Regularity Properties of Critical Invariant Circles of Twist Maps

Nikola P. Petrov, University of Oklahoma
Arturo Olvera, IIMAS-UNAM

November 28, 2008

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori	$\longrightarrow$	Topological barrier in the phase space	$\longrightarrow$	Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability :$\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori	$\longrightarrow$	Topological barrier in the phase space	$\longrightarrow$	Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability $:\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori $\longrightarrow$ Topological barrier in the phase space $\longrightarrow$ Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability $:\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori $\longrightarrow$ Topological barrier in the phase space $\longrightarrow$ Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability :$\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori $\longrightarrow$ Topological barrier in the phase space $\longrightarrow$ Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability :$\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori $\longrightarrow$ Topological barrier in the phase space $\longrightarrow$ Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability :$\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori	$\longrightarrow$	Topological barrier in the phase space	$\longrightarrow$	Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability :$\Rightarrow$ Critical Invariant Circle

Global Stability of Mechanical Systems

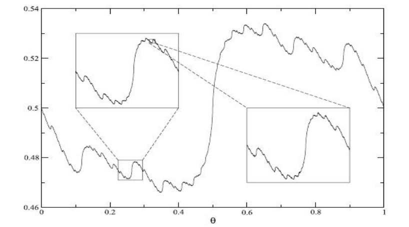
- Two degree of freedom Hamiltonian System (2DFHS):

KAM tori	$\longrightarrow$	Topological barrier in the phase space	$\longrightarrow$	Global stability

Hamiltonian flow $\Longleftrightarrow$ Area Preserving Twist Map (APTM)

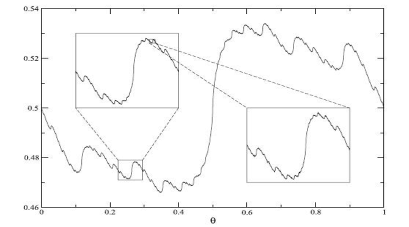
- Poincaré section: (2DFHS) $\longrightarrow$(APTM)
- KAM torus (2DFHS) $\longrightarrow$Invariant Circle (APTM)
- One parameter family of APTM
- Border of stability $:\Rightarrow$ Critical Invariant Circle

- Critical Invariant Circles (CIC) exhibit remarkable scaling properties at the boundary of chaos (Shenker & Kadanoff, 82)



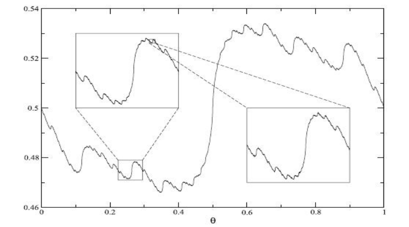
- Renormalization Group Analysis explains scaling properties (MacKay, 83)
- Scale invariances determine the regularity of the CIC
- CIC ⟶Fractional regularity ⟶Universal Property

- Critical Invariant Circles (CIC) exhibit remarkable scaling properties at the boundary of chaos (Shenker & Kadanoff, 82)



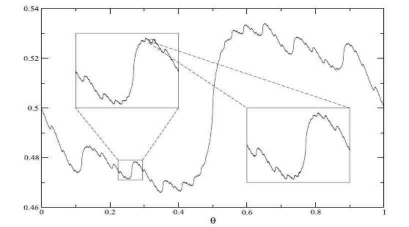
- Renormalization Group Analysis explains scaling properties (MacKay, 83)
- Scale invariances determine the regularity of the CIC
- CIC ⟶Fractional regularity ⟶Universal Property

- Critical Invariant Circles (CIC) exhibit remarkable scaling properties at the boundary of chaos (Shenker & Kadanoff, 82)



- Renormalization Group Analysis explains scaling properties (MacKay, 83)
- Scale invariances determine the regularity of the CIC
- CIC ⟶Fractional regularity ⟶Universal Property

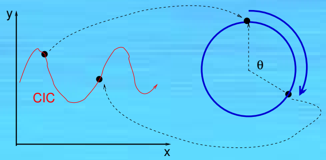
- Critical Invariant Circles (CIC) exhibit remarkable scaling properties at the boundary of chaos (Shenker & Kadanoff, 82)



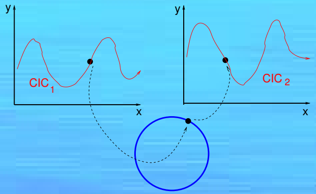
- Renormalization Group Analysis explains scaling properties (MacKay, 83)
- Scale invariances determine the regularity of the CIC
- CIC $\longrightarrow$Fractional regularity $\longrightarrow$Universal Property

Our goal: Compute the regularity of CIC

- Regularity of the conjugation:
  - CIC $\longrightarrow$rigid rotation

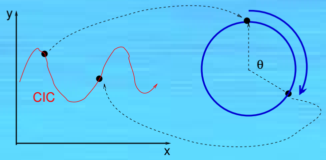


- Regularity between the conjugations:
  - $\mathrm{CIC}_1 \longrightarrow \mathrm{CIC}_2$

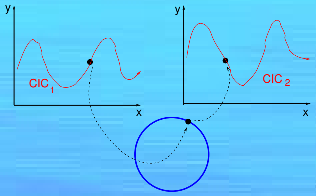


Our goal: Compute the regularity of CIC

- Regularity of the conjugation:
  - CIC $\longrightarrow$rigid rotation

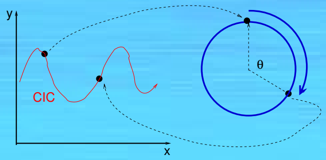


- Regularity between the conjugations:
  - $\text{CIC}_1 \longrightarrow \text{CIC}_2$

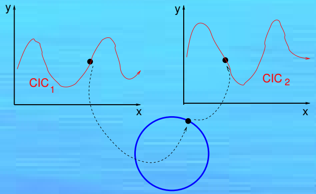


Our goal: Compute the regularity of CIC

- Regularity of the conjugation:
  - CIC $\longrightarrow$rigid rotation

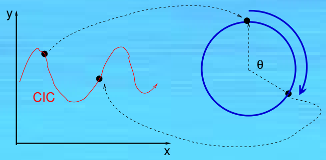


- Regularity between the conjugations:
  - $\mathrm{CIC}_1 \longrightarrow \mathrm{CIC}_2$

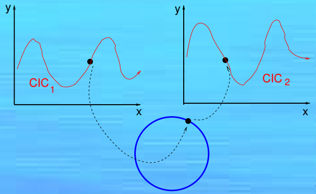


Our goal: Compute the regularity of CIC

- Regularity of the conjugation:
  - CIC $\longrightarrow$rigid rotation



- Regularity between the conjugations:
  - $\text{CIC}_1 \longrightarrow \text{CIC}_2$



How to compute fractional regularity?

