Shadowing Orbits for Transition Chains of Invariant Tori

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Joint work with Marian Gidea

Arnold's Paper

Context is Arnold's article on diffusion (1964)

- He assumed
  (i) a perturbation that was a coupling of a rotor with a saddle connection in a pendulum type system;
  (ii) all whiskered tori on the center manifold were assumed to survive the perturbation, and
  (iii) stable and unstable manifolds of nearby tori intersect transversely off the center manifold.
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Generic perturbation:

- Results in some large gaps of size $O(\epsilon^{1/2})$ between tori.
- The splitting of stable and unstable manifolds is $O(\epsilon)$.

Objectives

We use topologically correctly aligned windows:

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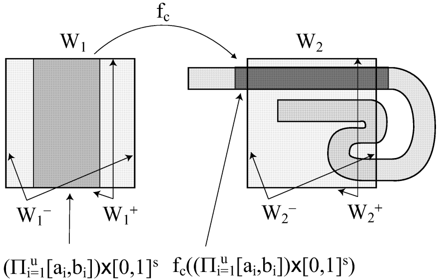
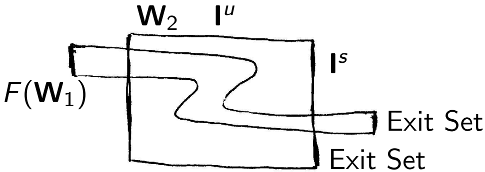
Some of the treatments with large gaps:

- Using variational methods: Mather (2002), Xia (1998), Chen & Yan (2002)
- Using secondary tori and normal forms near the tori: Delshams, de la Llave, & Seara (2003)
- Estimate the time: Gidea & de la Llave (2005, 2007, 2008)

Topologically Correctly Aligned Windows

- A window – a homeomorphic copy of a multi-dimensional rectangle $\mathbf{I}^u \times \mathbf{I}^s$, where the dimensions are split between "expanding" $\mathbf{I}^u$ and "contracting" $\mathbf{I}^s$
- $(\partial \mathbf{I}^u) \times \mathbf{I}^s$ is the exiting set
- One window correctly aligns with another – degree of the projection onto the stretching direction is non-zero:
  $\pi_u f(x, y_0)$ has $\neq 0$ degree on (∂I^u) by homology.
  Exiting directions are consistent.





$\mathbf{W}_2$
$\mathbf{I}^u$
$\mathbf{I}^s$
$F(\mathbf{W}_1)$
Exit Set
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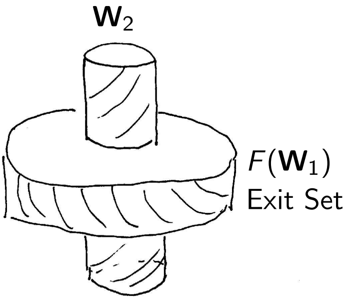

$\mathbf{W}_2$
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Exit Set

Sequence of aligned windows

Theorem

$F : M \to M$ and $\mathbf{B}_i$ a sequence of windows with "expanding direction" chosen for each such that $F(\mathbf{B}_i)$ is correctly aligned with $\mathbf{B}_{i+1}$. Then there exist $\mathbf{x}_i \in \mathbf{B}_i$ such that $F(\mathbf{x}_i) = \mathbf{x}_{i+1}$. The orbit is not necessarily unique.

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The intersection $\bigcap_{i \geq 0} F^i(\mathbf{B}_{-i})$ spans the "expanding" directions
$\bigcap_{i \geq 0} F^i(\mathbf{B}_{-i})$ spans the "contracting" directions.
They must intersect, so $\mathbf{x}_0 \in \bigcap_{i=-\infty}^{\infty} F^i(\mathbf{B}_{-i}) \neq \emptyset$.

