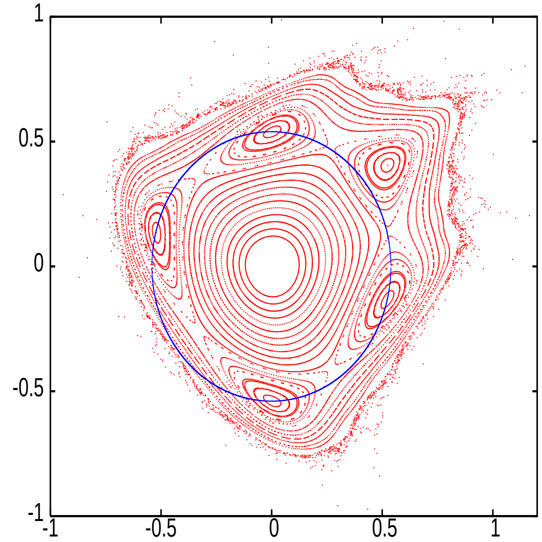


Application

Computation of 1st. and 2nd. Birkhoff coefficient.



$\alpha = 0.21$, $b_1 \approx -0.034166965929 5153$ and $r_* \approx 0.540999411522355$.



Affected by the near-the-identity change of variables of the normal form computation.

A model around a generic resonance

For a generic APM such that $\alpha = q/m + \delta,\ \delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the m-resonance strip $(m \geq 5)$ can be modelled, after suitable scaling, by the time one map of the flow generated by Hamiltonian

$$\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where

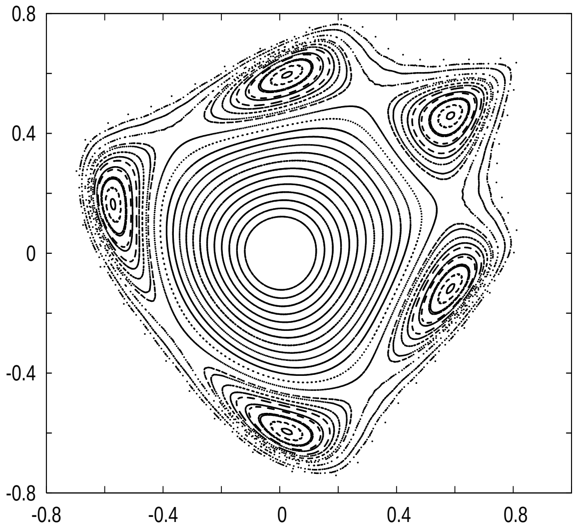
$$c \approx \frac{b_2}{\sqrt{m\pi}\, b_1^{\frac{6+m}{4}}}\, |\delta|^{\frac{m}{4}}, \quad d \approx \frac{\sqrt{m}}{2\sqrt{\pi}\, b_1^{\frac{m-2}{4}}}\, |\delta|^{\frac{m}{4}-1}.$$

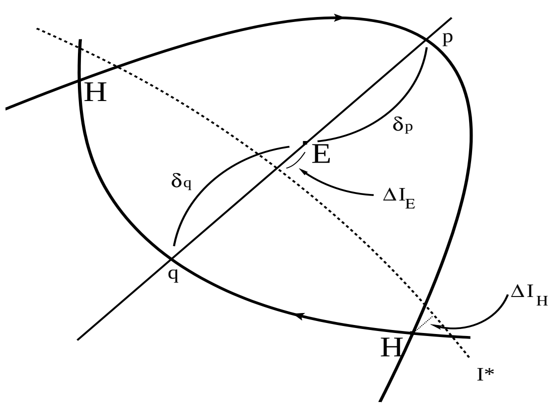
In an annulus domain centred at the radius I_* of width $\mathcal{O}(I_*^{m/4})$ the above approximation gives an error $\mathcal{O}(I_*^{\sigma})$, $\sigma = \min\{m/2 - 2, (m+2)/4\}$.

Map vs flow: inner and outer splittings

- We have described dynamics by terms of a Hamiltonian flow, and hence, by an **integrable approximation**.
- An estimation of how far is an APM to be integrable is given by the splitting of separatrices in a resonance of the phase space. Clearly, this **“distance-to-integrable” depends on the zone** we are studying the map.

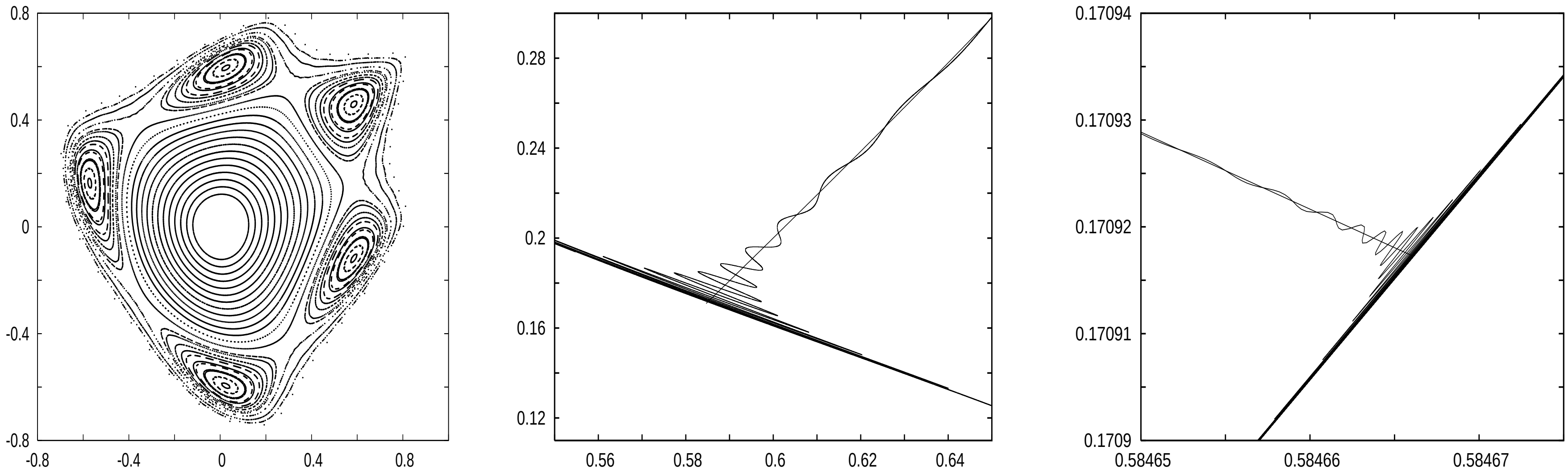
In particular, in a resonant chain of islands there are **two splittings** to be considered: the inner σ_- and the outer σ_+ splittings.



