Low-thrust Trajectory Optimization

Yuan Ren

Departament de Matemtica Aplicada I

Universitat Politcnica de Catalunya

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- The Action of Trajectory Optimization
- Calculus of Variations & Pontryagin’s Minimum Principle
- Dynamical Models
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Tsiolkovsky formula

Tsiolkovsky Formula

$$\Delta V = u \ln \frac{M_0}{M_k}$$

M_0 is the initial total mass, including propellant in kg

M_k is the final total mass in kg

u is the effective exhaust velocity in m/s $u = I_{sp} g_0$

ΔV is the delta-v in m/s

Chemical rocket engine

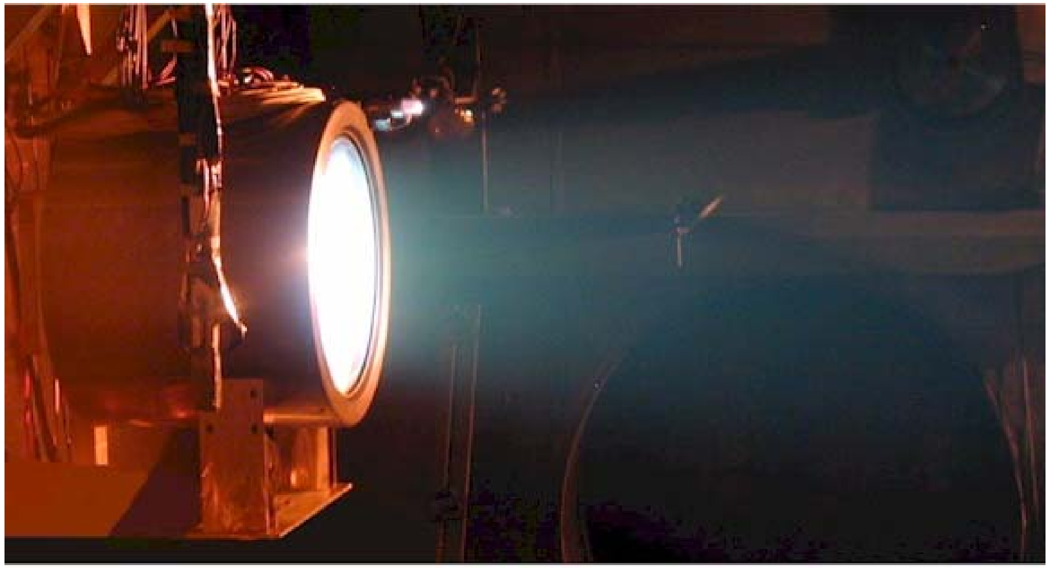
Chemical rocket engine used on carrier rocket

Thrust: 10-50tons, Specific Impulse: 300s

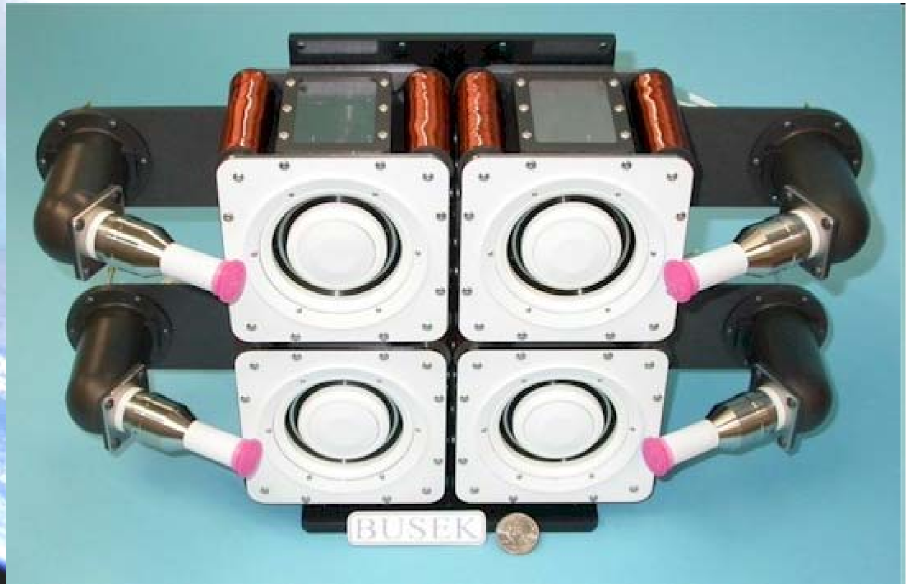
Chemical rocket engine used on satellite

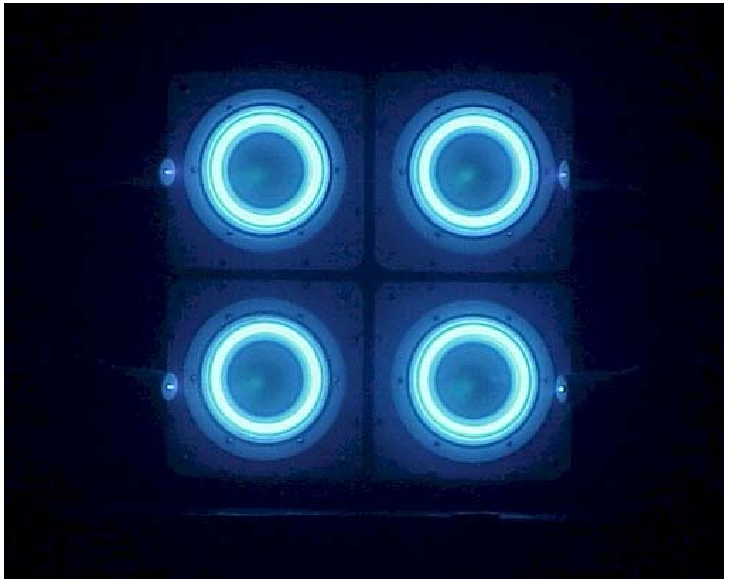
Thrust: 490 N, Specific Impulse: 290s

Low-thrust engine



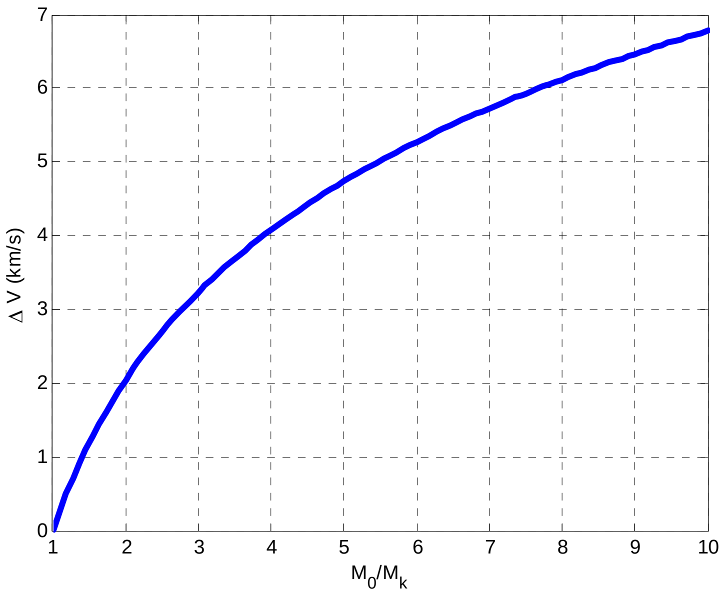
6.9 kW, LM 4 40 cm Ion Thruster 1.1 - 6.9 kW, specific impulse 2210 – 4100 s, thrust 50 - 237 mN.