Partial History of Correctly Aligned Windows

- Conley (and Conley index)
- Easton (1975, 1978, 1981)
- Easton & McGehee (1979)
- Churchill & Rod (1976, 1980)
- Burns & Weiss (1995): apply to Riemannian geometry
- Kennedy & Yorke (2001): general types of intersections in 2 dimensions
- Robinson: (2002) Apply to transition chains of tori.
- Gidea & Robinson (2003)
- Zgliczynski and Gidea (2004): without (co-)homology

Topologically transverse homoclinic point

Theorem

If F has a hyperbolic fixed point with a topologically transverse homoclinic point, then there is an invariant set Λ and a semiconjugacy $h : \Lambda \to \Sigma_A$ where Σ_A is a subshift of finite type. The map h is onto but not necessarily one-to-one. More that one point can have the same symbol sequence. Complexity of a horseshoe.

Burns and Weiss (1995)
Mischaikow & Mrozek (1995)

A local topologically transverse intersection of $W^s(\mathbf{p}) \cap W^u(\mathbf{p})$ with intersection number 2 in $\mathbb{R}^4 \approx \mathbb{C}^2$ can be like $\{(z, 0)\}$ & $\{(z, z^2)\}$.

Assumptions:Invariant Tori

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For $\epsilon = 0$, W^c_0 twist filled with invariant circles $\mathbb{T}_{0,\alpha}$ &
Hyperbolic in other $2n - 2$ directions.
A priori hyperbolic or unstable
$W^u_0(W^c_0) = W^s_0(W^c_0)$ and $W^u_0(\mathbb{T}_{0,\alpha}) = W^s_0(\mathbb{T}_{0,\alpha})$.

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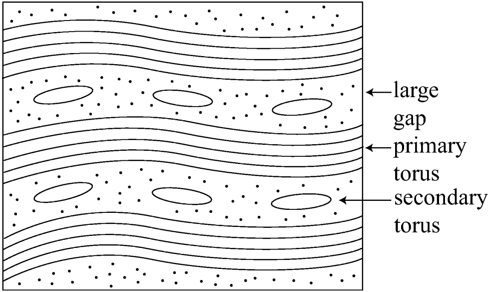
For $\epsilon \neq 0$, on center W^c_ϵ a Cantor set $\mathcal{C}$ of invariant tori $\{\, \mathbb{T}_{\epsilon,\alpha} \,\}_{\alpha \in \mathcal{C}}$.

- Each $\mathbb{T}_{\epsilon,\alpha}$ is topologically transverse with irrational rotation number.
- The family is uniformly Lipschitz.
- Assume that there are no isolated tori.
- An "interior" torus is accumulated on both sides by other tori.
- Assume that the differentiable interior tori are dense (KAM).
- "Boundary" tori are boundaries of a BZI.

Birkhoff Zone of Instability

A Birkhoff Zone of Instability, BZI, is a region in two dimensional twist map with boundary Lipschitz tori $\mathbb{T}_{\epsilon,\alpha_0}$ and $\mathbb{T}_{\epsilon,\alpha_1}$ with no essential invariant closed curves between.



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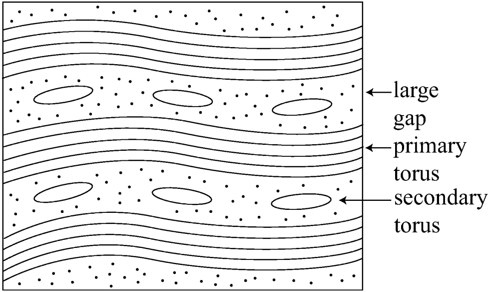
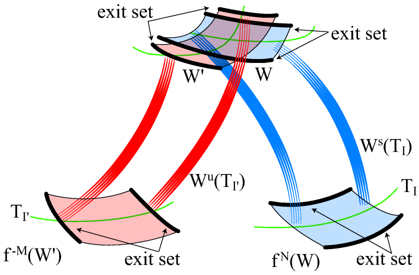


Birkhoff: There is an orbit that goes from arbitrarily near $\mathbb{T}_{\epsilon,\alpha_0}$ to arbitrarily near $\mathbb{T}_{\epsilon,\alpha_1}$.