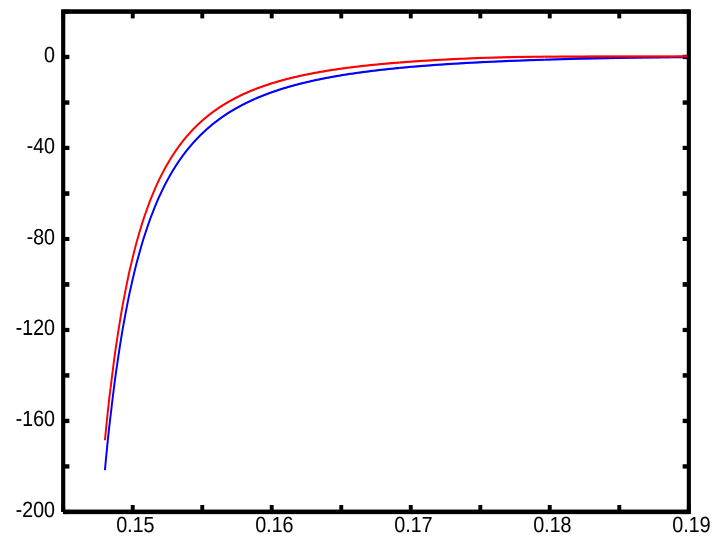


Difference inner-outer splittings

$\alpha = 0.212$, 1:5 resonant chain, Hénon map



Decimal logarithm of the inner (blue) and outer (red) splittings of the 1:7 resonance of the Hénon map.



The splittings characterisation

Assumption: $\sigma \sim A(\log\lambda)^B \exp(-C_r/\log(\lambda))\cos(C_i/\log(\lambda))$, where $C = 2\pi i\tau$, with $\tau \in \mathbb{C}$ the nearest singularity to the real axis of the separatrix $\{s(t), t \in \mathbb{C}\}$, of the interpolating Hamiltonian.

F APM, $\alpha = q/m + \delta$, δ sufficiently small, $b_1 \neq 0$, $m \geq 5$.

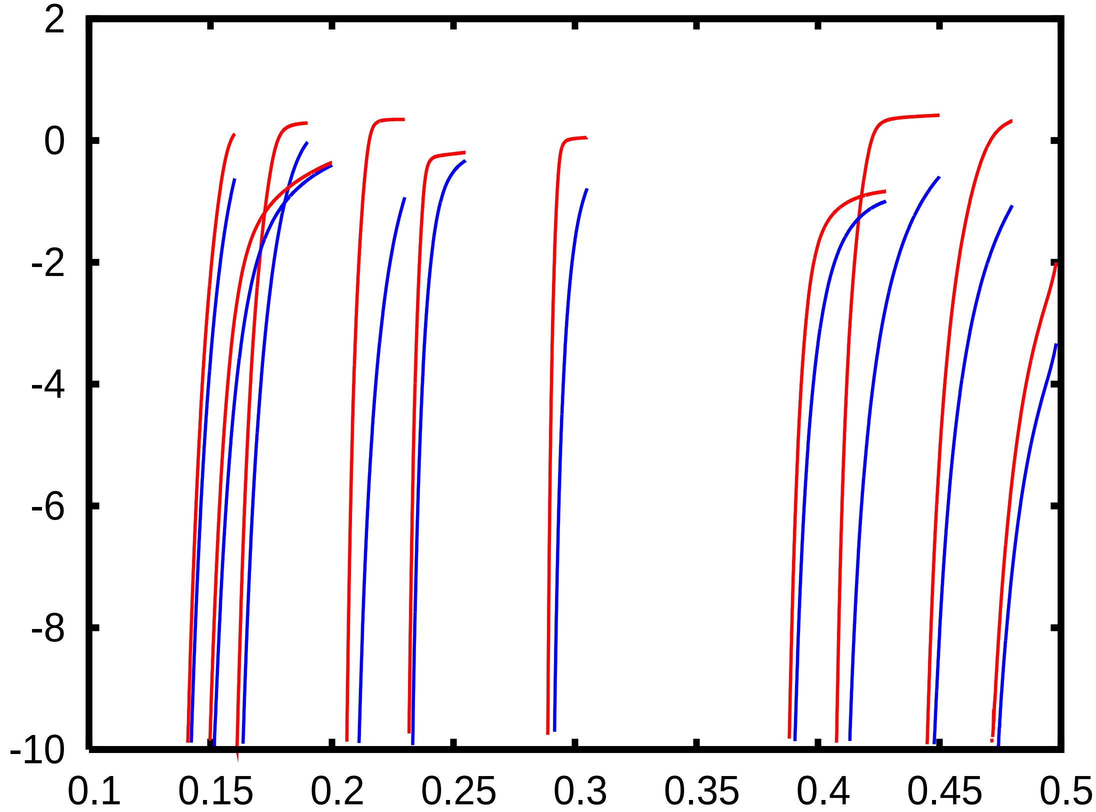
$\longrightarrow$ Then, the m-chain of resonant islands, located at a distance $\mathcal{O}(\delta)$, verifies:

a) The islands of the resonance have, generically, **both splittings different**.

b) The **outer** splitting is **larger** than the **inner** one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

c) Neither the inner nor the outer splittings oscillate.

Inner and outer splittings: Hénon map

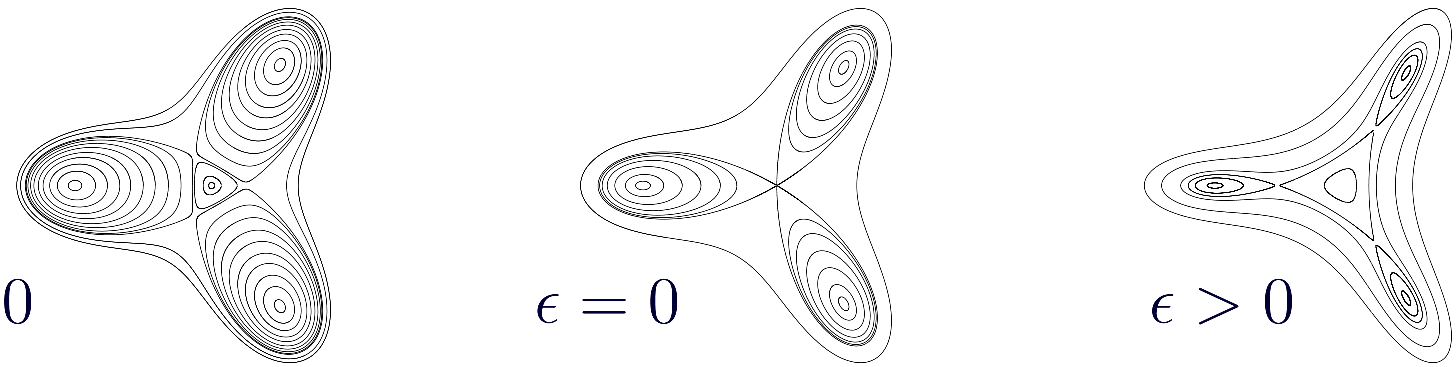


From left to right, it is represented the decimal logarithm of the splitting of the resonances 1:9, 1:8, 1:7, 1:5, 2:9, 2:7, 3:8, 2:5, 3:7 and 4:9, respectively. Each pair of red and blue lines corresponds to the outer and inner splitting, respectively, of a different resonance. Note that in all the cases shown the outer splitting (red) is greater than the inner one (blue). In the x-axis it is represented the value of α.

Strong resonances (I)

The description of the resonant structure by means of the interpolating Hamiltonian does not hold if $m \leq 4$.

1:3 resonance: $\mathcal{H}(I, \varphi) = \epsilon I + I^2 + I^{\frac{3}{2}} \cos(3\varphi)$



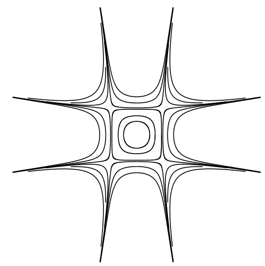
$\epsilon < 0$ $\epsilon = 0$ $\epsilon > 0$

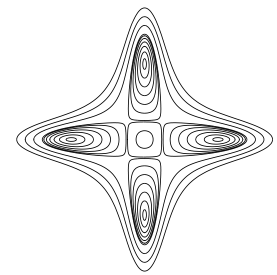
- Hyperbolic points at a distance $\mathcal{O}(\epsilon^2)$. Elliptic points at a **finite** distance.
- Outer splitting non-perturvative since the separatrices remain at a finite distance.
- Inner splitting behaves as described in the generic case $m > 4$.

Strong resonances (II)

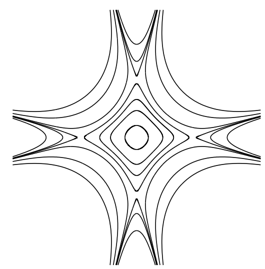
1:4 resonance: $\mathcal{H}(I,\varphi) = \epsilon I + I^2 + \xi I^2 \cos(4\varphi), \quad \xi < 0.$

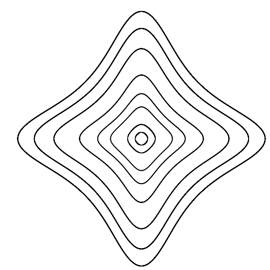
$\epsilon < 0$, $\xi < -1$ left,
$-1 < \xi < 0$ right