- De la Llave and Petrov used Harmonic Analysis Methods to determine the regularity of Critical Circles Maps, $\mathbb{T} \mapsto \mathbb{T}$ (Llave & Petrov,02)

- We extend this methodology to the study of CIC of APTM

How to compute fractional regularity?

- De la Llave and Petrov used Harmonic Analysis Methods to determine the regularity of Critical Circles Maps, $\mathbb{T} \mapsto \mathbb{T}$ (Llave & Petrov,02)

- We extend this methodology to the study of CIC of APTM

Area Preserving Twist Maps (APTM)

- One parameter family of APTM
  $F_\lambda : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$:

  $$y_{n+1} = y_n + \lambda V(x_n)$$
  $$x_{n+1} = x_n + y_{n+1}$$

  where $V(x) = V(x+1)$ and has zero-average.

- Rotation number: $\rho = \lim_{n \to \pm\infty} \frac{x_n - x_0}{n}$

Area Preserving Twist Maps (APTM)

- One parameter family of APTM
  $F_\lambda : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$:

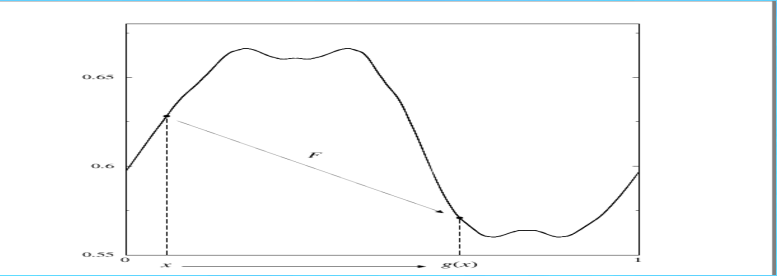
$$\begin{aligned} y_{n+1} &= y_n + \lambda V(x_n) \\ x_{n+1} &= x_n + y_{n+1} \end{aligned}$$

  where $V(x) = V(x+1)$ and has zero-average.

- Rotation number: $\rho = \lim_{n \to \pm\infty} \dfrac{x_n - x_0}{n}$

Area Preserving Twist Maps (APTM)

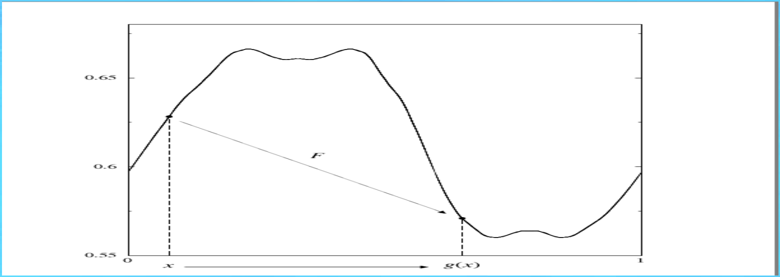
- Invariant Circle of rotation number ρ, IC_ρ is the graph of a Lipschitz function (Birkhoff,)



- If ρ is a Diophantine number $\longrightarrow \mathrm{IC}_\rho$ depends analytically on λ
- Golden IC_ρ $\longrightarrow \rho = \sigma_G = [1, 1, 1, \ldots]$

Area Preserving Twist Maps (APTM)

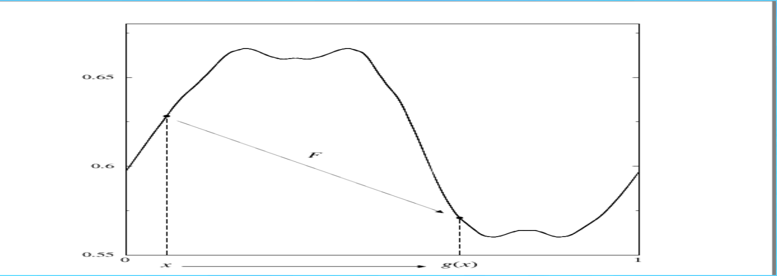
- Invariant Circle of rotation number ρ, IC_ρ is the graph of a Lipschitz function (Birkhoff,)



- If ρ is a Diophantine number $\longrightarrow IC_\rho$ depends analytically on λ
- Golden IC_ρ $\longrightarrow \rho = \sigma_G = [1, 1, 1, \dots]$

Area Preserving Twist Maps (APTM)

- Invariant Circle of rotation number ρ, IC_ρ is the graph of a Lipschitz function (Birkhoff,)



- If ρ is a Diophantine number $\longrightarrow \mathrm{IC}_\rho$ depends analytically on λ
- Golden IC_ρ $\longrightarrow \rho = \sigma_G = [1, 1, 1, \ldots]$

Existence of Invariant Circles

- If $\lambda \sup_x |V(x)| > 1 \longrightarrow \nexists$ any IC_ρ
- If $\lambda > 4/3 \longrightarrow \nexists$ Golden IC_ρ
- Conjecture: For Diophantine ρ exists $\bar{\lambda}_\rho$ such that:

$$\exists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| < \bar{\lambda}_\rho$$

and

$$\nexists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| > \bar{\lambda}_\rho$$

- Critical Invariant Circle (CIC)

$$\lambda \longrightarrow \bar{\lambda}_\rho \Longrightarrow \mathrm{IC}_\rho \longrightarrow \mathrm{CIC}$$

Existence of Invariant Circles

- If $\lambda \sup_x |V(x)| > 1 \longrightarrow \nexists$ any IC_ρ
- If $\lambda > 4/3 \longrightarrow \nexists$ Golden IC_ρ
- Conjecture: For Diophantine ρ exists $\bar{\lambda}_\rho$ such that:

$$\exists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| < \bar{\lambda}_\rho$$

and

$$\nexists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| > \bar{\lambda}_\rho$$

- Critical Invariant Circle (CIC)

$$\lambda \longrightarrow \bar{\lambda}_\rho \Longrightarrow \mathrm{IC}_\rho \longrightarrow \mathrm{CIC}$$

Existence of Invariant Circles

- If $\lambda \sup_x |V(x)| > 1 \longrightarrow \nexists$ any IC_ρ
- If $\lambda > 4/3 \longrightarrow \nexists$ Golden IC_ρ
- Conjecture: For Diophantine ρ exists $\bar{\lambda}_\rho$ such that:

$$\exists \; IC_\rho \qquad \text{if} \qquad |\lambda| < \bar{\lambda}_\rho$$

and

$$\nexists \; IC_\rho \qquad \text{if} \qquad |\lambda| > \bar{\lambda}_\rho$$

- Critical Invariant Circle (CIC)

$$\lambda \longrightarrow \bar{\lambda}_\rho \Longrightarrow IC_\rho \longrightarrow \text{CIC}$$