Assumptions: Transversality and Scattering Map

For $\epsilon \neq 0$, assume $W^u_\epsilon(W^c_\epsilon)$ and $W^s_\epsilon(W^c_\epsilon)$ transverse off W^c_ϵ.
$W^u_\epsilon(pts)$ transverse to $W^s_\epsilon(W^c_\epsilon)$.

Defines a scattering map $\mathcal{S}$ from W^c_ϵ to itself by going out along $W^u_\epsilon(pts)$ and back along $W^s_\epsilon(pts)$.



Sequence of Tori

For our theorem, we assume that there is a a sequence of tori $\{\mathbb{T}_i = \mathbb{T}_{\epsilon,\alpha_i}\}$ from Cantor set, $\alpha_i \in \mathcal{C}$, such that the following hold:
(Not necessarily a perturbation so drop ϵ and α):

(i) There is a subsequence i_k such that the region in W^c between $\mathbb{T}_{i_k}$ and $\mathbb{T}_{i_k+1}$ is a BZI.

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(**ii**) For $i_{k-1} + 1 < i < i_k$, the tori $\{\mathbb{T}_i\}$ are not on the boundary of a BZI, are interior tori of $\mathcal{C}$, and are differentiable.

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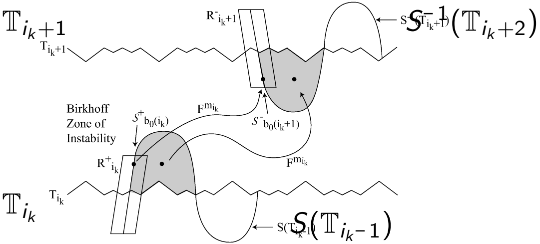
(**iv**) Each $\mathbb{T}_i$ is topologically transitive, including those on boundary of a BZI.

(**v**) For the Lipschitz boundaries of a BZI, $\{\mathbb{T}_{i_k}, \mathbb{T}_{i_k+1}\}$.
the image of $\mathcal{S}(\mathbb{T}_{i_k-1})$ topologically crosses $\mathbb{T}_{i_k}$ 3 times in an interval of definition of scattering map, and preimage $\mathcal{S}^{-1}(\mathbb{T}_{i_k+2})$ topologically crosses $\mathbb{T}_{i_k+1}$ 3 times.

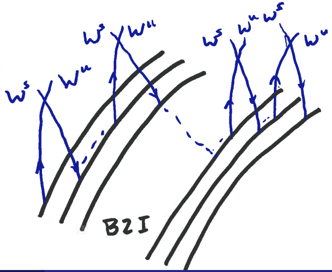


Main Theorem

Theorem

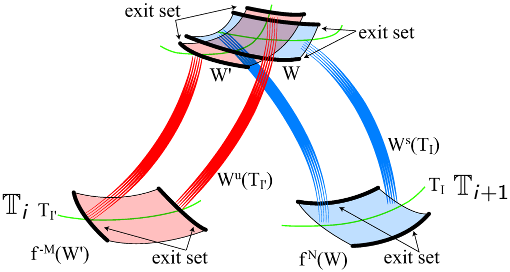
Assume there is a sequence of tori $\{\mathbb{T}_i\}$, such that the image $\mathcal{S}(\mathbb{T}_i)$ using the scattering map is topologically transverse to $\mathbb{T}_{i+1}$, with 3 points of intersection at the boundaries of a BZI.
Then there is an orbit which comes near the successive $\mathbb{T}_i$.

The orbit that we should exists is like the one using variational methods and not the one found using secondary tori as found by de la Llave et al.