$\epsilon > 0$, $\xi < -1$ left,
$-1 < \xi < 0$ right

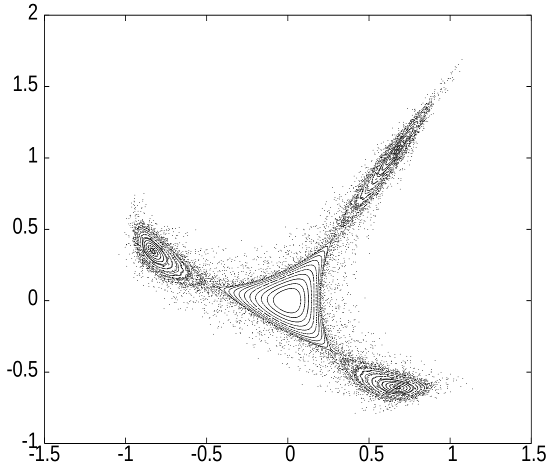




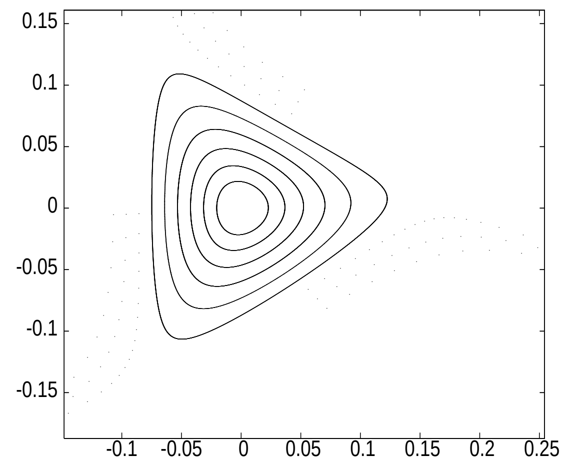
- Elliptic and hyperbolic points located at a distance $\mathcal{O}(\epsilon)$.
- Cases with $\xi < -1$: The splitting **oscillates** and behaves as expected in magnitude in the generic case.
- Case $\epsilon < 0$, $\xi > -1$: The splittings behave as expected in the generic case.

Strong resonances of the Hénon map (I)

1:3 resonance:



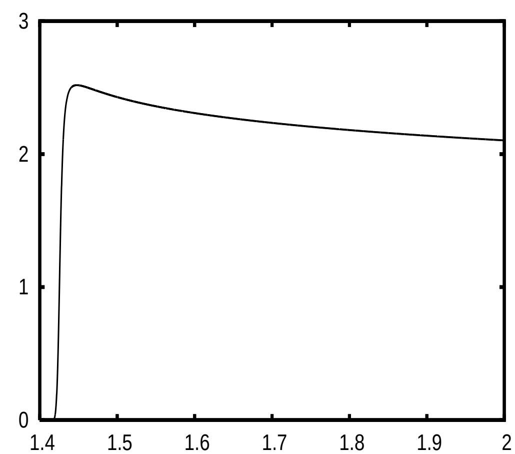
$\alpha = 0.32$



$\alpha = 0.34$

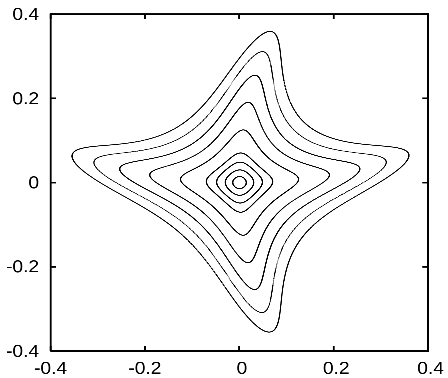
The outer splitting remains finite

($\alpha = 1/3$ corresponds to $c = \sqrt{2}$):

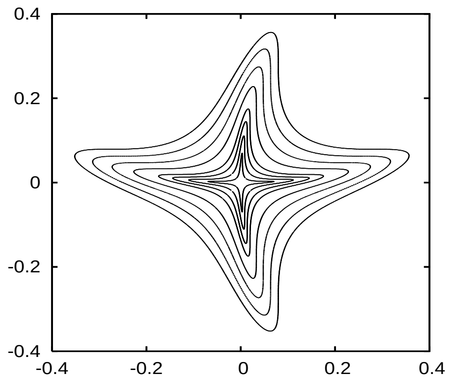


Strong resonances of the Hénon map (II)