Existence of Invariant Circles

- If $\lambda \sup_x |V(x)| > 1 \longrightarrow \nexists$ any IC_ρ
- If $\lambda > 4/3 \longrightarrow \nexists$ Golden IC_ρ
- Conjecture: For Diophantine ρ exists $\bar{\lambda}_\rho$ such that:

$$\exists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| < \bar{\lambda}_\rho$$

and

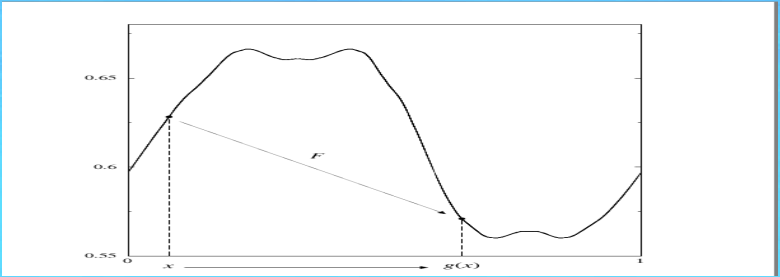
$$\nexists \ \mathrm{IC}_\rho \qquad \text{if} \qquad |\lambda| > \bar{\lambda}_\rho$$

- Critical Invariant Circle (CIC)

$$\lambda \longrightarrow \bar{\lambda}_\rho \Longrightarrow \mathrm{IC}_\rho \longrightarrow \mathrm{CIC}$$

Description of CIC:

- $R : \mathbb{T} \mapsto \mathbb{R}$ is the graph of IC_ρ

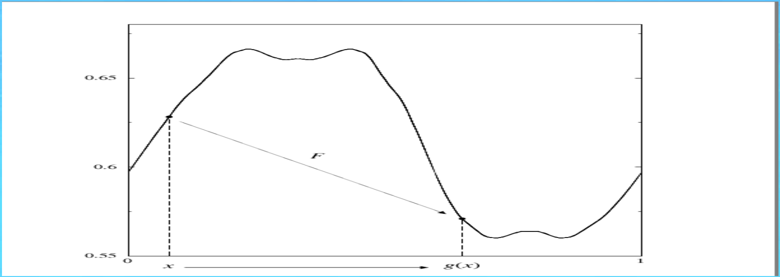


- Advance Map $g : \mathbb{T} \mapsto \mathbb{T}$ defined by

$$F(x, R(x)) = (g(x), R \circ g(x))$$

Description of CIC:

- $R : \mathbb{T} \mapsto \mathbb{R}$ is the graph of IC_ρ



- Advance Map $g : \mathbb{T} \mapsto \mathbb{T}$ defined by

$$F(x, R(x)) = (g(x), R \circ g(x))$$

Description of CIC:

- Hull Map $\Psi : \mathbb{T} \mapsto \mathbb{T} \times \mathbb{R}$ such that:

$$F \circ \Psi(x) = \Psi(x + \rho)$$

- Conjugation function $h : \pi_1 \circ \Psi : \mathbb{T} \mapsto \mathbb{T}$ where:
$$g \circ h(x) = h(x + \rho)$$

Description of CIC:

- Hull Map $\Psi : \mathbb{T} \mapsto \mathbb{T} \times \mathbb{R}$ such that:

$$F \circ \Psi(x) = \Psi(x + \rho)$$

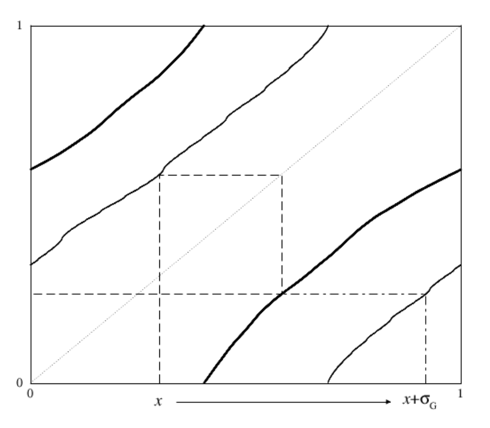
- Conjugation function $h : \pi_1 \circ \Psi : \mathbb{T} \mapsto \mathbb{T}$ where:

$$g \circ h(x) = h(x + \rho)$$

Description of CIC:

- Conjugating g to a rotation by σ_{G}:

$$g \circ h(x) = h(x + \sigma_{\mathrm{G}})$$



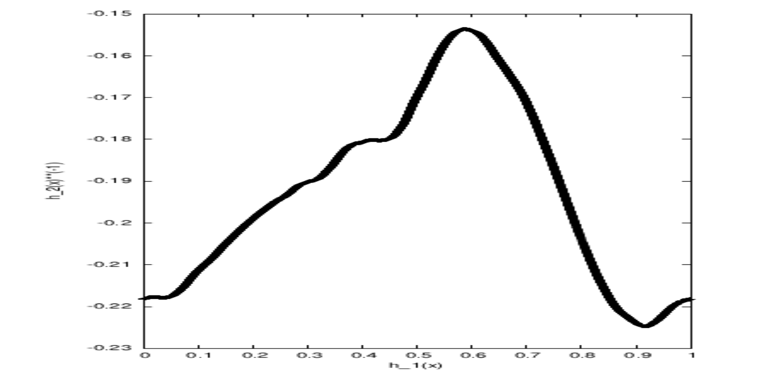
(g = thick line, h = thin line)

Big Conjugacies

- Conjugation of two CIC, γ_1 and γ_2:

$$G^{\gamma_1,\gamma_2} = g_{\gamma_1} \circ g_{\gamma_2}^{-1}$$

$$H^{\gamma_1,\gamma_2} = h_{\gamma_1} \circ h_{\gamma_2}^{-1}$$



HÖLDER REGULARITY

For $\kappa = n + \xi$ with $n \in \mathbb{Z}$ and $\xi \in (0, 1)$:

The function $K : \mathbb{T} \to \mathbb{R}$ has global Hölder exponent κ ($K \in \Lambda_\kappa(\mathbb{T})$) when K is n time differentiable and, for some constant $C > 0$:

$$|D^n K(\theta_1) - D^n K(\theta_0)| \leq C\,|\theta_1 - \theta_0|^\xi$$

$\kappa(K) :=$ Is the Hölder regularity of K

HÖLDER REGULARITY

For $\kappa = n + \xi$ with $n \in \mathbb{Z}$ and $\xi \in (0,1)$:

The function $K : \mathbb{T} \to \mathbb{R}$ has global Hölder exponent κ ($K \in \Lambda_\kappa(\mathbb{T})$) when K is n time differentiable and, for some constant $C > 0$:

$$|D^n K(\theta_1) - D^n K(\theta_0)| \leq C\, |\theta_1 - \theta_0|^\xi$$

$\kappa(K) :=$ Is the Hölder regularity of K

HÖLDER REGULARITY

For $\kappa = n + \xi$ with $n \in \mathbb{Z}$ and $\xi \in (0, 1)$:

The function $K : \mathbb{T} \to \mathbb{R}$ has global Hölder exponent κ ($K \in \Lambda_\kappa(\mathbb{T})$) when K is n time differentiable and, for some constant $C > 0$:

$$|D^n K(\theta_1) - D^n K(\theta_0)| \leq C \, |\theta_1 - \theta_0|^\xi$$

$\kappa(K) :=$ Is the Hölder regularity of K

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - $\exists!$ Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number ($\kappa(R)$)
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - $\exists!$ Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number ($\kappa(R)$)
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - $\exists!$ Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number $(\kappa(R))$
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - ∃! Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number ($\kappa(R)$)
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - ∃! Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number ($\kappa(R)$)
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

CIC and Universality

- Universality: A characteristic is universal when it takes the same value in a open set of functions
- Conjectures:
  - $\exists!$ Nontrivial fixed point of renormalization operator $\Rightarrow$ Universal property
  - The regularity of R is a universal number $(\kappa(R))$
  - The regularity of g, h and h^{-1} are universal numbers
  - Regularity of "Big" conjugacies:

$$\kappa(h) < \kappa(R)$$
$$\kappa(h) < \kappa(H)$$
$$\kappa(g) < \kappa(G)$$

Poisson kernel method

- Poisson kernel (periodic case):

$$P_s(x) = \sum_{k\in\mathbb{Z}} s^{|k|}\, \mathrm{e}^{2\pi i k x} = \frac{1-s^2}{1-2s\cos 2\pi x + s^2}\,, \qquad s\in[0,1)$$

-

$$\left(\mathrm{e}^{-t\sqrt{-\Delta}}\, h\right)(x) = \left(P_{\exp(-2\pi t)} * h\right)(x) = \sum_{k\in\mathbb{Z}} \hat{h}_k\, \mathrm{e}^{-2\pi t|k|}\, \mathrm{e}^{2\pi i k x}\,.$$

- Theorem ("Poisson kernel method"):
  $h\in\Lambda_\alpha(\mathbb{T})$ if and only if $\forall\,\eta\geq 0$

$$\left\|\left(\frac{\partial}{\partial t}\right)^{\eta} \mathrm{e}^{-t\sqrt{-\Delta}}\, h\right\|_{L^\infty} \leq C\, t^{\alpha-\eta}\,.$$

Poisson kernel method

- Poisson kernel (periodic case):

$$P_s(x) = \sum_{k\in\mathbb{Z}} s^{|k|}\, \mathrm{e}^{2\pi i k x} = \frac{1-s^2}{1-2s\cos 2\pi x + s^2}\,, \qquad s\in[0,1)$$

- $$\left(\mathrm{e}^{-t\sqrt{-\Delta}}\, h\right)(x) = \left(P_{\exp(-2\pi t)} * h\right)(x) = \sum_{k\in\mathbb{Z}} \hat{h}_k\, \mathrm{e}^{-2\pi t|k|}\, \mathrm{e}^{2\pi i k x}\,.$$

- Theorem ("Poisson kernel method"):
  $h\in\Lambda_\alpha(\mathbb{T})$ if and only if $\forall\,\eta\geq 0$

$$\left\|\left(\frac{\partial}{\partial t}\right)^{\eta} \mathrm{e}^{-t\sqrt{-\Delta}}\, h\right\|_{L^\infty} \leq C\, t^{\alpha-\eta}\,.$$

Poisson kernel method

- Poisson kernel (periodic case):

$$P_s(x) = \sum_{k\in\mathbb{Z}} s^{|k|}\, \mathrm{e}^{2\pi i k x}$$

$$= \frac{1-s^2}{1-2s\cos 2\pi x + s^2}\,, \qquad s\in[0,1)$$

-

$$\left(\mathrm{e}^{-t\sqrt{-\Delta}}\, h\right)(x) = \left(P_{\exp(-2\pi t)} * h\right)(x)$$

$$= \sum_{k\in\mathbb{Z}} \hat{h}_k\, \mathrm{e}^{-2\pi t|k|}\, \mathrm{e}^{2\pi i k x}\,.$$

- Theorem ("Poisson kernel method"):
$h\in\Lambda_\alpha(\mathbb{T})$ if and only if $\forall\,\eta\geq 0$

$$\left\| \left(\frac{\partial}{\partial t}\right)^{\eta} \mathrm{e}^{-t\sqrt{-\Delta}}\, h \right\|_{L^\infty} \leq C\, t^{\alpha-\eta}\,.$$

Advantages of the "Poisson kernel method"

$$\log \left\| \left(\frac{\partial}{\partial t} \right)^{\eta} e^{-t\sqrt{-\Delta}} \, h \right\|_{L^\infty} \leq \text{const} + (\alpha - \eta) \log t$$

- the number of values of t is not limited;
- all known Fourier coefficients taken into account in calculating each point;
- different η values $\rightarrow$ numerical tests.

Advantages of the "Poisson kernel method"

$$\log \left\| \left(\frac{\partial}{\partial t} \right)^{\eta} \mathrm{e}^{-t\sqrt{-\Delta}}\, h \right\|_{L^\infty} \leq \text{const} + (\alpha - \eta) \log t$$

- the number of values of t is not limited;
- all known Fourier coefficients taken into account in calculating each point;
- different η values $\to$ numerical tests.

Advantages of the "Poisson kernel method"

$$\log \left\| \left(\frac{\partial}{\partial t} \right)^{\eta} \mathrm{e}^{-t\sqrt{-\Delta}}\, h \right\|_{L^\infty} \leq \mathrm{const} + (\alpha - \eta) \log t$$

- the number of values of t is not limited;
- all known Fourier coefficients taken into account in calculating each point;
- different η values $\to$ numerical tests.

