On the parallel computation of invariant tori

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Setting

We focus on dynamical systems of the form

$$\begin{aligned} \bar{x} &= f(x, \theta), \\ \bar{\theta} &= \theta + \omega, \end{aligned}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{T}^r$.

The frequency vector $\omega \in \mathbb{R}^r$ is supposed to be irrational.

The autonomous case $\bar{x} = f(x)$ is included in this setting.

Setting

An invariant torus can be represented by a map

$$x : (\theta, \varphi) \in \mathbb{T}^r \times \mathbb{T}^s \mapsto \mathbb{R}^n,$$

and it must satisfy the invariance condition

$$f(x(\theta, \varphi)) = x(\theta + \omega, \varphi + \nu).$$

For simplicity, we will explain the methods for tori that do not depend on the "inner" angles φ, although in some of the examples used later on we will compute tori depending on inner angles.

The parametrization satisfies the equation

$$f(x(\theta)) = x(\theta + \omega).$$

Setting

Suppose that the map has an invariant curve with rotation number ω. The curve is given (in parametric form) by a map $x : \mathbb{T}^1 \to \mathbb{R}^n$. Let us write $x(\theta)$ as a real Fourier series,

$$x(\theta) = a_0 + \sum_{k>0} a_k \cos(k\theta) + b_k \sin(k\theta),$$

where $a_k, b_k \in \mathbb{R}^n,\ k \in \mathbb{N}$. As it is usual in numerical methods, we look for a truncation of this series.

So, let us fix in advance a truncation value N (the selection of N will be discussed later on), and let us try to determine an approximation to the $2N+1$ unknown coefficients a_0, a_k and b_k, $0 < k \leq N$.

Newton method

We consider the map

$$x(\theta) \mapsto F(x(\theta)) = f(x(\theta)) - x(\theta + \omega),$$

where $x(\theta)$ denotes the parametrization of a torus.

The main idea is to apply a Newton method to find $x(\theta)$ such that $F(x(\theta)) \equiv 0$. We note that F acts on a space of periodic functions.

First, let us define a mesh of $2N + 1$ points on $\mathbb{T}^1$,

$$\theta_j = \frac{2\pi j}{2N + 1}, \qquad 0 \leq j \leq 2N.$$

It is not difficult to compute $x(\theta_j)$, $f(\theta_j)$, $f(\theta_j + \omega)$ and, hence, $F(x(\theta_j))$. From these values, we can derive the Fourier coefficents of $F(x(\theta))$.

Newton method

Therefore, we have a procedure to compute the map F.

As this procedure can be easily differentiated, we can also obtain DF.

Then, a Newton method can be applied:

$$x_{m+1} = x_m - (DF(x_m))^{-1} F(x_m).$$

Error estimates

A natural question is about the size of the error of the obtained curve.

To measure such error we use

$$E(x, \omega) = \max_{\theta \in \mathbb{T}^1} \|f(x(\theta)) - x(\theta + \omega)\|.$$

We estimate $E(x, \omega)$ using a much finer mesh than the one used in the previous computations.

If this error is too big, we increase N and we apply the Newton process again.

Linearized normal behaviour

Let h represent a small displacement with respect to an arbitrary point $x(\theta)$ on the invariant curve. Then,

$$f(x(\theta)+h) = f(x(\theta)) + D_x f(x(\theta))h + O(\|h\|^2).$$

As $f(x(\theta)) = x(\theta+\omega)$, it follows that the linear normal behaviour is described by the following dynamical system,

$$\left.\begin{array}{rcl} \bar{x} & = & A(\theta)x \\ \bar{\theta} & = & \theta+\omega \end{array}\right\},$$

where $A(\theta) = D_x f(x(\theta))$.

Linearized normal behaviour

Definition

The previous linear system is called reducible iff there exists a (may be complex) change of variables $x = C(\theta)y$ such that it becomes

$$\left.\begin{array}{rcl} \bar{y} & = & By \\ \bar{\theta} & = & \theta + \omega \end{array}\right\},$$

where the matrix $B \equiv C^{-1}(\theta + \omega)A(\theta)C(\theta)$ does not depend on θ.