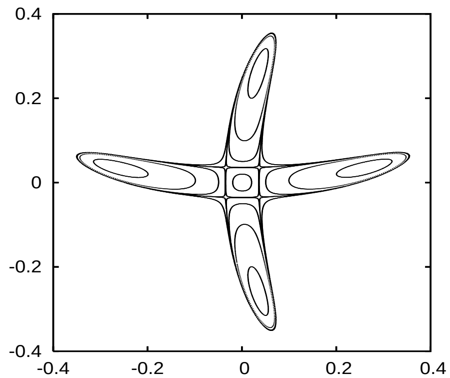
1:4 resonance: **Non-generic!!**



$\alpha = 0.2499$



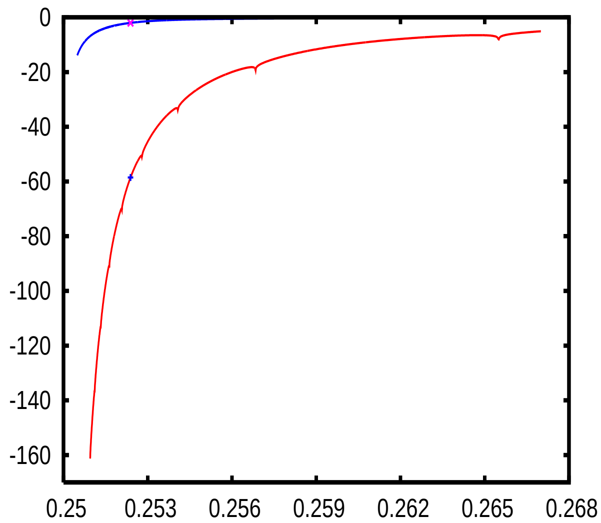
$\alpha = 0.2500$

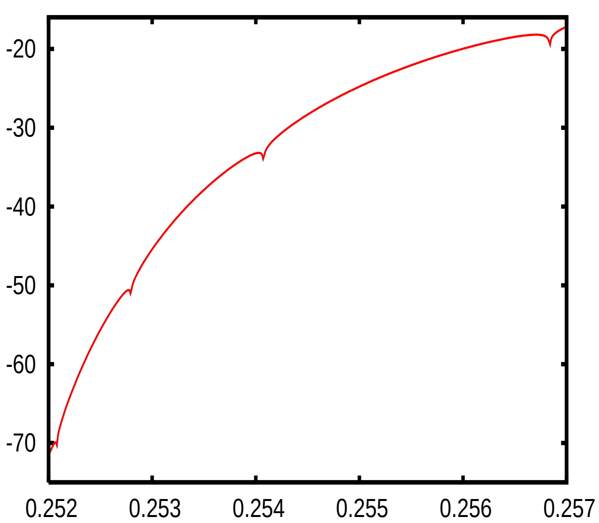


$\alpha = 0.2501$

- It corresponds to the case $\xi = -1$ in the Hamiltonian above.
- The elliptic point goes to a distance $\mathcal{O}(\epsilon^{1/2})$ instead $\mathcal{O}(\epsilon)$.
- $H(I, \varphi) = \epsilon I + I^2(1 - \cos(\psi)) + I^3(a + b\cos(\psi) + c\sin(\psi))$.
- Hénon corresponds to $\epsilon < 0$, $a + b > 0$. The inner splitting oscillates and the outer does not. There is a big difference inner-outer splitting magnitude (outer singularity at a distance $\mathcal{O}((\epsilon(a+b))^{1/4})$, inner singularity real part distance 2π).

Strong resonances of the Hénon map (III)

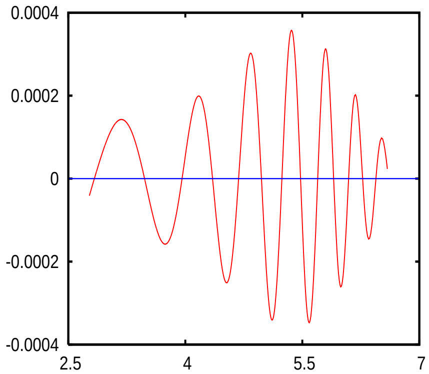




Decimal logarithm of the inner (red) and outer (blue) splittings as a function of α.

- Big difference in the order of the size of the splittings:
  For $\alpha \approx 0.25238741368$ it is $\sigma_+ \approx 2.5238741368 \times 10^{-1}$ and $\sigma_- \approx -2.986620731 \times 10^{-59}$.

- The inner splitting oscillates ("peaks").



Global analysis

Dynamics in a neighbourhood of any resonance.

Dynamics close to separatrices:

Separatrix map

Double separatrix map

} "Well-known" but...

Dynamics in Birkhoff zones: Biseparatrix map } "New"

What we mean by global?

From now on (unless the opposite is stated) it will be assumed that we are interested in dynamics within a region containing a resonant chain of islands.

It is not assumed that the resonance is located close to the elliptic fixed point (δ arbitrary).

A model away from E_0

Question: How global are the results obtained before?

Dynamics in an annulus containing a $q:m$ resonance far away of the elliptic point E_0 can be studied by means of a **perturbation of an integrable twist map**. After reduce the near integrable twist map to normal form and compute the m-th iterate to have a near-the-identity map it can be obtained an interpolating Hamiltonian flow. A straightforward computation gives

$$\mathcal{H}(J,\psi) = J^2/2 + cJ^3/3 - (1 + dJ)\cos(\psi)$$

that is, **the same Hamiltonian** as the one interpolating the m resonance when located in a neighbourhood of the elliptic fixed point E_0.

BUT the coefficients c and d are arbitrary (and maybe there are higher order (J) coefficients which play relevant role in dynamics).