Numerical computation of CIC

Area Preserving Twist Maps (APTM)

Let X_ω be an orbit with rotation number ω then:

- Birkhoff: For any rational number $\omega \in [\rho_1, \rho_2]$ exists at least a pair of periodic orbits with rotation number ω.
- Aubry Mather: Let $\{\omega_i\}_{i=0}^{\infty}$, $\omega_i \in \mathbb{Q}$, s.t.

$$\lim_{i \to \infty} \omega_i = \rho$$

then the limit set of $\{X_{\omega_i}\}_{i=0}^{\infty}$ converges to an IC_ρ (or a Cantorus)

Numerical computation of CIC

Area Preserving Twist Maps (APTM)

Let X_ω be an orbit with rotation number ω then:

- Birkhoff: For any rational number $\omega \in [\rho_1, \rho_2]$ exists at least a pair of periodic orbits with rotation number ω.
- Aubry Mather: Let $\{\omega_i\}_{i=0}^{\infty}$, $\omega_i \in \mathbb{Q}$, s.t.

$$\lim_{i \to \infty} \omega_i = \rho$$

then the limit set of $\{X_{\omega_i}\}_{i=0}^{\infty}$ converges to an IC_ρ (or a Cantorus)

Numerical computation of CIC

Area Preserving Twist Maps (APTM)
Let X_ω be an orbit with rotation number ω then:

- Birkhoff: For any rational number $\omega \in [\rho_1, \rho_2]$ exists at least a pair of periodic orbits with rotation number ω.
- Aubry Mather: Let $\{\omega_i\}_{i=0}^{\infty}$, $\omega_i \in \mathbb{Q}$, s.t.

$$\lim_{i \to \infty} \omega_i = \rho$$

  then the limit set of $\{X_{\omega_i}\}_{i=0}^{\infty}$ converges to an IC_ρ (or a Cantorus)

Numerical computation of CIC

Area Preserving Twist Maps (APTM)

Let X_ω be an orbit with rotation number ω then:

- Birkhoff: For any rational number $\omega \in [\rho_1, \rho_2]$ exists at least a pair of periodic orbits with rotation number ω.
- Aubry Mather: Let $\{\omega_i\}_{i=0}^{\infty}$, $\omega_i \in \mathbb{Q}$, s.t.

$$\lim_{i\to\infty} \omega_i = \rho$$

then the limit set of $\{X_{\omega_i}\}_{i=0}^{\infty}$ converges to an IC_ρ (or a Cantorus)

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists \quad \mathrm{IC}_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists \quad \mathrm{IC}_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists\ \ \mathrm{IC}_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists\ \ \mathrm{IC}_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists \ \mathrm{IC}_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists \ \mathrm{IC}_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists \ IC_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists \ IC_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists \quad \mathrm{IC}_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists \quad \mathrm{IC}_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Greene's residues method

Greene criterion to determine CIC with rotation number ρ:

- Let $\mathcal{R}_i$ be the residue of an hyperbolic periodic orbits $\{X_{\omega_i}\}_{i=0}^{\infty}$, such that

$$\lim_{i\to\infty} \omega_i = \rho$$

- X_{ω_i} are the approximants of an IC_ρ
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto 0$ then $\exists \quad \mathrm{IC}_\rho$
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -\infty$ then $\nexists \quad \mathrm{IC}_\rho$ (Cantorus)
- If $\lim_{i\to\infty} \mathcal{R}_i \mapsto -0.25542\ldots$ then IC_ρ is critical

Numerical Experiments

We studied six APTM:

$$\begin{cases} y_{n+1} = y_n + \lambda V(x_n) \\ x_{n+1} = x_n + y_{n+1} \end{cases}$$

- Standard map:

$$V(x) = \sin(2\pi x)$$

- Two harmonics map:

$$V(x) = \sin(2\pi x) + 0.03 \sin(6\pi x)$$

- Critical map:

$$V(x) = \sin(2\pi x) - 0.5 \sin(4\pi x)$$

Numerical Experiments

We studied six APTM:
$$\begin{cases} y_{n+1} = y_n + \lambda V(x_n) \\ x_{n+1} = x_n + y_{n+1} \end{cases}$$

- Standard map:

$$V(x) = \sin(2\pi x)$$

- Two harmonics map:

$$V(x) = \sin(2\pi x) + 0.03 \sin(6\pi x)$$

- Critical map:

$$V(x) = \sin(2\pi x) - 0.5 \sin(4\pi x)$$

Numerical Experiments

We studied six APTM: $$\begin{cases} y_{n+1} = y_n + \lambda V(x_n) \\ x_{n+1} = x_n + y_{n+1} \end{cases}$$

- Standard map:

$$V(x) = \sin(2\pi x)$$

- Two harmonics map:

$$V(x) = \sin(2\pi x) + 0.03\sin(6\pi x)$$

- Critical map:

$$V(x) = \sin(2\pi x) - 0.5\sin(4\pi x)$$

Numerical Experiments

We studied six APTM:
$$\begin{cases} y_{n+1} = y_n + \lambda V(x_n) \\ x_{n+1} = x_n + y_{n+1} \end{cases}$$

- Standard map:

$$V(x) = \sin(2\pi x)$$

- Two harmonics map:

$$V(x) = \sin(2\pi x) + 0.03\sin(6\pi x)$$

- Critical map:

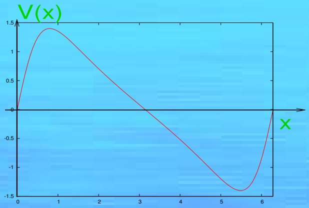
$$V(x) = \sin(2\pi x) - 0.5\sin(4\pi x)$$

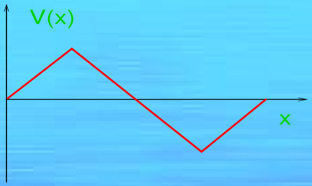
- Analytical map:

$$V(x) = \frac{\sin(2\pi x)}{1 - \beta \cos(2\pi x)} \qquad \beta = 0.2, 0.4$$

- Tent map:

$$V(x) = \sum_{j=1}^{17} c_j \sin(2\pi j x) \qquad c_j = \begin{cases} (-1)^{\frac{j+1}{2}} \frac{4}{\pi^2 j^2} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$



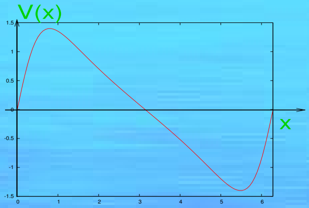


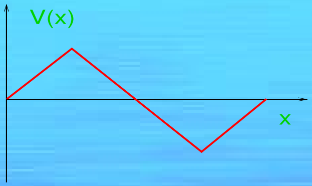
- Analytical map:

$$V(x) = \frac{\sin(2\pi x)}{1 - \beta\cos(2\pi x)} \qquad \beta = 0.2, 0.4$$

- Tent map:

$$V(x) = \sum_{j=1}^{17} c_j \sin(2\pi j x) \qquad c_j = \begin{cases} (-1)^{\frac{j+1}{2}} \frac{4}{\pi^2 j^2} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$