Linearized normal behaviour

We define the operator

$$T_\omega : \psi(\theta) \in C(\mathbb{T}^1, \mathbb{C}^n) \mapsto \psi(\theta + \omega) \in C(\mathbb{T}^1, \mathbb{C}^n),$$

and let us consider now the following generalized eigenvalue problem: to look for couples $(\lambda, \psi) \in \mathbb{C} \times C(\mathbb{T}^1, \mathbb{C}^n)$ such that

$$A(\theta)\psi(\theta) = \lambda T_\omega \psi(\theta).$$

Linearized normal behaviour

Definition

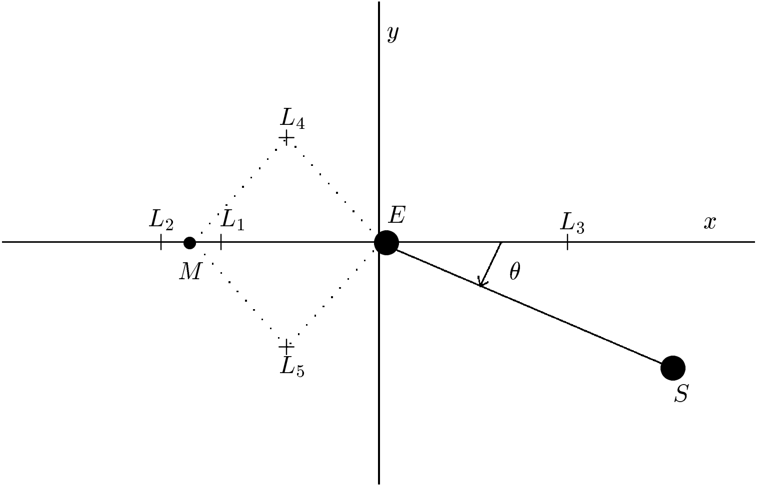
Two eigenvalues λ_1 and λ_2 are said to be unrelated iff $\lambda_1 \neq \exp(\mathrm{i}\, k\omega)\lambda_2$, $\forall k \in \mathbb{Z}$. Otherwise, we refer to them as related.

Lemma

Assume that there exist n unrelated eigenvalues $\lambda_1, \ldots, \lambda_n$ for the previous eigenproblem. Then, the linear skew product can be reduced to constant coefficients, where $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

The Bicircular problem

It is a model for the study of the dynamics of a small particle in the Earth-Moon-Sun system.



The Bicircular problem

The BCP can be described by the Hamiltonian system,

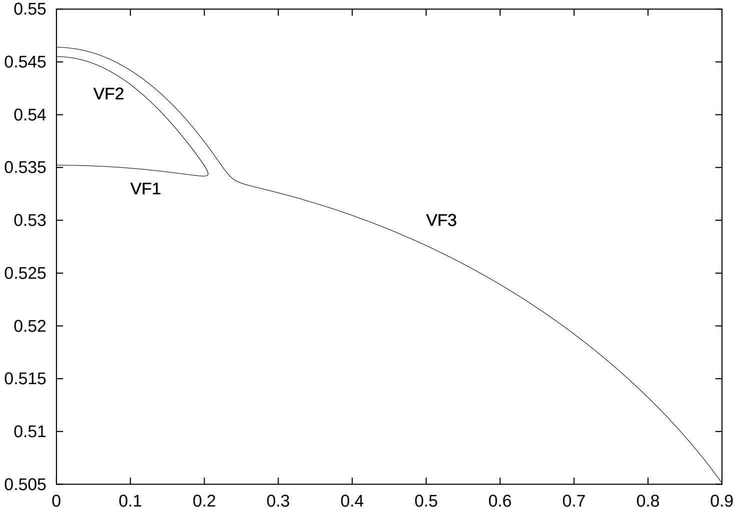
$$
\begin{aligned}
H_{BCP} &= \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \frac{1-\mu}{r_{PE}} - \frac{\mu}{r_{PM}} - \frac{m_S}{r_{PS}} \\
&\quad - \frac{m_S}{a_S^2}\left(y\sin\theta - x\cos\theta\right),
\end{aligned}
$$