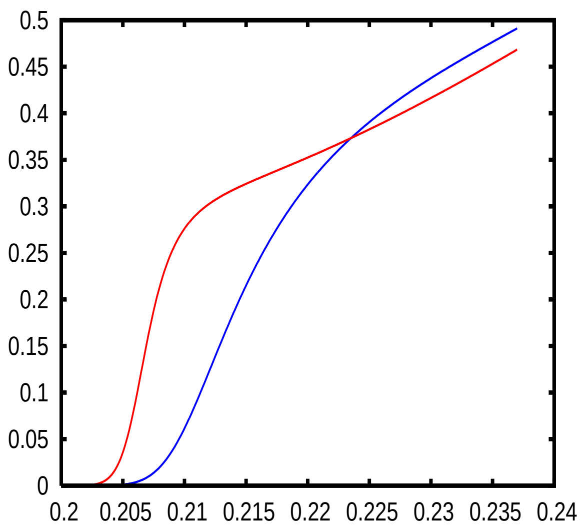
A model away from E_0: splittings

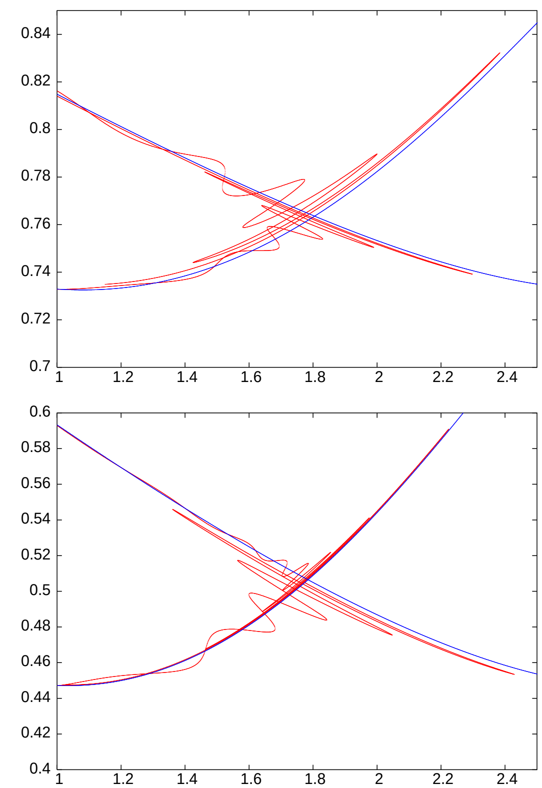
In particular, it cannot be assumed the outer splitting to be larger than the inner when far from the elliptic point.

$$T_\epsilon(I, \theta) = (I + \epsilon \cos(\theta + \alpha(I)), \theta + \alpha(I))$$

$$\alpha(I) = b_1 I + b_2 I^2$$

2:11 Hénon map





$b_1 = 0.2$

$b_2 = 4$

$\epsilon = 0.14$

$b_1 = 6$

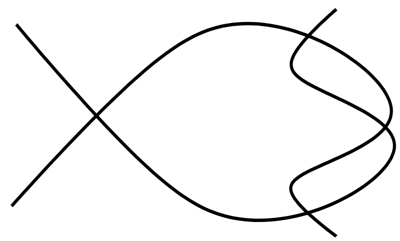
$b_2 = -2$

$\epsilon = 0.14$

Dynamics close to separatrices

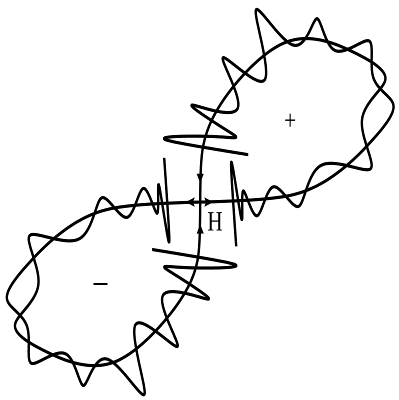
We distinguish two cases:

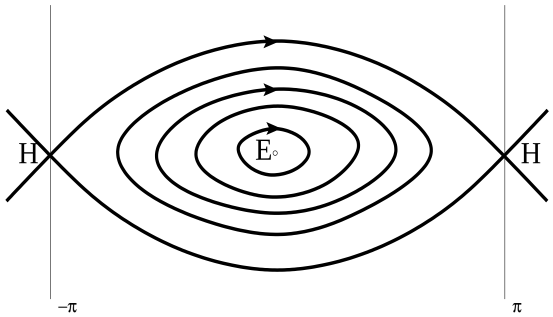
Open map.



Separatrix map

Figure eight.





Double separatrix map

Separatrix map

$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b\log(y') \\ y + \sin(2\pi x) \end{pmatrix}$$

- Describes the dynamics in a close neighbourhood of the separatrices emanating from a hyperbolic point H.
- a is related with a shift needed to get the image in the fundamental domain ("no dynamical relevance").
- $b = -1/\log(\lambda)$, λ is the eigenvalue of modulus greater than one of H.
- The y variable is rescaled by the amplitude of the splitting.

SM: invariant curves and islands

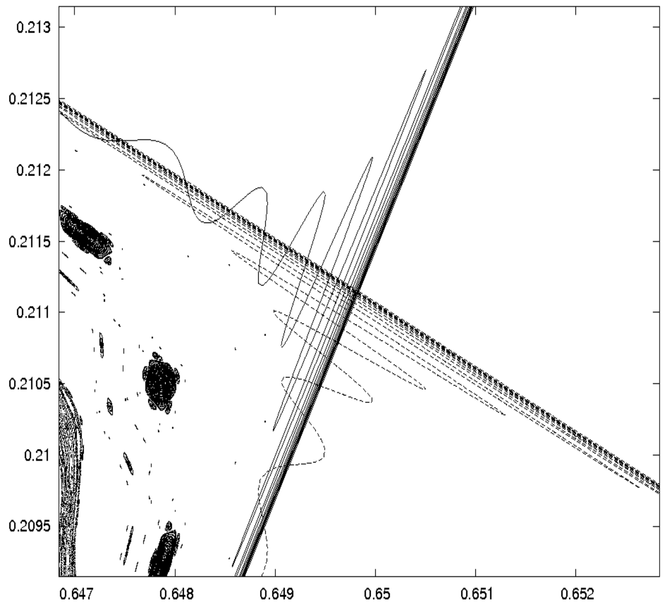
Approximating SM by the Chirikov standard map it is obtained:

$\longrightarrow$ distance to expect rotational invariant curves:

$\longrightarrow$ from the stable separatrix: $d_c \sim |b|/k^*$, k^* Greene

$\longrightarrow$ from the hyperbolic point: $d_c^h \sim \sqrt{|b|/k^*}$

$\longrightarrow$ distance to expect islands from the hyperbolic point: $d_i \sim \sqrt{b\pi/2}$



Hénon map ($\alpha = 0.1$), hyperbolic fixed point.