Numerical results

- Rotation number(CIC) = Golden mean
- Rotation number of the approximants $\rho = 832040/1346269$

- CIC max error: 10^{-23}, Residue max diff: 10^{-10}
- Fourier uniformly spaced grid $\rightarrow 2^{20}$ points
- CPL algorithm test: $\eta = 1, 2, 3, 4, 5$

Numerical results

- Rotation number(CIC) = Golden mean
- Rotation number of the approximants $\rho = 832040/1346269$

- CIC max error: 10^{-23}, Residue max diff: 10^{-10}
- Fourier uniformly spaced grid $\rightarrow 2^{20}$ points
- CPL algorithm test: $\eta = 1, 2, 3, 4, 5$

Numerical results

- Rotation number(CIC) = Golden mean
- Rotation number of the approximants $\rho = 832040/1346269$

- CIC max error: 10^{-23}, Residue max diff: 10^{-10}
- Fourier uniformly spaced grid $\rightarrow 2^{20}$ points
- CPL algorithm test: $\eta = 1, 2, 3, 4, 5$

Numerical results

- Rotation number(CIC) = Golden mean
- Rotation number of the approximants $\rho = 832040/1346269$

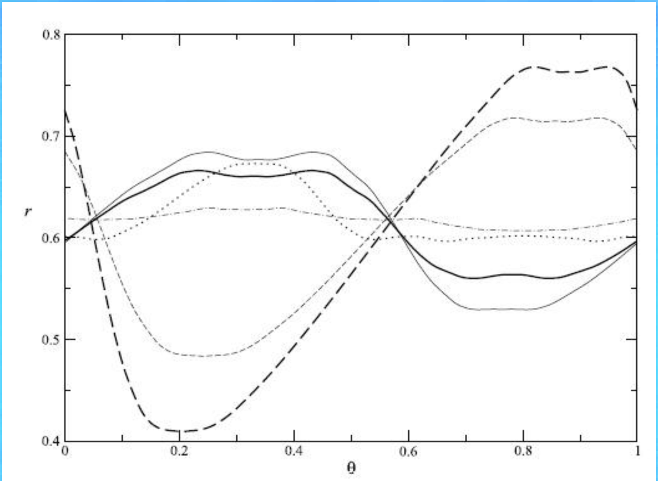
- CIC max error: 10^{-23}, Residue max diff: 10^{-10}
- Fourier uniformly spaced grid $\rightarrow 2^{20}$ points
- CPL algorithm test: $\eta = 1, 2, 3, 4, 5$

Numerical results

- Rotation number(CIC) = Golden mean
- Rotation number of the approximants $\rho = 832040/1346269$

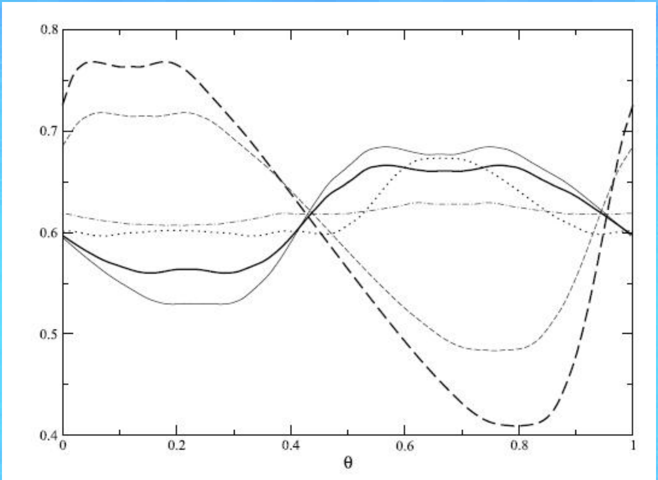
- CIC max error: 10^{-23}, Residue max diff: 10^{-10}
- Fourier uniformly spaced grid $\rightarrow 2^{20}$ points
- CPL algorithm test: $\eta = 1, 2, 3, 4, 5$

CIC: $R(\theta)$



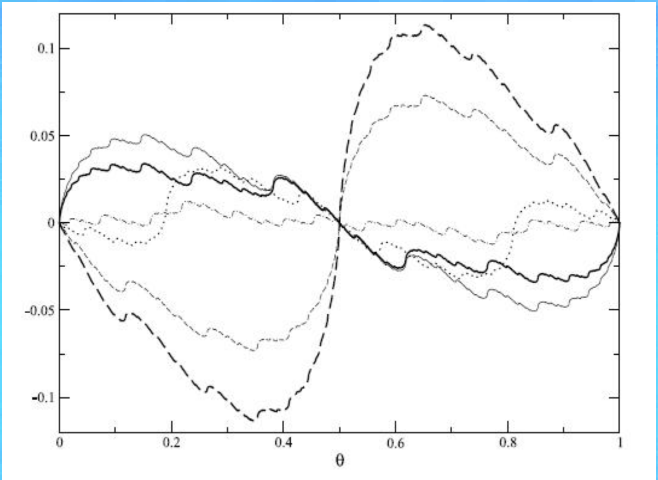
Std → thin solid. 2Har → thick solid. CritMp → dotted. Ana2 → thin dashed. Ana4 → thick dashed. Tent → dotted dashed.

Advance map: $g(\theta)$



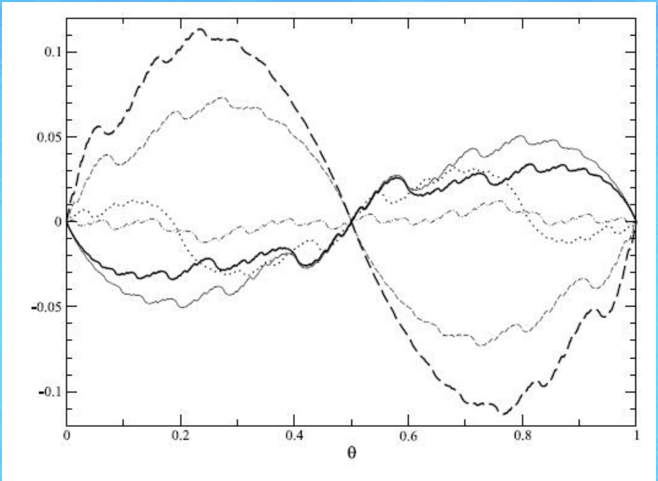
Std → thin solid. 2Har → thick solid. CritMp → dotted. Ana2 → thin dashed. Ana4 → thick dashed. Tent → dotted dashed.