where

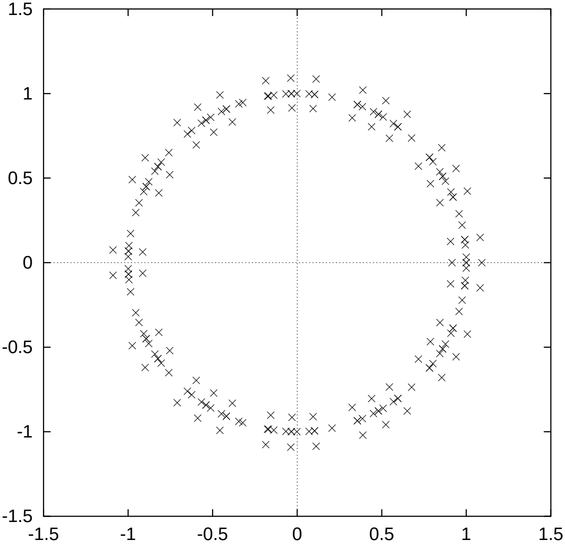
$$
\begin{aligned}
r_{PE}^2 &= (x-\mu)^2 + y^2 + z^2, \\
r_{PM}^2 &= (x-\mu+1)^2 + y^2 + z^2, \\
r_{PS}^2 &= (x-x_S)^2 + (y-y_S)^2 + z^2,
\end{aligned}
$$

being $x_S = a_S\cos\theta$, $y_S = -a_S\sin\theta$ and $\theta = \omega_S t$.

The Bicircular problem



The Bicircular problem



$N = 16$ (total dimension: 198).

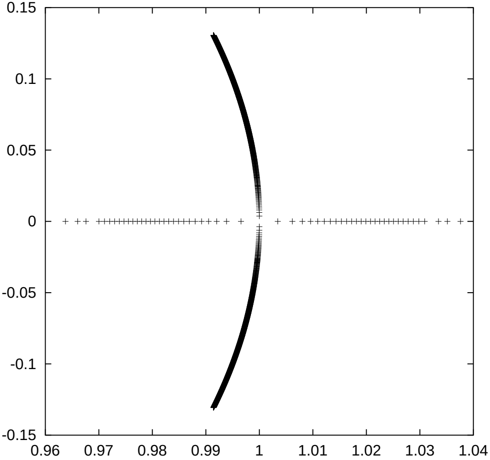
The Bicircular problem

	Modulus	Argument
λ_1	1.091942641437887	0.000000000000000
λ_2	0.915799019152856	0.000000000000000
λ_3	0.999999999999985	2.035517841801725
λ_4	0.999999999999985	-2.035517841801725

Normal eigenvalues around an unstable invariant curve of the family VF1. The rotation number is $\omega = 0.535033339385478$, and the value of the $\dot{z}$ coordinate when $z = 0$ is $\dot{z} = 0.080508698608030$.

We can check that $|\lambda_1\lambda_2 - 1| \approx 4 \times 10^{-15}$.

The Bicircular problem



Motion of one of the couples of eigenvalues in the complex plane, near the change of stability in the families VF1 and VF2.

Computing the unstable manifold

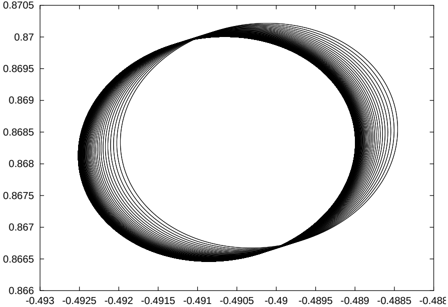
We call ψ_j the eigenfunction corresponding to λ_j, $j = 1, \ldots, 4$, and we focus on the couple $(\lambda_1, \psi_1) \in \mathbb{R} \times C(\mathbb{T}^1, \mathbb{R}^n)$.
The linearized unstable manifold is given by $x(\theta) + h\psi_1(\theta)$. To estimate a suitable value for h, we note that

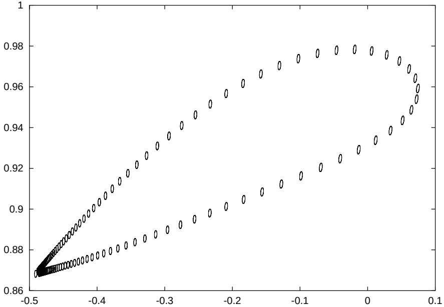
$$\begin{aligned} f(x(\theta) + h\psi_1(\theta)) &= f(x(\theta)) + hD_x f(x(\theta))\psi_1(\theta) + O(h^2) \\ &= x(\theta + \omega) + h\lambda_1\psi_1(\theta + \omega) + O(h^2). \end{aligned}$$