Observed: $d_c^h \approx 3.2 \times 10^{-3}$, $d_i \approx 2 \times 10^{-3}$

Formulas above:

$P_h \approx (0.64983939, 0.21114562)$

$\lambda_+ \approx 1.83785279$

$\sigma \approx 1.19 \times 10^{-5}$

$d_c^h \approx 1.12 \times 10^{-2}$, $d_i \approx 5.5 \times 10^{-3}$

Double separatrix map

$$DSM: \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a + b \log|\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix}$$

- a i b parameters defined as before.
- $s = 1$ outer separatrix domain U and $s = -1$ inner separatrix domain D.
- $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, where A_1 and A_{-1} are the amplitudes of the outer and inner splittings respectively of the resonant island.

DSM: invariant curves

- **Inv. curves outside island:** DSM reduce to SM and above formulas hold.
- **Inv curves inside island:**

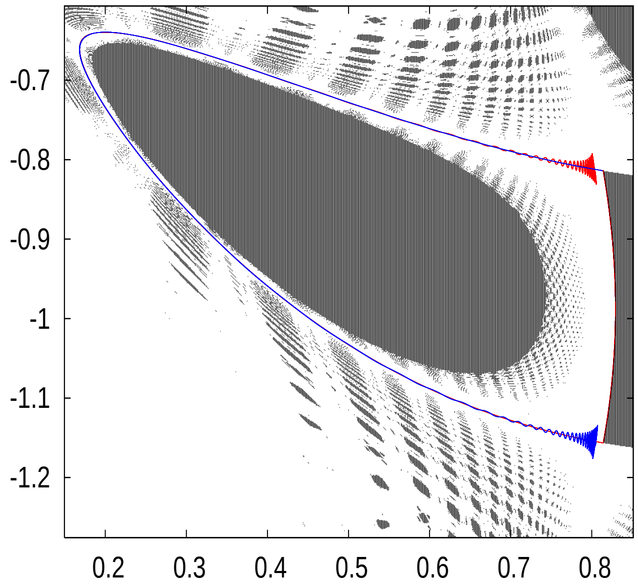
IDEA: Both inner and outer separatrices play a role.

Assume that the dynamics of F inside the "pendulum" like island is modelled by the time one flow of an interpolating Hamiltonian $\mathcal{H}(J, \phi)$. Let $J = J_m$ be the action on the separatrix in the inner domain and J_M be the action on the separatrix in the outer domain. Put
$f = \nabla H(J_m)/\nabla H(J_M)$.

$\longrightarrow$ Then, a distance d measured with respect the outer separatrix becomes a distance fd with respect the inner one.

DSM: example

1:4 resonance Hénon ($H_c(x, y) = ((1 - x^2)c + 2x + y, -x), c = 1.015$).



$\lambda \approx 1.1284291, \sigma_+ \sim 10^{-54}, \sigma_- \sim 10^{-1}$.

Outside (inner) island: $d_c \approx 10^{-52}$.

Inside. Interpolating flow of H_c^4 given by $H(x, y) = H_0 + \delta H_1 + \delta^2 H_2$, with $\delta = 2\pi\alpha - \pi/2$ and

$$H_0 = x^2y^2 - x^4y - xy^4 + x^6/3 + 2x^3y^3 + y^6/3 - x^5y^2 - x^2y^5 - 5x^4y^4/6,$$
$$H_1 = -2x^2 - 2y^2 + 2x^2y + 2xy^2 - x^4 - 2x^3y - y^4 + x^5 - 2x^3y^2 + 2x^2y^3 + 2x^5y - 5x^4y^2/3 + 13x^2y^4/3,$$
$$H_2 = -2x^3 + 4xy^2 - x^4/3 - 4x^3y + x^2y^2/2 - 4y^4/3.$$

The value of ∇H in the maximum (outer zone) of the separatrices oscillates between 0.0086 and 0.0098 depending on the island considered. On the other hand, the corresponding value in the minimum (inner zone) is ≈ 0.00066. Then f is between 13 and 15 which coincides with what is observed in the figure.

DSM: invariant curves, δ small (close to E_0)

Let F be an APM having an elliptic fixed point with rotation number $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$, $\delta \in \mathbb{R} \setminus \mathbb{Q}$.

Denote by $b_1 \in \mathbb{R}$ the first Birkhoff coefficient of the normal form of F around the elliptic point and assume $b_1\delta < 0$.

Then, for $|\delta|$ small enough, the width of the **chaotic outer zone is larger** than the width of the inner one if, and only if, *sign* $b_1 \cdot$ *sign* $b_2 > 0$. Both amplitudes of the stochastic layer are of the **same order of magnitude of the outer splitting.**

Birkhoff zones of instability

Let F be an APM. A **Birkhoff zone of instability** is a rotational non-contractile annulus without rotational invariant curves.