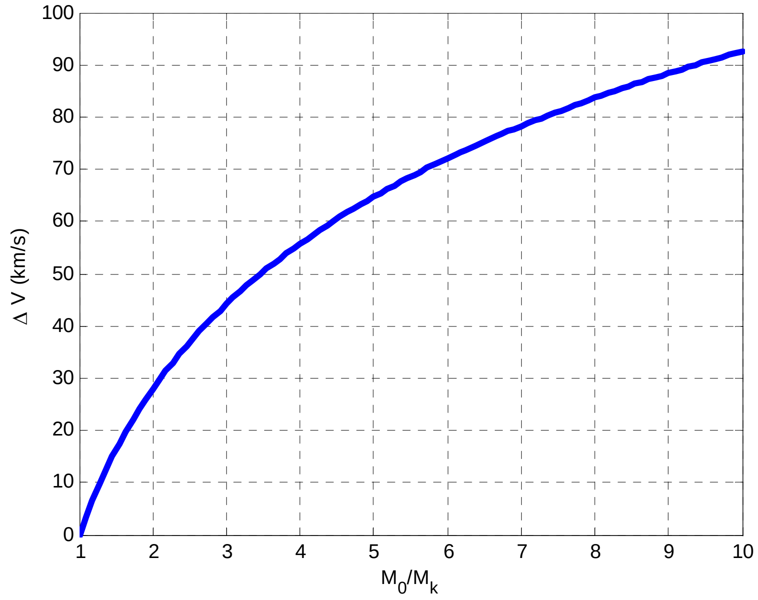


Busek 600 W Hall Thruster Cluster

Different design methods



Chemical rocket Isp=300s



Low-thrust engine Isp=4100s

Orbit transfer using chemical rocket:

The orbit maneuvers are assumed as Impulse of the Velocity. The whole trajectory is consisted by a series of unpowered arcs, which are connected with each other by these 'impulses'.

Orbit transfer using low-thrust engine:

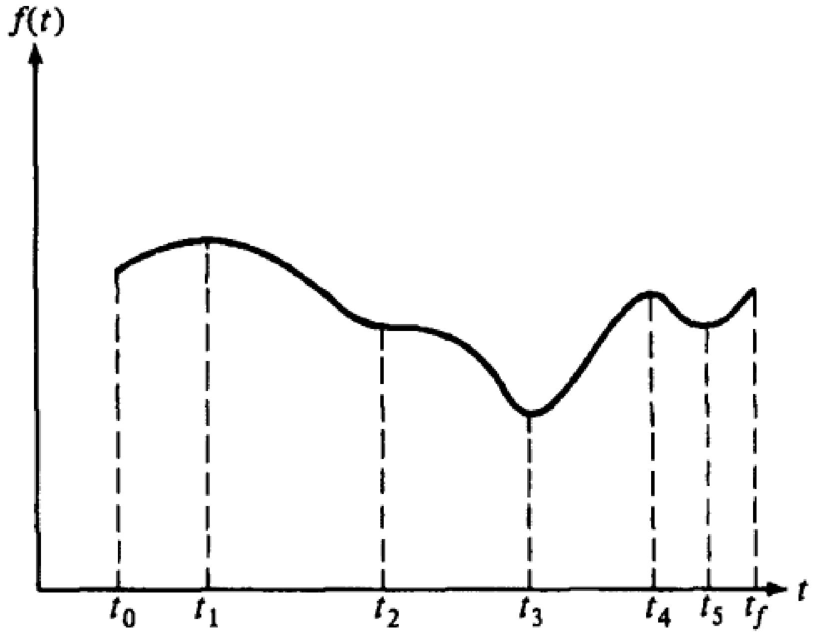
The orbit maneuver needs a long time. So the Impulse Assumption can not be used in the low-thrust orbit design and optimization. We need some new methods to deal with this kind of problem.

Functional and its variations

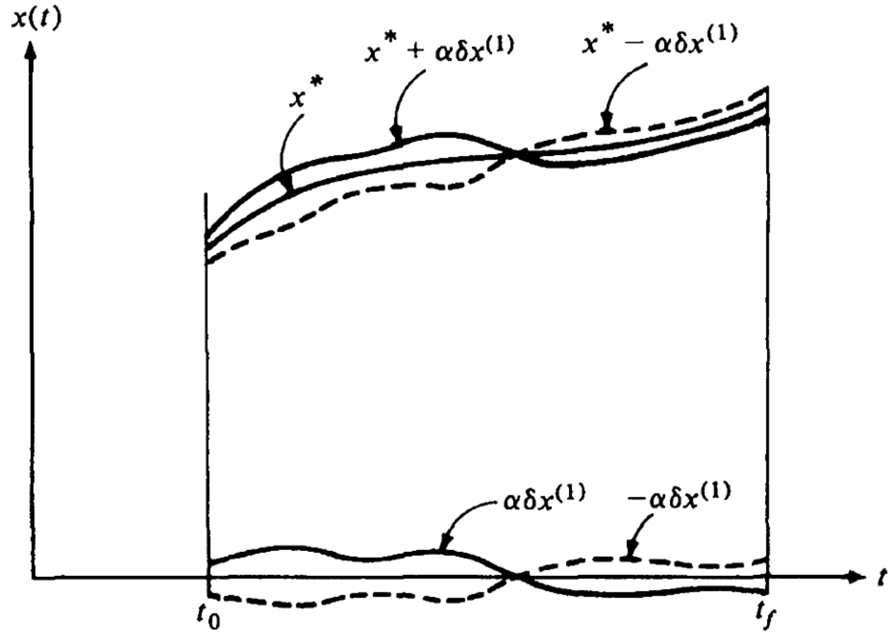
Suppose that x is a continuous function of t defined in the interval [***t0, tf***] and

$$J(x) = \int_{t_0}^{t_f} x(t)dt$$

J(x) is a functional of the function x(t). Roughly speaking, the functional is the function of function. The variation plays the same role in determining extreme values of functional as the differential does in finding maxima and minima of functions



The extremum of function



The extremum of functional

Necessary conditions for optimal control (I)

There is a dynamical equation of a system.

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

The problem is that to find an admissible control $\mathbf{u}^*$ that causes the system to follow an admissible trajectory $\mathbf{x}^*$ that minimizes the performance index:

$$J(\mathbf{u}) = \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

We shall initially assume that the admissible state and control regions are not bounded, and that the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and the initial time t_0 are specified.

If Φ is a differentiable function, we can write

$$\Phi(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[\Phi(\mathbf{x}(t), t)]dt + \Phi(\mathbf{x}(t_0), t_0)$$

Necessary conditions for optimal control (II)

So that the performance index can be expressed as

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \{L(\mathbf{x}(t),\mathbf{u}(t),t) + \frac{d}{dt}[\Phi(\mathbf{x}(t),t)]\}dt + \Phi(\mathbf{x}(t_0),t_0)$$

Since $\mathbf{x}(t_0)$ and t_0 are fixed, the minimization does not affect the $\Phi(\mathbf{x}(t_0),t_0)$ term, and using the chain rule of differentiation, we find that $J(\mathbf{u})$ becomes

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ L(\mathbf{x}(t),\mathbf{u}(t),t) + \left[\frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{x}(t),t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial\Phi}{\partial t}(\mathbf{x}(t),t) \right\} dt$$

To include the differential equation constraints, we form the augmented functional

$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ L(\mathbf{x}(t),\mathbf{u}(t),t) + \left[\frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{x}(t),t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial\Phi}{\partial t}(\mathbf{x}(t),t) + \boldsymbol{\lambda}^T(t)\left[\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t) - \dot{\mathbf{x}}(t)\right] \right\} dt$$

Necessary conditions for optimal control (III)