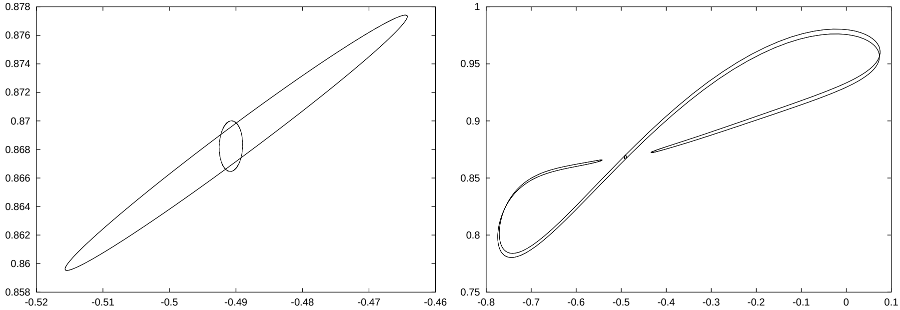
Hence, the size of the term $O(h^2)$ can be estimated by a numerical evaluation of

$$E(h) = \max_{\theta \in \mathbb{T}^1} \| f(x(\theta) + h\psi_1(\theta)) - x(\theta + \omega) - h\lambda_1\psi_1(\theta + \omega) \|_2.$$

It follows that $h = 10^{-7}$ is enough to have $E(h) < 10^{-13}$.
We define the curve $C_1 \subset \mathbb{R}^n$ as the image of the map $\theta \mapsto x(\theta) + h\psi_1(\theta)$ and, for $j > 1$, $C_j = f(C_{j-1})$.


0.8705
0.87
0.8695
0.869
0.8685
0.868
0.8675
0.867
0.8665
0.866
-0.493
-0.4925
-0.492
-0.4915
-0.491
-0.4905
-0.49
-0.4895
-0.489
-0.4885


1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1


0.878
0.876
0.874
0.872
0.87
0.868
0.866
0.864
0.862
0.86
0.858
-0.52
-0.51
-0.5
-0.49
-0.48
-0.47
-0.46
1
0.95
0.9
0.85
0.8
0.75
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1

Tori of higher dimensions

Now we assume that

$$x : (\theta, \varphi) \in \mathbb{T}^r \times \mathbb{T}^s \mapsto \mathbb{R}^n,$$

with $r > 1$.

We use multidimensional truncated Fourier series to approximate the torus.

DF is a full matrix of large dimension and, hence,

- it requires a large amount of memory,
- solving the linear system $(DF)h_m = f_m$ requieres a lot of CPU time.

Parallelization (I)

We assume we have a system with distributed memory (for instance, a cluster).

- The matrix can be distributed in different machines
- The linear system is solved in parallel

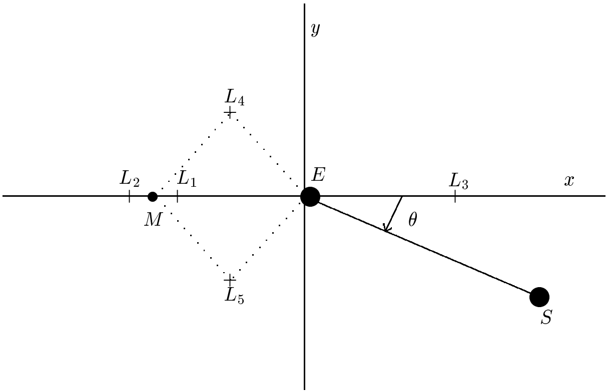
For this scheme, we have coded a (parallel) QR factorization using Householder reflections.

Advantages: We can deal with Fourier expansions with many terms, the computations are distributed

Inconveniences: When the number of machines increases, the communications become the new bottleneck.

Example: The EBCP

The Elliptic Bicircular Problem



Here we assume that the motion of Earth and Moon is elliptical.