Assume we are interested in the dynamics between two concentric chains of islands. Let d denote the distance between them. A simple model is given by the **biseparatrix map**

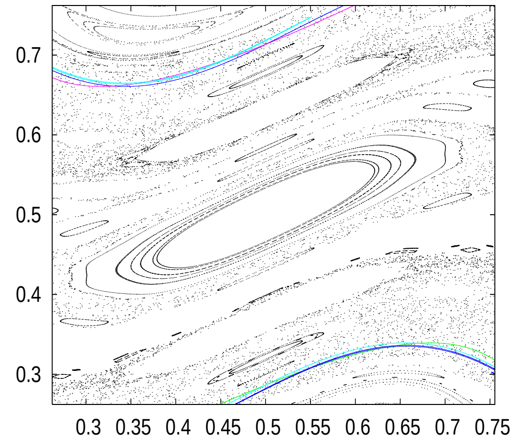
$$BSM : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}$$

$\beta_1 = 1/\log(\lambda)$, $\beta_2 = 1/\log(\mu)$, λ and μ eigenvalues of modulus greater than one associated to the hyperbolic points of each chain of islands.

Just qualitative but...

BSM: twist case

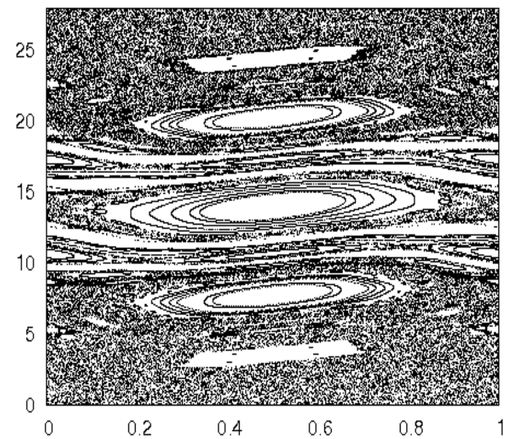
Chirikov standard map with $k = 0.16$.

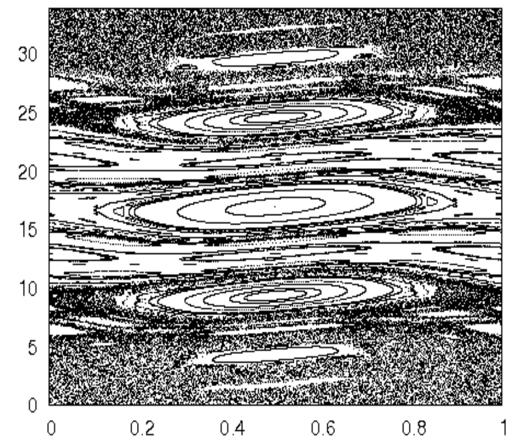


For the corresponding BSM model:

$\beta_1 = \beta_2 \approx 1.0365$

$(\lambda = \mu \approx 2.624248)$





Amplitude splitting outer island $\approx 1.2 \times 10^{-2}$

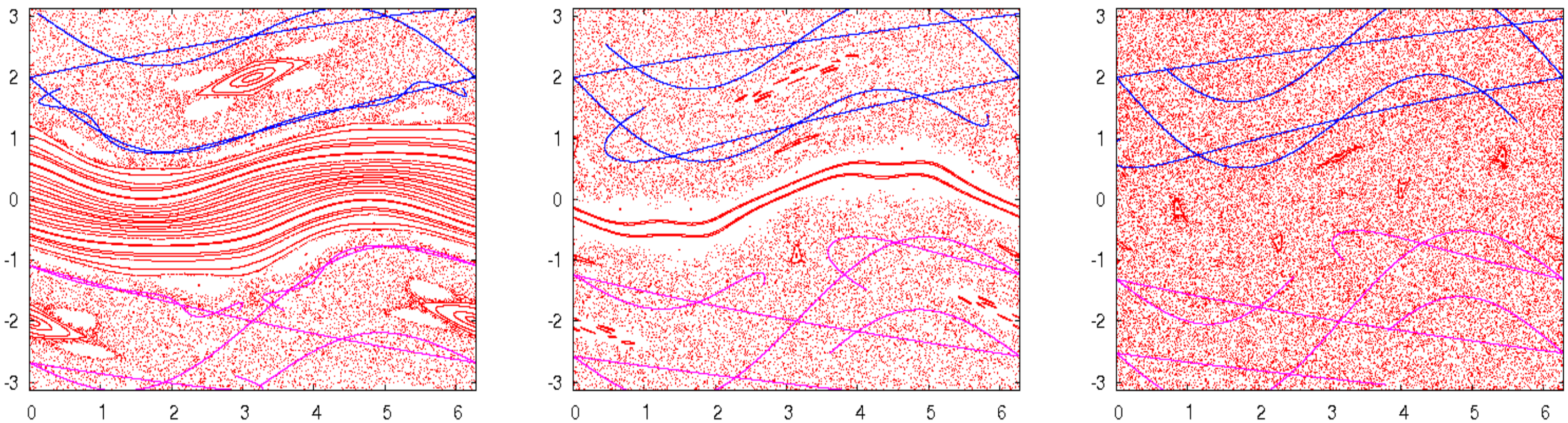
Amplitude splitting inner island $\approx 1.5 \times 10^{-2}$

Distance between the islands ≈ 0.424

$\Rightarrow d$ between 28.2 and 35.4

BSM: Non-twist case

For APM it can be zones without rotational invariant curves but where the twist vanished. $F_b(x,y) = (\bar{x},\bar{y}) = (x+\epsilon(\bar{y}^2 - b), y+\epsilon \sin x)$



$$BSM : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') + \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}$$

The End

Thank you!