Let's define

$$L_a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t) = L(\mathbf{x}(t),\mathbf{u}(t),t) + \left[\frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x}(t),t)\right]^T \dot{\mathbf{x}}(t) +$$

$$\frac{\partial \Phi}{\partial t}(\mathbf{x}(t),t) + \boldsymbol{\lambda}^T(t)\left[\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t) - \dot{\mathbf{x}}(t)\right]$$

We shall assume that the end points at $t = t_f$ can be specified or free. To determine the variation of J_a, we introduce the variations $\delta\mathbf{x}, \delta\dot{\mathbf{x}}, \delta\mathbf{u}, \delta\boldsymbol{\lambda}$ and δt_f.

$$\delta J_a(\mathbf{u}^*) = 0 = \left.\frac{\partial L_a}{\partial \dot{\mathbf{x}}}\right|_{t_f} \delta\mathbf{x}_f + \left[L_a\big|_{t_f} - \left.\frac{\partial L_a}{\partial \dot{\mathbf{x}}}\right|_{t_f} \dot{\mathbf{x}}(t_f)\right]\delta t_f +$$

$$\int_{t_0}^{t_f} \left\{ \left[\frac{\partial L_a}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial L_a}{\partial \dot{\mathbf{x}}}\right]\delta\mathbf{x}(t) + \frac{\partial L_a}{\partial \mathbf{u}}\delta\mathbf{u}(t) + \frac{\partial L_a}{\partial \boldsymbol{\lambda}}\delta\boldsymbol{\lambda}(t)\right\} dt$$

Notice that the above result is obtained because $\dot{\mathbf{u}}(t)$ and $\dot{\boldsymbol{\lambda}}(t)$ do not appear in L_a.

Necessary conditions for optimal control (IV)

If it assumed that the second partial derivatives are continuous, the order of differentiation can be interchanged. In the integral term we have, then,

$$\int_{t_0}^{t_f}\left\{\left[\frac{\partial L}{\partial \mathbf{x}}+\boldsymbol{\lambda}(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}+\frac{d}{dt}\boldsymbol{\lambda}(t)\right]\delta\mathbf{x}(t)+\left[\frac{\partial L}{\partial \mathbf{u}}+\boldsymbol{\lambda}(t)\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right]\delta\mathbf{u}(t)+\left[\mathbf{f}-\dot{\mathbf{x}}(t)\right]\delta\boldsymbol{\lambda}(t)\right\}dt$$

The integral must vanish on an extremal regardless of the boundary conditions. If we define $H = L + \boldsymbol{\lambda}^T\mathbf{f}$, we can observe that the constraints

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\\ \dot{\boldsymbol{\lambda}} = -\dfrac{\partial H}{\partial \mathbf{x}}\end{cases}$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0$$

Necessary conditions for optimal control (V)

There are still the terms outside the integral to deal with; since the variation must be zero, we have

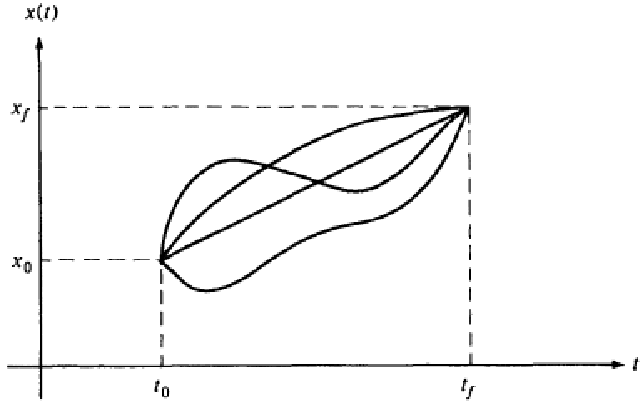
$$\left[\frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f) - \boldsymbol{\lambda}(t_f)\right]\delta \mathbf{x}_f + \left[H\Big|_{t_f} + \frac{\partial \Phi}{\partial t}(\mathbf{x}^*(t_f),t_f)\right]\delta t_f = 0$$

So we obtain the boundary condition.

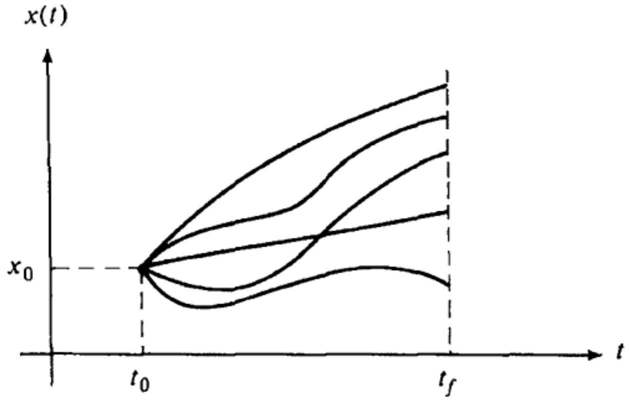
$$H\Big|_{t_f} = -\frac{\partial \Phi}{\partial t}(\mathbf{x}^*(t_f),t_f)$$