Example: The EBCP

The equations of motion can be written in Hamiltonian form,

$$H = H_{RTBP}(x, y) + \hat{H}(x, y, \theta) + \langle I_\theta, \omega \rangle \,,$$

where $x \in \mathbb{R}^3$, $y \in \mathbb{R}^3$ and $\theta \in \mathbb{T}^2$.

We are interested in computing the 1-parametric family of 3-D tori corresponding to the vertical family of periodic orbits of the RTBP near L_5.

These tori are parametrized by 3 angles, the two of the perturbation and an inner angle coming from the vertical periodic orbit considered.

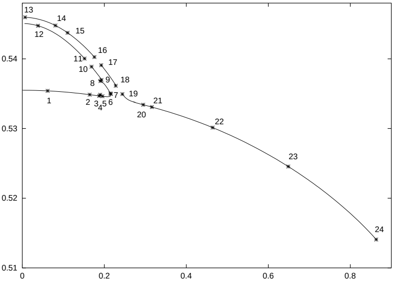
Example: The EBCP

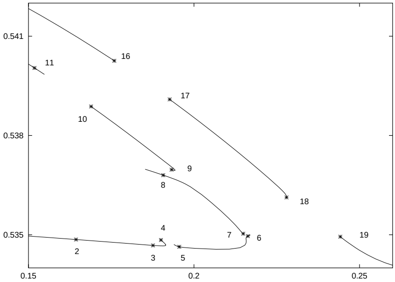
We use the Poincaré section $\theta_1 = 0 \pmod{2\pi}$.

When applying the Newton method, we obtained linear systems of dimension up to 20,000.

We used up to 16 nodes to solve the problem.