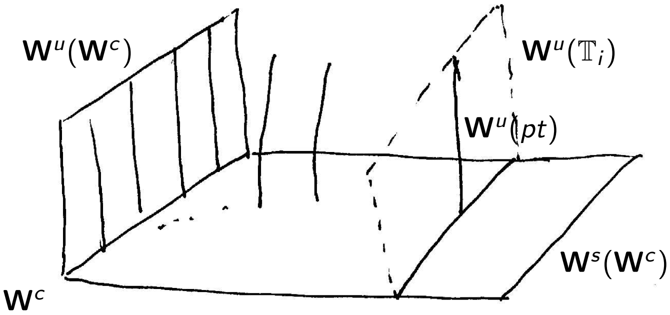


Proof: Windows for an interior tori

For two interior tori $\mathbb{T}_i$ and $\mathbb{T}_{i+1}$, we get the correctly aligned windows as follows:



Proof: Stable and unstable directions of windows



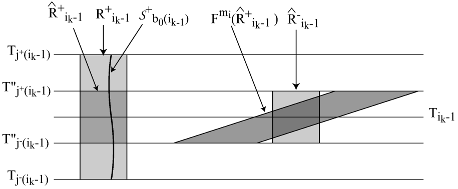
The iterates of the unstable manifolds of a point $\mathbf{W}^u(pt)$ crosses the stable manifold $\mathbf{W}^s(\mathbf{W}^c)$ transversely.

Its iterates converge in a C^1 fashion toward the unstable manifold of a point in $\mathbf{W}^c$ (that changes with each iterate). A Lambda Lemma.

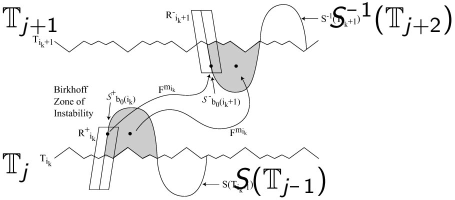
Proof: Adjustment on an interior tori

$W^s(\mathbb{T}_i)$ and $W^u(\mathbb{T}_i)$ are not transverse along $\mathbb{T}_i$.

- But twist on W^c and topologically transitivity along torus allows iterate of entering window along $\mathbb{T}_i$ to be correctly aligned with exiting window for $\mathbb{T}_i$.
- Not a boundary tori so can use nearby tori to control the top and bottom edges of window in W^c.



Consider a BZI with boundary $\mathbb{T}_j \cup \mathbb{T}_{j+1}$ where $j = i_k$.
$S(\mathbb{T}_{j-1})$ and $\mathbb{T}_j$ form one region in the BZI and $S^{-1}(\mathbb{T}_{j+2})$ and $\mathbb{T}_{j+1}$ form another region in BZI. (Shaded regions in figure.)



Proof: Crossing a BZI

Consider a BZI with boundary $\mathbb{T}_j \cup \mathbb{T}_{j+1}$ where $j = i_k$.
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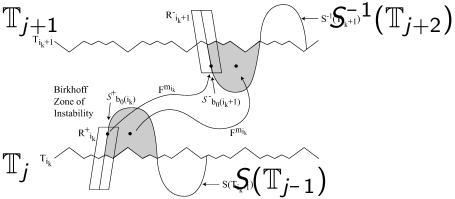

By the proof of orbit crossing the BZI, there is a point inside the boundary region near $\mathbb{T}_j$ going inside the boundary region near $\mathbb{T}_{j+1}$. Thus the orbit of $S(\mathbb{T}_{j-1})$ intersects $S^{-1}(\mathbb{T}_{j+2})$.

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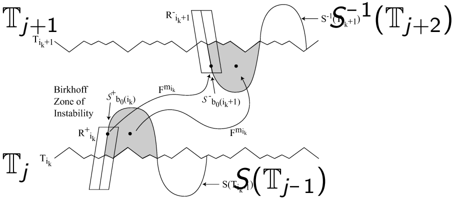

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A thin window along $S(\mathbb{T}_{j-1})$ is correctly aligned with $S^{-1}(\mathbb{T}_{j+2})$.