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)$$

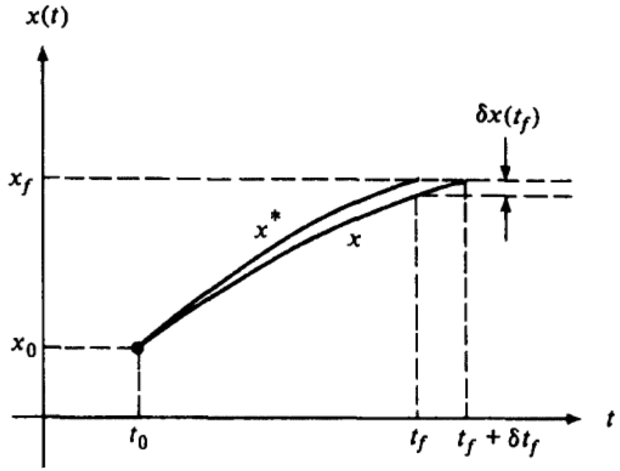
Boundary conditions



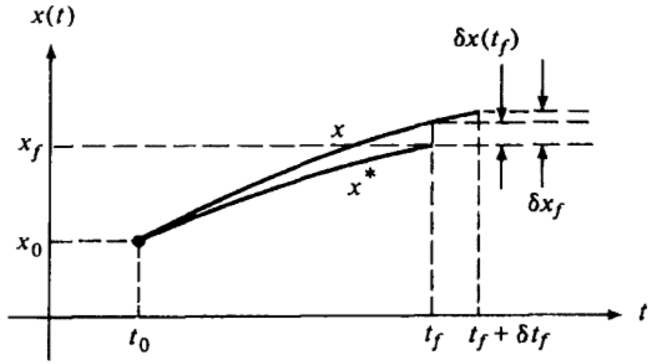
Fixed time and end point



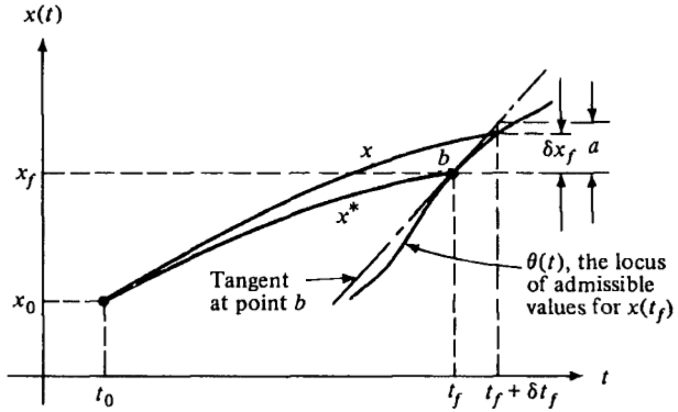
Fixed time, free end point



Free time and fixed end point



Free time and free end point



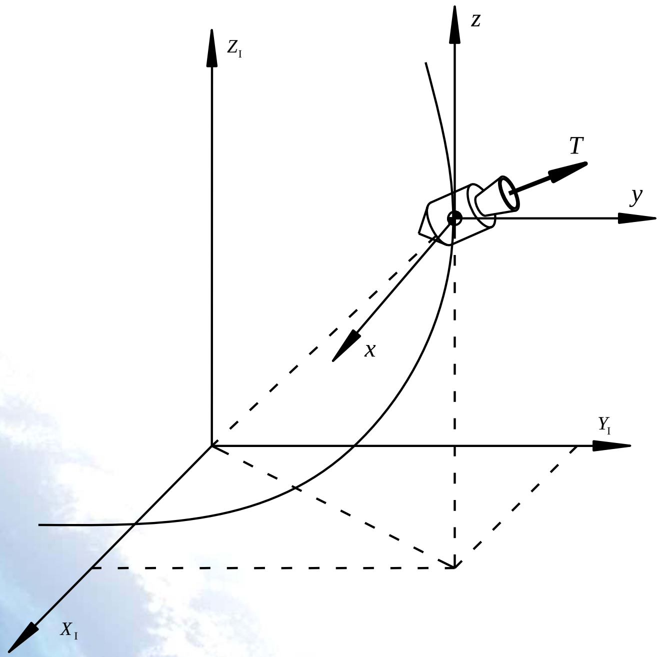
Final states on the moving point

Boundary conditions

Problem	Description	Variations	Boundary-condition
t_f fixed	$\mathbf{x}(t_f)=\mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f=\delta\mathbf{x}(t_f)=0$ $\delta t_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\mathbf{x}_f$
	$\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f=\delta\mathbf{x}(t_f)$ $\delta t_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{x}^*(t_f))-\boldsymbol{\lambda}^*(t_f)=0$
t_f free	$\mathbf{x}(t_f)=\mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\mathbf{x}_f$ $H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\boldsymbol{\lambda}^*(t_f),t_f)+\frac{\partial\Phi}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$
	$\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{x}^*(t_f))-\boldsymbol{\lambda}^*(t_f)=0$ $H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\boldsymbol{\lambda}^*(t_f),t_f)+\frac{\partial\Phi}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$
	$\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta\mathbf{x}_f=\left[\frac{\partial\boldsymbol{\theta}}{\partial t}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\boldsymbol{\theta}(t_f)$ $H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\boldsymbol{\lambda}^*(t_f),t_f)+\frac{\partial\Phi}{\partial t}(\mathbf{x}^*(t_f),t_f)$ $+\left[\frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\boldsymbol{\lambda}^*(t_f)\right]\left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]=0$

Dynamical Equations of Classic Orbital Elements

Dynamical equations of classic orbital elements can be written as follow:



$$
\begin{cases}
\dot{a} = \dfrac{2a^2 e \sin\upsilon}{h} f_r + \dfrac{2a^2 p}{hr} f_t \\
\dot{e} = \dfrac{1}{h} p \sin\upsilon f_r + \dfrac{1}{h}\left[(p+r)\cos\upsilon + re\right] f_t \\
\dot{i} = \dfrac{r\cos(\omega+\upsilon)}{h} f_n \\
\dot{\Omega} = \dfrac{r\sin(\omega+\upsilon)}{h\sin i} f_n \\
\dot{\omega} = -\dfrac{p\cos\upsilon}{he} f_r + \dfrac{(p+r)\sin\upsilon}{he} f_t - \dfrac{r\sin(\omega+\upsilon)\cos i}{h\sin i} f_n \\
\dot{M} = n + \dfrac{1}{aeh}\left[(p\cos\upsilon - 2er) f_r - (p+r)\sin\upsilon f_t\right]
\end{cases}
$$

We can find that when ‘i’ or ‘e’ equal to zero, the dynamical equations will become singularity. In low-thrust transfer, the orbital elements is changed continuous, so the singularity could hardly be avoided.

Modified equinoctial elements

In order to avoid the singularity, we need to use a set of no singularity elements. The modified equinoctial elements is a good choice. The modified equinoctial elements and its dynamical equations are defined as follows:

$$
\begin{aligned}
p &= a\left(1-e^2\right) \\
f &= e\cos\left(\omega+\Omega\right) \\
g &= e\sin\left(\omega+\Omega\right) \\
h &= \tan\left(i/2\right)\cos\Omega \\
k &= \tan\left(i/2\right)\sin\Omega \\
L &= \Omega+\omega+\upsilon
\end{aligned}
$$

$$
\begin{cases}
\dot{p} = \sqrt{\dfrac{p}{\mu}}\dfrac{2p}{w} f_{\mathrm{t}} \\
\dot{f} = \sqrt{\dfrac{p}{\mu}}\left\{ f_{\mathrm{r}}\sin L + \left[\left(1+w\right)\cos L + f\right]\dfrac{f_{\mathrm{t}}}{w} - \left(h\sin L - k\cos L\right)\dfrac{g\cdot f_{\mathrm{n}}}{w}\right\} \\
\dot{g} = \sqrt{\dfrac{p}{\mu}}\left\{ -f_{\mathrm{r}}\cos L + \left[\left(1+w\right)\sin L + g\right]\dfrac{f_{\mathrm{t}}}{w} + \left(h\sin L - k\cos L\right)\dfrac{f\cdot f_{\mathrm{n}}}{w}\right\} \\
\dot{h} = \sqrt{\dfrac{p}{\mu}}\dfrac{s^2 f_{\mathrm{n}}}{2w}\cos L \\
\dot{k} = \sqrt{\dfrac{p}{\mu}}\dfrac{s^2 f_{\mathrm{n}}}{2w}\sin L \\
\dot{L} = \sqrt{\mu p}\left(\dfrac{w}{p}\right)^2 + \dfrac{1}{w}\sqrt{\dfrac{p}{\mu}}\left(h\sin L - k\cos L\right) f_{\mathrm{n}}
\end{cases}
$$

Where $w = 1 + f\cos L + g\sin L$

$$s^2 = 1 + h^2 + k^2$$

Necessary conditions for this model

The dynamical equations can be describe in matrix form.

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{B}_{6\times 3}\left(\dfrac{T}{m}\boldsymbol{\alpha} + \boldsymbol{f}_{\mathrm{p}}\right) + \boldsymbol{D}_{6\times 1} \\ \dot{m} = -\dfrac{T}{g_0 I_{\mathrm{sp}}} \end{cases}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & \dfrac{2p}{w}\sqrt{\dfrac{p}{\mu}} & 0 \\ \sqrt{\dfrac{p}{\mu}}\sin L & \sqrt{\dfrac{p}{\mu}}[(w+1)\cos L + f]\dfrac{1}{w} & -\sqrt{\dfrac{p}{\mu}}[h\sin L - k\cos L]\dfrac{g}{w} \\ -\sqrt{\dfrac{p}{\mu}}\cos L & \sqrt{\dfrac{p}{\mu}}[(w+1)\sin L + g]\dfrac{1}{w} & \sqrt{\dfrac{p}{\mu}}[h\sin L - k\cos L]\dfrac{f}{w} \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2}{2w}\cos L \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2}{2w}\sin L \\ 0 & 0 & \dfrac{1}{w}\sqrt{\dfrac{p}{\mu}}(h\sin L - k\cos L) \end{bmatrix}$$

$$\boldsymbol{D} = [0\ 0\ 0\ 0\ 0\ d]^T$$

$$d = \sqrt{\mu p}\left(\frac{w}{p}\right)^2$$

I_{sp} - specific impulse of the thruster

g_0 - acceleration of gravity at sea level

The Hamiltonian can be defined as:

$$H = \boldsymbol{\lambda}^T \boldsymbol{B}\boldsymbol{\alpha}\frac{T}{m} + \lambda_{\mathrm{L}} d - \lambda_{\mathrm{m}}\frac{T}{I_{\mathrm{sp}} g_0} + \gamma\left(1 - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}\right)$$