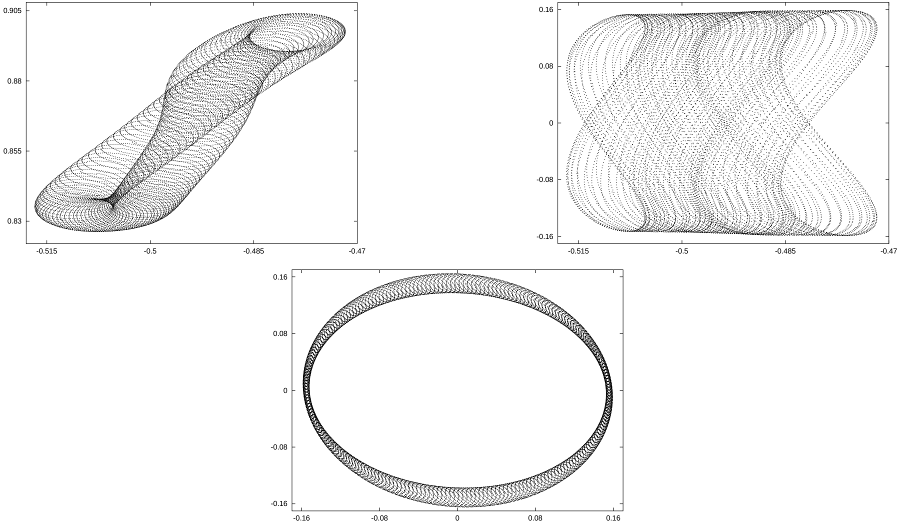
Example: The EBCP





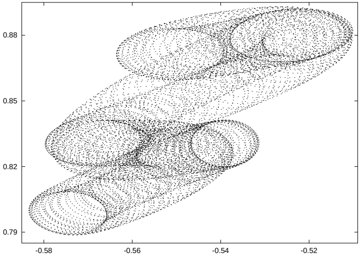
Example: The EBCP

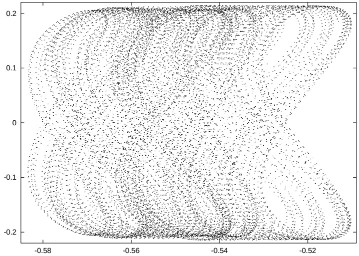
Torus number 2

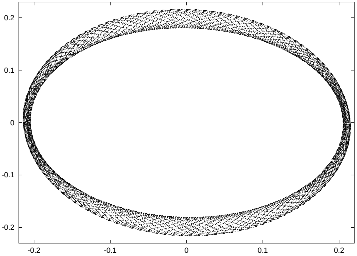


Example: The EBCP

Torus number 6

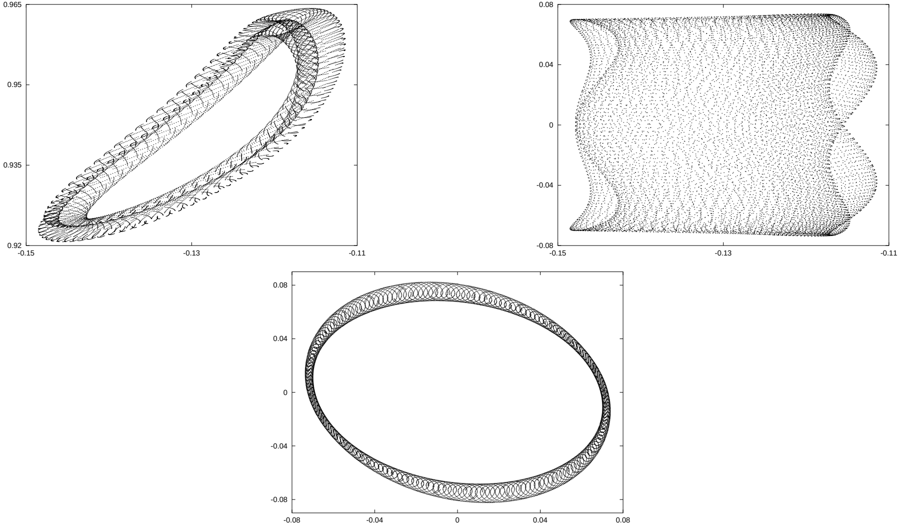






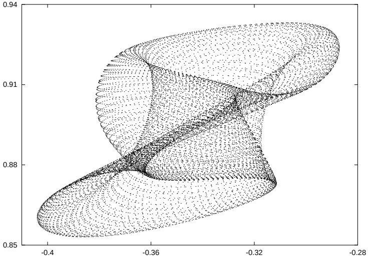
Example: The EBCP

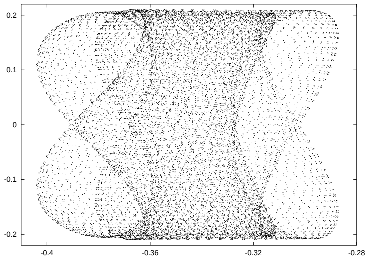
Torus number 14

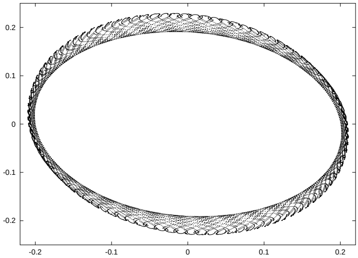


Example: The EBCP

Torus number 18







A different Newton scheme

The proof of several KAM-related results for lower dimensional tori involve Newton methods.

Most of them are not based on inverting an operator, but on reducing (to constant coefficients) the linearized dynamics around the torus. This is done iteratively, as a part of the Newton process.

These schemes are explicit and constructive, so they can be used as an efficient numerical method (see also recent papers by A. Haro and R. de la Llave).

A different Newton scheme

Initial approximations:

$$y_0 = \bar{x}_0 - f(x_0, \theta) \qquad \|y_0\| \approx \varepsilon$$

$$\left.\begin{array}{rcl} \bar{x} & = & A_0(\theta)x \\ \bar{\theta} & = & \theta + \omega \end{array}\right\} \xrightarrow{x = C_0(\theta)y} \left\{\begin{array}{rcl} \bar{y} & = & (B_0 + Q_0(\theta))y \\ \bar{\theta} & = & \theta + \omega \end{array}\right.$$

$$A_0(\theta) = D_x f(x_0(\theta), \theta), \qquad \|Q_0(\theta)\| \approx \varepsilon$$

Another Newton scheme

After one step:

$$y_1 = \bar{x}_1 - f(x_1, \theta), \qquad \|y_1\| \approx \varepsilon^2$$

$$\left.\begin{array}{rcl} \bar{x} & = & A_1(\theta)x \\ \bar{\theta} & = & \theta + \omega \end{array}\right\} \xrightarrow{x = C_1(\theta)y} \left\{\begin{array}{rcl} \bar{y} & = & (B_1 + Q_1(\theta))y \\ \bar{\theta} & = & \theta + \omega \end{array}\right.$$

$$A_1(\theta) = D_x f(x_1(\theta), \theta), \qquad \|Q_1(\theta)\| \approx \varepsilon^2$$