Hull map: $h(\theta)$



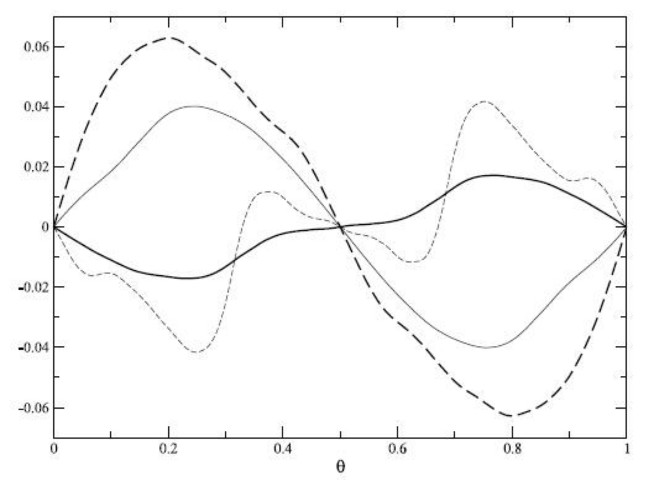
Std → thin solid. 2Har → thick solid. CritMp → dotted. Ana2 → thin dashed. Ana4 → thick dashed. Tent → dotted dashed.

Inverse hull map: $h^{-1}(\theta)$

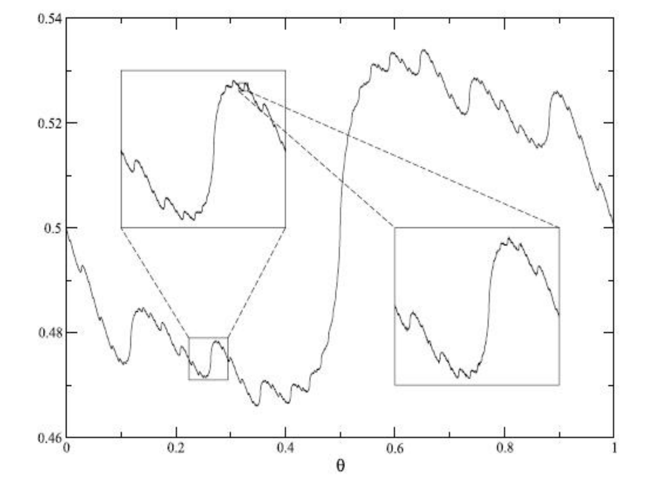


Std → thin solid. 2Har → thick solid. CritMp → dotted. Ana2 → thin dashed. Ana4 → thick dashed. Tent → dotted dashed.

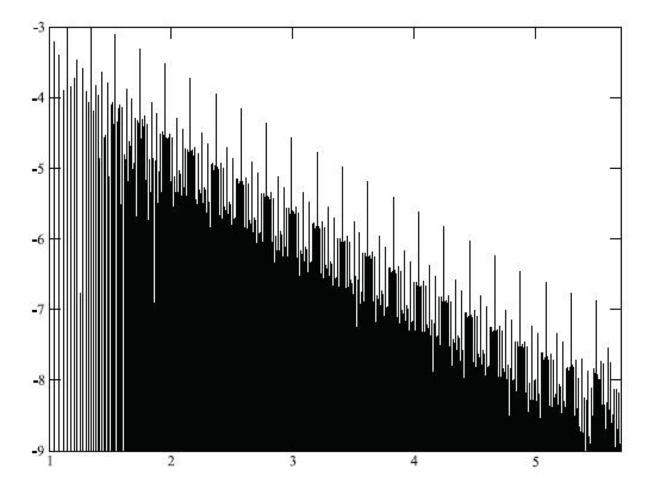
Big conjugacies: $H(\theta)$



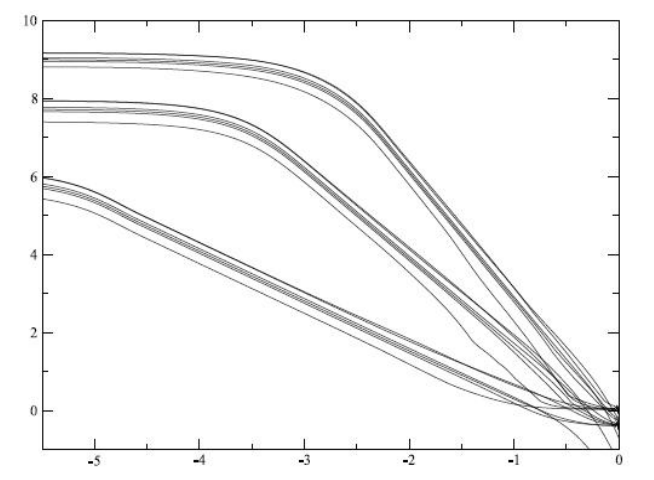
Self similarity of h



Self similarity of h – Fourier spectrum



CLP analysis



$$\log_{10} \left\| \left(\frac{\partial}{\partial t} \right)^{\eta} \mathbf{e}^{-t\sqrt{-\Delta}} K \right\|_{L^\infty(T)} \qquad \text{versus} \qquad \log_{10}(t)$$

Hölder regularities ⟶ Numerical results

Map	$\kappa(R)$	$\kappa(g)$	$\kappa(h)$	$\kappa(h^{-1})$
Standart	1.83 ± 0.09	1.83 ± 0.09	0.772 ± 0.001	0.92 ± 0.01
Two harmonics	1.79 ± 0.06	1.75 ± 0.09	0.721 ± 0.001	0.92 ± 0.01
Critical	1.83 ± 0.04	1.84 ± 0.09	0.724 ± 0.002	0.93 ± 0.02
Analytic 0.2	1.86 ± 0.08	1.86 ± 0.08	0.722 ± 0.001	0.92 ± 0.01
Analytic 0.4	1.85 ± 0.05	1.85 ± 0.05	0.724 ± 0.002	0.93 ± 0.01
Tent	1.85 ± 0.15	1.88 ± 0.12	0.726 ± 0.003	0.93 ± 0.02

Hölder regularities of "Big" Conjugacies

- We compute the regularities of all big conjugacies H between each of the six functions h_i
- We have thirty functions H
- Applying CLP method:

$$\kappa(H) = 1.80 \pm 0.15$$

Hölder regularities of "Big" Conjugacies

- We compute the regularities of all big conjugacies H between each of the six functions h_i
- We have thirty functions H
- Applying CLP method:

$$\kappa(H) = 1.80 \pm 0.15$$

Hölder regularities of "Big" Conjugacies

- We compute the regularities of all big conjugacies H between each of the six functions h_i
- We have thirty functions H
- Applying CLP method:

$$\kappa(H) = 1.80 \pm 0.15$$

Hölder regularities for rotation number silver mean

- Silver mean $= \sigma_S = [2, 2, 2, 2, \dots]$
- Maps: Standard and Two harmonics

$$
\begin{aligned}
\kappa(R_S) &= 1.70 \pm 0.15 \\
\kappa(g_S) &= 1.75 \pm 0.15 \\
\kappa(h_S) &= 0.715 \pm 0.015 \\
\kappa(h_S^{-1}) &= 0.87 \pm 0.02 \\
\kappa(H_S) &= 1.80 \pm 0.15
\end{aligned}
$$