Necessary conditions for this model

By using the control equation, we can obtain the optimal control:

$$\frac{\partial H}{\partial \boldsymbol{\alpha}^*} = 0 \quad \rightarrow \quad \boldsymbol{\alpha}^* = \frac{\left(\boldsymbol{\lambda}^T \boldsymbol{B}\right)^T}{\left\|\boldsymbol{\lambda}^T \boldsymbol{B}\right\|}$$

By using Pontryagin minimal principle, the Hamiltonian should always be minimized. So we can get a switch function.

$$\Gamma = \frac{\partial H}{\partial T} = \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{B} \frac{\boldsymbol{\alpha}^*}{m} - \frac{\lambda_{\mathrm{m}}}{g_0 I_{\mathrm{sp}}} = \frac{\left\|\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{B}\right\|}{m} - \frac{\lambda_{\mathrm{m}}}{g_0 I_{\mathrm{sp}}}$$

$$\begin{cases} T = 0 & \Gamma > 0 \\ T = T_{\max} & \Gamma < 0 \\ 0 < T < T_{\max} & \Gamma = 0 \end{cases}$$

The co-state equations are determined as follow:

$$\begin{cases} \dot{\boldsymbol{\lambda}} = -\dfrac{\partial H}{\partial \boldsymbol{x}} = \left(-\boldsymbol{\lambda}^T \dfrac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}} \boldsymbol{\alpha} \dfrac{T}{m} - \lambda_L \dfrac{\partial d}{\partial \boldsymbol{x}}\right) + \left(-\boldsymbol{\lambda}^T \dfrac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}} \boldsymbol{f}_{\mathrm{p}} - \lambda_L \boldsymbol{B} \dfrac{\partial \boldsymbol{f}_{\mathrm{p}}}{\partial \boldsymbol{x}}\right) \\ \dot{\lambda}_m = -\dfrac{\partial H}{\partial m} = \boldsymbol{\lambda}^T \boldsymbol{B} \boldsymbol{\alpha} \dfrac{T}{m^2} = \left\|\boldsymbol{\lambda}^T \boldsymbol{B}\right\| \dfrac{T}{m^2} \end{cases}$$

The boundary condition:

$$\boldsymbol{\lambda}(t_{\mathrm{f}}) = \left.\frac{\partial \phi}{\partial \boldsymbol{x}}\right|_{t=t_{\mathrm{f}}} + \boldsymbol{\gamma}^{\mathrm{T}} \left.\frac{\partial \psi}{\partial \boldsymbol{x}}\right|_{t=t_{\mathrm{f}}} \qquad \left[\frac{\partial \phi}{\partial t} + \left(\frac{\partial \psi}{\partial t}\right)^{\mathrm{T}} \boldsymbol{\gamma} + \tilde{H}\right]_{t=t_{\mathrm{f}}} = 0$$

Optimal trajectory in three-body model

Dynamical equations

$$\begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{\boldsymbol{v}} = \boldsymbol{M}\boldsymbol{v} + \dfrac{\partial U}{\partial \boldsymbol{r}} + \dfrac{T}{m}\boldsymbol{\alpha} \\ \dot{m} = -\dfrac{T}{g_0 I_{sp}} \end{cases}$$

Conjugate equations

$$\begin{cases} \dot{\boldsymbol{\lambda}}_r = -\dfrac{\partial H}{\partial \boldsymbol{r}} = -\boldsymbol{\lambda}_v^T \dfrac{\partial U}{\partial \boldsymbol{r}^2} \\ \dot{\boldsymbol{\lambda}}_v = -\dfrac{\partial H}{\partial \boldsymbol{v}} = -\boldsymbol{\lambda}_r - \boldsymbol{\lambda}_v^T \boldsymbol{M} \\ \dot{\lambda}_m = -\dfrac{\partial H}{\partial m} = \left\|\boldsymbol{\lambda}_v\right\| \dfrac{T}{m^2} \end{cases}$$

Direction of thrust

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\lambda}_v}{\left\|\boldsymbol{\lambda}_v\right\|}$$

Switch function

$$\Gamma = \frac{\partial \tilde{H}}{\partial T} = \frac{\boldsymbol{\lambda}_v^T \boldsymbol{\alpha}^*}{m} - \frac{\lambda_m}{g_0 I_{sp}} = \frac{\left\|\boldsymbol{\lambda}_v\right\|}{m} - \frac{\lambda_m}{g_0 I_{sp}}$$

$$\begin{cases} T = 0 & \Gamma > 0 \\ T = T_{max} & \Gamma < 0 \\ 0 < T < T_{max} & \Gamma = 0 \end{cases}$$

Hamiltonian

$$H = \boldsymbol{\lambda}_r^T \boldsymbol{v} + \boldsymbol{\lambda}_v^T \left(\boldsymbol{M}\boldsymbol{v} + \frac{\partial U}{\partial \boldsymbol{r}} + \frac{T}{m}\boldsymbol{\alpha} \right) - \lambda_m \frac{T}{I_{sp} g_0} + \gamma\left(1 - \boldsymbol{\alpha}^T \boldsymbol{\alpha}\right)$$

Transversality condition

$$\boldsymbol{\lambda}(t_f) = \left.\frac{\partial \phi}{\partial \boldsymbol{x}}\right|_{t=t_f} + \boldsymbol{\gamma}^T \left.\frac{\partial \psi}{\partial \boldsymbol{x}}\right|_{t=t_f}$$

$$\left[\frac{\partial \phi}{\partial t} + \left(\frac{\partial \psi}{\partial t} \right)^T \boldsymbol{\gamma} + H \right]_{t=t_f} = 0$$

Numerical methods

Indirect Method

Solve the Two-point Boundary-value Problem.

Advantage: the solution is smooth and accurate

Disadvantage: Difficult to converge, Difficult to give reasonable a Initial guesses

Direct Method

Transform the optimal control problem into parameter optimization by discretization.

Advantage: Easy to converge than indirect method, Some other optimization parameters can be added

Disadvantage: Time consuming, The solution is not smooth

Hybrid Method

Neglect the transversality condition, adjust the co-states by using parameter optimization method (SQP)

It is a tradeoff method between indirect and direct method.

Hybrid method

Differential equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \\ \dot{\boldsymbol{\lambda}} = -\dfrac{\partial H}{\partial \mathbf{x}} \end{cases}$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0$$

Boundary condition

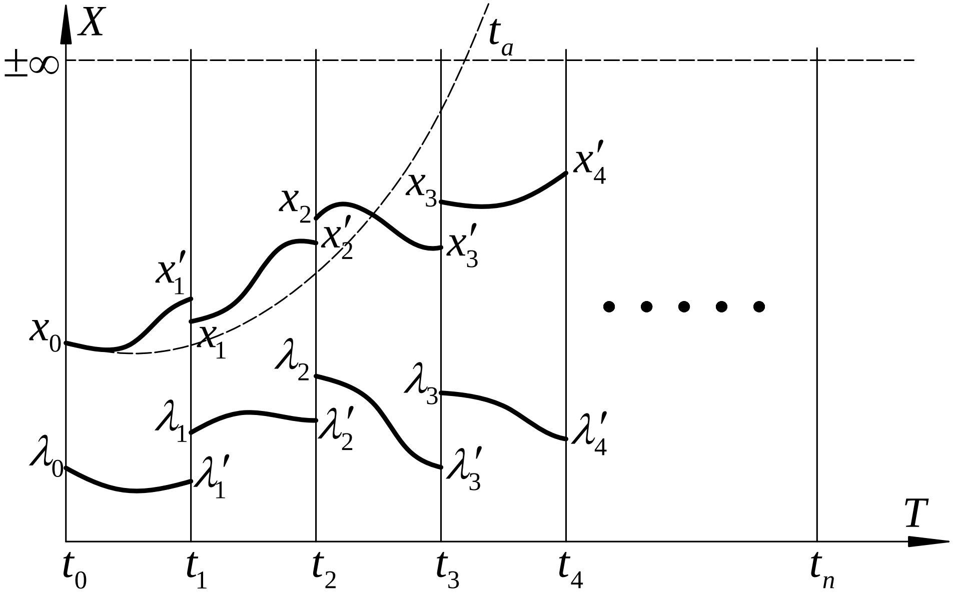
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$$\mathbf{x}(t_f) = \mathbf{x}_f$$