Improving the solution

We want

$$x_1(\theta) = x_0(\theta) + h(\theta) \text{ with } h(\theta) \text{ small}$$

$h(\theta)$ must satisfy:

$$h(\theta + \omega) = A_0(\theta)h(\theta) - y_0(\theta)$$

We apply:

$$h(\theta) = C_0(\theta)u(\theta)$$

We obtain:

$$u(\theta + \omega) = B_0 u(\theta) + r(\theta)$$

We solve:

$$2N + 1 \text{ independent linear systems } n \times n$$

Improving the Floquet matrix

We want

$$C_1(\theta) = C_0(\theta)(I + H(\theta)) \text{ with } H(\theta) \text{ small}$$

We want to reduce up to order ε^2

$$\bar{x} = A_1(\theta)x, \qquad \bar{\theta} = \theta + \omega$$

We apply

$$x(\theta) = C_0(\theta)(I + H(\theta))z(\theta)$$

We obtain

$$H(\theta + \omega)B_1 = B_1 H(\theta) + R(\theta)$$

We solve

$$N \text{ independent linear systems } 2n^2 \times 2n^2$$

Example II

"Toy" example:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\alpha \sin x + \frac{\varepsilon}{d + 2 + \sum_{i=0}^{d} \cos \theta_i}\end{aligned}$$

with $\alpha = 0.8$, $d = 4$, $\theta_i = \omega_i t + \theta_i^0$, θ_i^0 are the initial phases and

$$\omega_0 = 1,\ \omega_1 = \sqrt{2},\ \omega_2 = \sqrt{3},\ \omega_3 = \sqrt{5},\ \omega_4 = \sqrt{7}.$$

Example II

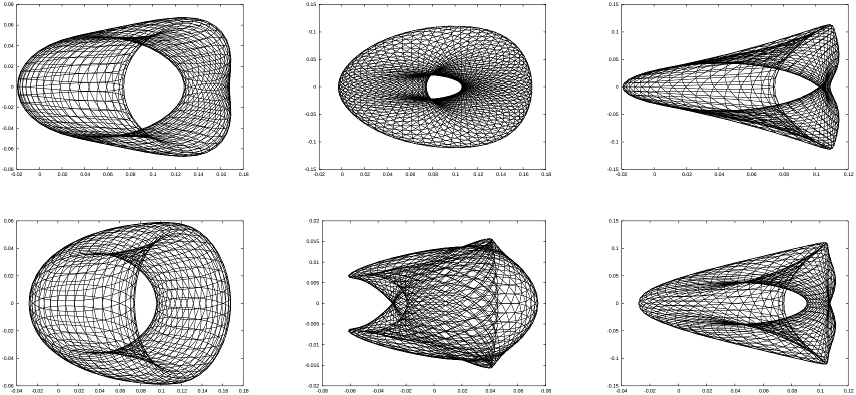
There is a 5-D dimensional torus that branches off from the origin when the perturbation is added. This torus is of the saddle type.

We take the section $\theta_0 = 0 \pmod{2\pi}$ and we apply the previous scheme, to compute a 4-D torus for the Poincaré map.

The accuracy to compute the torus is 10^{-10}.

Example II

Several 2-D slices of the torus



Example II

Number of Fourier coefficients: 1,975,467.

p	Total
1	109m 45.770s
2	58m 5.449s
4	32m 7.966s
6	23m 28.204s
8	19m 8.460s
10	16m 55.771s
12	16m 43.602s

Example III

We use a model by Gomez, Masdemont & Mondelo for the motion of a particle in the Earth-Moon system. The model includes effects from the Sun and the noncircular motion of the Moon.