Hölder regularities for rotation number silver mean

- Silver mean $= \sigma_S = [2, 2, 2, 2, \ldots]$
- Maps: Standard and Two harmonics

$$\begin{aligned} \kappa(R_S) &= 1.70 \pm 0.15 \\ \kappa(g_S) &= 1.75 \pm 0.15 \\ \kappa(h_S) &= 0.715 \pm 0.015 \\ \kappa(h_S^{-1}) &= 0.87 \pm 0.02 \\ \kappa(H_S) &= 1.80 \pm 0.15 \end{aligned}$$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den}$$

and

$$\Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

  - $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

    where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den}$$

and

$$\Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

- $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

  where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den}$$

and

$$\Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

  - $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

    where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den} \quad \text{and} \quad \Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

  - $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

    where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den} \quad \text{and} \quad \Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

- $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

  where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\begin{array}{c} \Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den} \\ \text{and} \\ \Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den}) \end{array} \quad \text{where } F_i = \text{Fibonacci numbers}$$

  - $$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

    where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Shenker & Kadanoff (82):
  - Let $\theta_{den} \in \mathbb{T}$ stand the value around which the iterates of the function G are most dense.
  - Iteration of $p_{den} = (\theta_{den}, R(\theta_{den}))$ are more dense around p_{den}.
  - Asymptotic invariant behaviour:

$$\Delta_i\theta := g^{F_{n+3}}(\theta_{den}) - \theta_{den}$$

and

$$\Delta_i r := R(g^{F_{n+3}}(\theta_{den})) - R(\theta_{den})$$

where $F_i =$ Fibonacci numbers

-
$$\frac{\Delta_{i+3}\theta}{\Delta_i\theta} \sim \alpha_3^{-1} \qquad \frac{\Delta_{i+3}r}{\Delta_i r} \sim \beta_3^{-1}$$

where $\alpha_3 \sim -4.84581$ and $\beta_3 \sim -16.8597$

Hölder regularity and scaling factors

- Hölder regularity of $R \longrightarrow |\Delta r| \sim |\Delta\theta|^{\kappa}$
- Asymptotical scaling: $|\beta_3 \Delta r| \sim |\alpha_3 \Delta\theta|^{\kappa}$
- $k(R) \leq \frac{\log(\beta_3)}{\log(\alpha_3)} \sim 1.7901$
- This bound is saturated.

Hölder regularity and scaling factors

- Hölder regularity of $R \longrightarrow |\Delta r| \sim |\Delta\theta|^{\kappa}$
- Asymptotical scaling: $|\beta_3 \Delta r| \sim |\alpha_3 \Delta\theta|^{\kappa}$
- $k(R) \leq \frac{\log(\beta_3)}{\log(\alpha_3)} \sim 1.7901$
- This bound is saturated.

Hölder regularity and scaling factors

- Hölder regularity of $R \longrightarrow |\Delta r| \sim |\Delta\theta|^{\kappa}$
- Asymptotical scaling: $|\beta_3 \Delta r| \sim |\alpha_3 \Delta\theta|^{\kappa}$
- $k(R) \leq \frac{\log(\beta_3)}{\log(\alpha_3)} \sim 1.7901$
- This bound is saturated.

Hölder regularity and scaling factors

- Hölder regularity of $R \longrightarrow |\Delta r| \sim |\Delta\theta|^{\kappa}$
- Asymptotical scaling: $|\beta_3 \Delta r| \sim |\alpha_3 \Delta\theta|^{\kappa}$
- $k(R) \leq \frac{\log(\beta_3)}{\log(\alpha_3)} \sim 1.7901$
- This bound is saturated.

Conclusions

- We accurately compute de golden critical invariant circles of six twist maps
- We obtain the Hölder regularity of R, g, h, h^{-1} and H
- Our numerical experiments lend credibility to our Conjetures concerning the universality of the regularities of R, g, h, h^{-1} and H
- Our results seem to indicate that the regularities of R, h, h^{-1} saturate the upper bounds coming from previous studies of scaling exponents
- $\kappa(H)$ is greater than $\kappa(h)$ and $\kappa(h^{-1})$ by a confortable margin

Conclusions

- We accurately compute de golden critical invariant circles of six twist maps
- We obtain the Hölder regularity of R, g, h, h^{-1} and H
- Our numerical experiments lend credibility to our Conjetures concerning the universality of the regularities of R, g, h, h^{-1} and H
- Our results seem to indicate that the regularities of R, h, h^{-1} saturate the upper bounds coming from previous studies of scaling exponents
- $\kappa(H)$ is greater than $\kappa(h)$ and $\kappa(h^{-1})$ by a confortable margin

Conclusions

- We accurately compute de golden critical invariant circles of six twist maps
- We obtain the Hölder regularity of R, g, h, h^{-1} and H
- Our numerical experiments lend credibility to our Conjetures concerning the universality of the regularities of R, g, h, h^{-1} and H
- Our results seem to indicate that the regularities of R, h, h^{-1} saturate the upper bounds coming from previous studies of scaling exponents
- $\kappa(H)$ is greater than $\kappa(h)$ and $\kappa(h^{-1})$ by a confortable margin

Conclusions

- We accurately compute de golden critical invariant circles of six twist maps
- We obtain the Hölder regularity of R, g, h, h^{-1} and H
- Our numerical experiments lend credibility to our Conjetures concerning the universality of the regularities of R, g, h, h^{-1} and H
- Our results seem to indicate that the regularities of R, h, h^{-1} saturate the upper bounds coming from previous studies of scaling exponents
- $\kappa(H)$ is greater than $\kappa(h)$ and $\kappa(h^{-1})$ by a confortable margin

Conclusions

- We accurately compute de golden critical invariant circles of six twist maps
- We obtain the Hölder regularity of R, g, h, h^{-1} and H
- Our numerical experiments lend credibility to our Conjetures concerning the universality of the regularities of R, g, h, h^{-1} and H
- Our results seem to indicate that the regularities of R, h, h^{-1} saturate the upper bounds coming from previous studies of scaling exponents
- $\kappa(H)$ is greater than $\kappa(h)$ and $\kappa(h^{-1})$ by a confortable margin

Thank you

Gràcies