Transversality condition

$$H\big|_{t_f} = -\frac{\partial \Phi}{\partial t}(\mathbf{x}^*(t_f), t_f)$$

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)$$



Performance Index

$$J(\mathbf{u}) = \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)\,dt$$

1. Neglect the transversality condition;
2. Discretization of the states and the co-states;(optional)
3. Using constrained parameter optimization method to minimize the performance index.

Sequential Quadratic Programming (SQP).
Software package named SNOPT.
Genetic Algorithm (GA), Simulated annealing (SA)

Orbit transfer from LEO to GEO

Optimization Parameter:

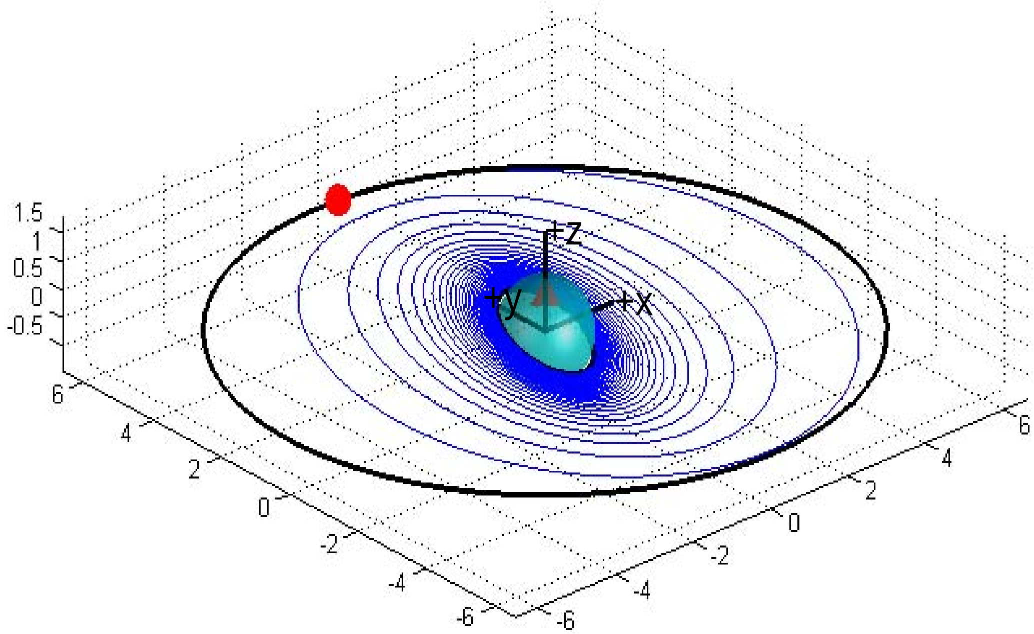
$$\boldsymbol{Z} = [t_{\mathrm{TOF}}, \lambda_{\mathrm{p}}, \lambda_{\mathrm{f}}, \lambda_{\mathrm{g}}, \lambda_{\mathrm{h}}, \lambda_{\mathrm{k}}, \lambda_{\mathrm{L}}, \lambda_{\mathrm{m}}]$$

Constraints:

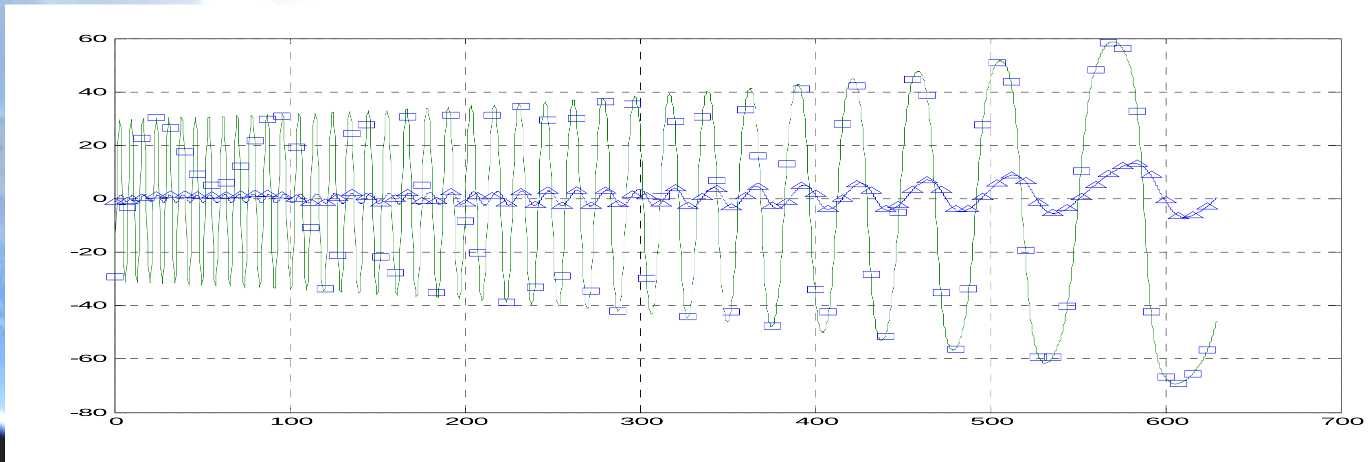
$$C = \zeta_{\mathrm{T}}\left(t_{\mathrm{TOF}}\right) - \zeta_{\mathrm{GEO}} = 0, \quad \left(\zeta = p, f, g, h, k\right)$$

Performance Index:

$$J = t_{\mathrm{TOF}}$$



Transfer trajectory



Thruster angles in orbit frame

Green: yaw thruster angle

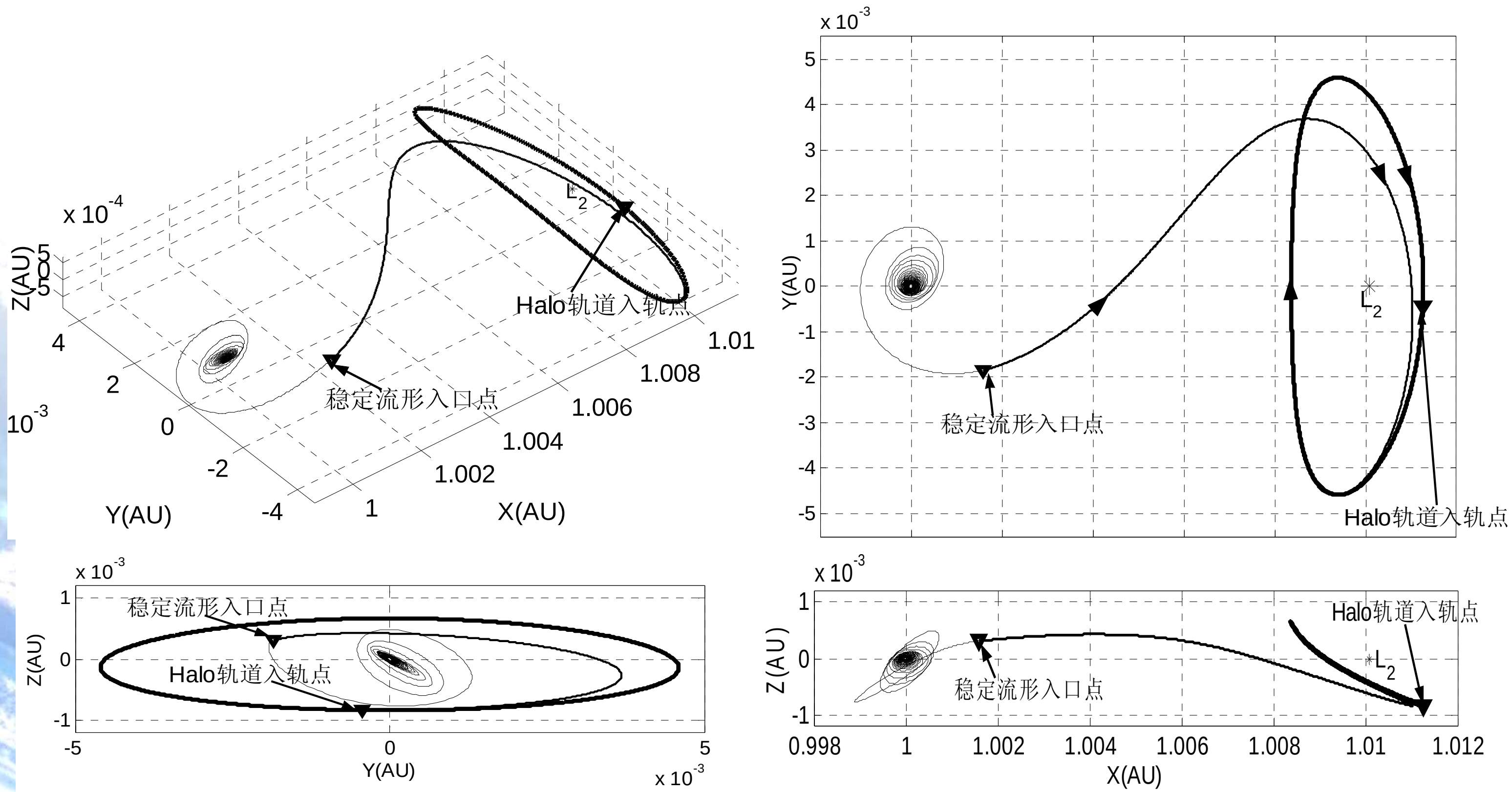
Blue: pitch thruster angle

Low-thrust transfer from parking orbit to halo orbit

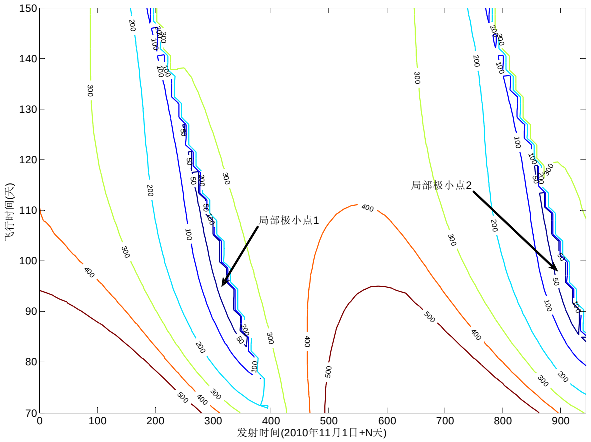
Optimization Parameter: $\boldsymbol{Z}=[t_{\mathrm{T}},\lambda_{\mathrm{p}},\lambda_{\mathrm{f}},\lambda_{\mathrm{g}},\lambda_{\mathrm{h}},\lambda_{\mathrm{k}},\lambda_{\mathrm{L}},\lambda_{\mathrm{m}},t_{\mathrm{p}},t_{\mathrm{c}}]$

Constraints: $C=\zeta_{\mathrm{BACK}}(t_{\mathrm{T}})-\zeta_{\mathrm{LEO}}=0,\ \ (\zeta=p,f,g,h,k)$

Performance Index: $J=t_{\mathrm{T}}$



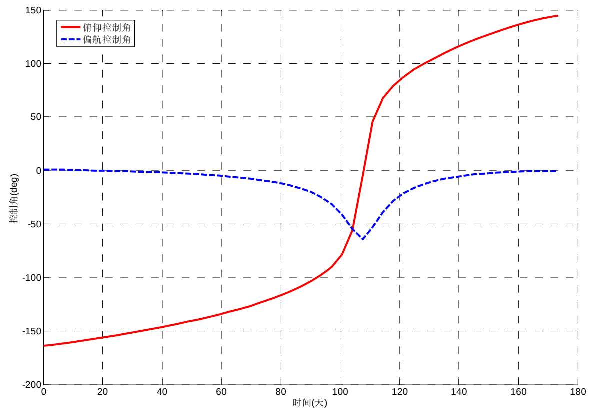
Direct transfer from the Earth to the Venus

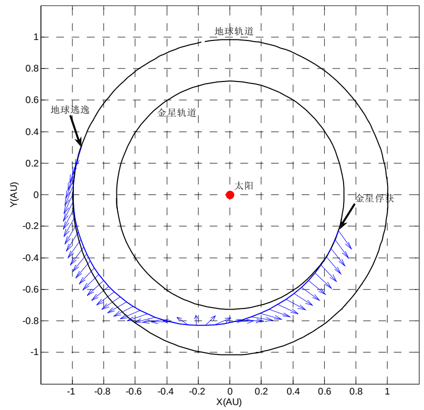


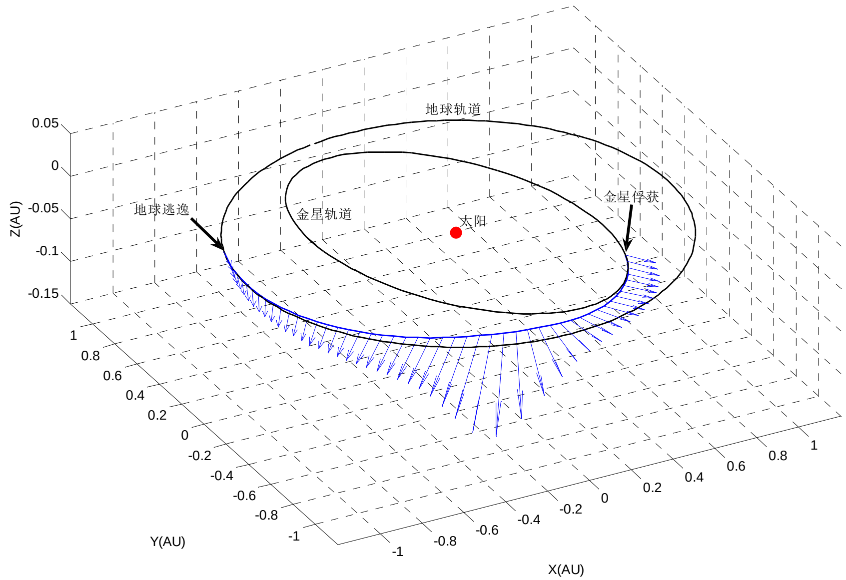
Launch opportunity search

$$
\begin{aligned}
&\boldsymbol{Z} = [t_L, t_A, \lambda_p, \lambda_f, \lambda_g, \lambda_h, \lambda_k, \lambda_L, \lambda_m] \\
&C = \zeta_T(t_A) - \zeta_{Venus}(t_A) = 0, \quad (\zeta = p, f, g, h, k, L) \\
&J = t_A - t_L
\end{aligned}
$$