These models are written as a perturbation of the RTBP

$$\begin{aligned} H &= \frac{1}{2}\{p_x^2 + p_y^2 + p_z^2\} + yp_x - xp_y - \frac{1-\mu}{\{(x-\mu)^2 + y^2 + z^2\}^{(1/2)}} \\ &\quad - \frac{\mu}{\{(x-\mu+1)^2 + y^2 + z^2\}^{(1/2)}} + \varepsilon g_i(x, y, z, p_x, p_y, p_z, \theta). \end{aligned}$$

When $\varepsilon = 0$, the equations are the equations of RTBP and when $\varepsilon = 1$, we have model $SSSM_1$ if $i = 1$ with $\theta \in \mathbb{T}$; $SSSM_2$ if $i = 2$, with $\theta \in \mathbb{T}^2$; and $SSSM_3$ if $i = 3$, with $\theta \in \mathbb{T}^3$.

Example III

We focus on the computation of the invariant tori that substitute $L_{1,2}$.

Due to the instability of this region, we use a parallel shooting method.

This means that, in the process of reducing the flow to a map, we will increase the dimension of the phase space.

Example III

For instance, to compute the substitute of L_1 we have used 4 intermediate Poincaré sections. This means that we look for a torus of the map

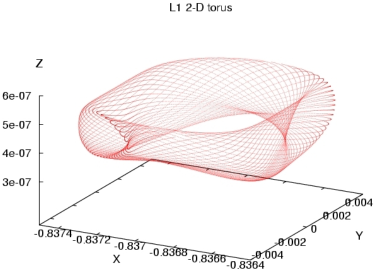
$$\begin{array}{ccc} \mathbb{R}^{24} \times \mathbb{T}^d & \longrightarrow & \mathbb{R}^{24} \times \mathbb{T}^d \\ (x, \theta) & \mapsto & (P(x), \theta + \rho), \end{array}$$

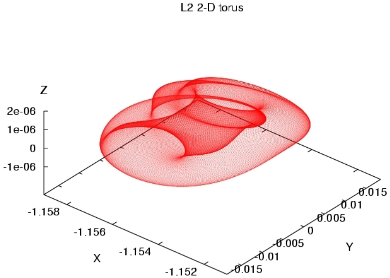
To compute the substitute of L_2 we have used 3 sections:

$$\begin{array}{ccc} \mathbb{R}^{18} \times \mathbb{T}^d & \longrightarrow & \mathbb{R}^{18} \times \mathbb{T}^d \\ (x, \theta) & \mapsto & (P(x), \theta + \rho), \end{array}$$

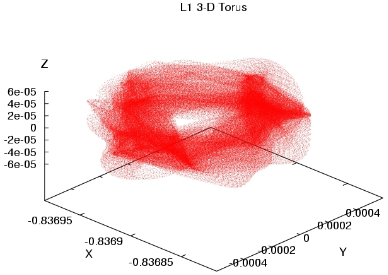
The tori have been computed with an accuracy of 10^{-10}.

Example III

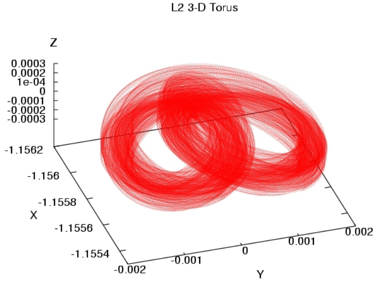

L1 2-D torus
Z
6e-07
5e-07
4e-07
3e-07
-0.8374
-0.8372
-0.837
-0.8368
-0.8366
-0.8364
X
0.004
0.002
0
-0.002
-0.004
Y


L2 2-D torus
Z
2e-06
1e-06
0
-1e-06
-1.158
-1.156
-1.154
-1.152
X
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
Y

Example III



L1 3-D Torus
Z
6e-05
4e-05
2e-05
0
-2e-05
-4e-05
-6e-05
-0.83695
-0.8369
X
-0.83685
-0.0004
-0.0002
0
0.0002
0.0004
Y



L2 3-D Torus
Z
0.0003
0.0002
1e-04
0
-0.0001
-0.0002
-0.0003
-1.1562
-1.156
-1.1558
X
-1.1556
-1.1554
-0.002
-0.001
0
0.001
0.002
Y

Conclusions

It is possible to approximate invariant tori of moderate dimension (2,3, 4 or even 5) in a phase space of moderate dimension.

Computing the Floquet transformation at the same time as the torus increases the degree of parallelism, but sometimes the number of Fourier modes to approximate the Fourier transformation becomes very large. So, if the transformation needed to reduce the torus is much more complex than the torus, this method could be a bad option.