Trajectory optimization





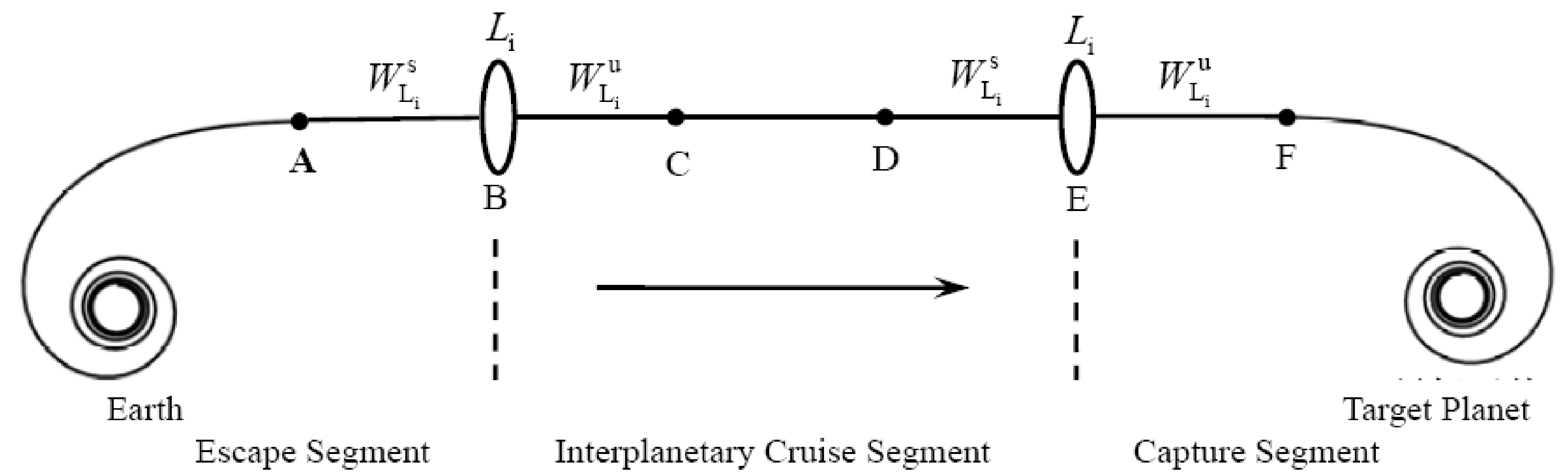


low-thrust IPS transfer

Low-thrust & Interplanetary SuperHighway(IPS) have the same aim ---- Increase the percentage of payload

Low-thrust ---- Increase the availability factor of working substance

IPS ---- Decrease the total delta V



Orbit structure of the low-thrust IPS transfer

One Thrust Arc IPS transfer

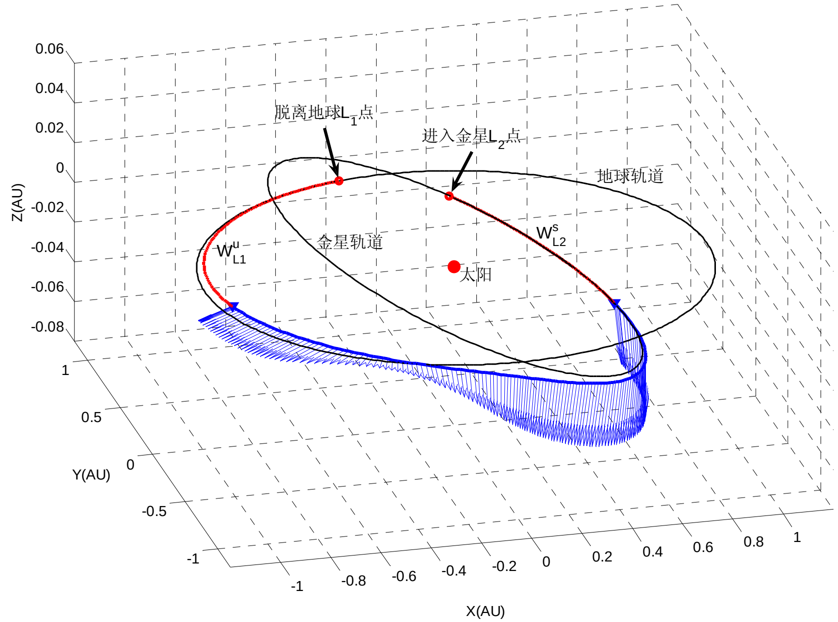
$$\boldsymbol{Z}=[t_{\mathrm{L}},t_{\mathrm{A}},t_{\mathrm{wu}},t_{\mathrm{ws}},\lambda_{\mathrm{p}},\lambda_{\mathrm{f}},\lambda_{\mathrm{g}},\lambda_{\mathrm{h}},\lambda_{\mathrm{k}},\lambda_{\mathrm{L}},\lambda_{\mathrm{m}},\delta v_{\mathrm{L}},\delta v_{\mathrm{A}}]$$

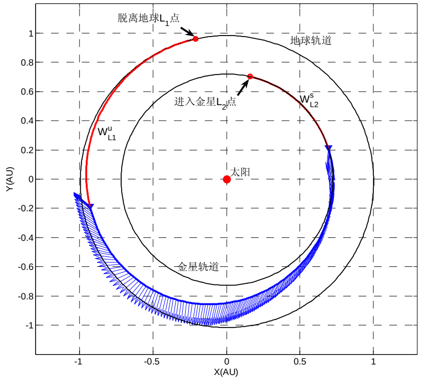
$$\delta v_{\mathrm{L}}\in\left[\delta v_{\mathrm{L}}^{\mathrm{l}},\delta v_{\mathrm{L}}^{\mathrm{u}}\right],\quad \delta v_{\mathrm{A}}\in\left[\delta v_{\mathrm{A}}^{\mathrm{l}},\delta v_{\mathrm{A}}^{\mathrm{u}}\right]$$

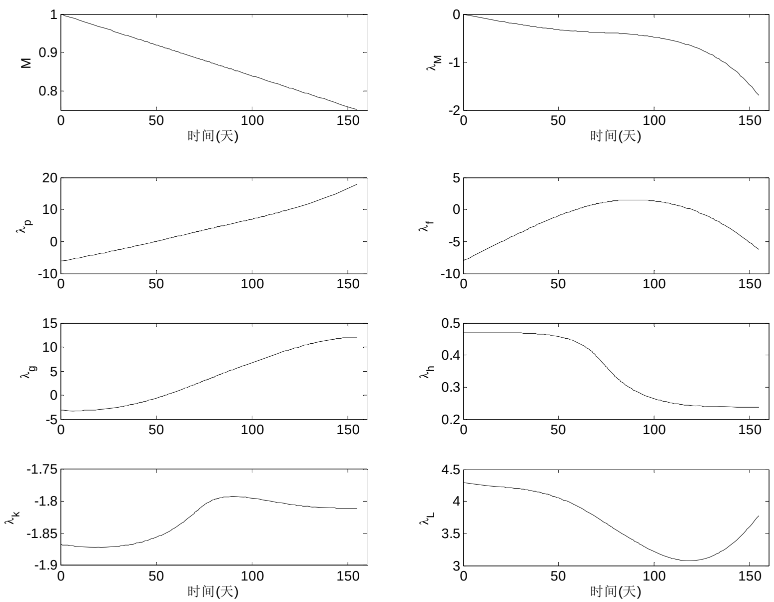
$$T_{\mathrm{ThrustArc}}=\left(t_{\mathrm{A}}-t_{\mathrm{L}}\right)\times 24\times 3600/\mathrm{TU}-t_{\mathrm{wu}}-t_{\mathrm{ws}}\geq 0$$

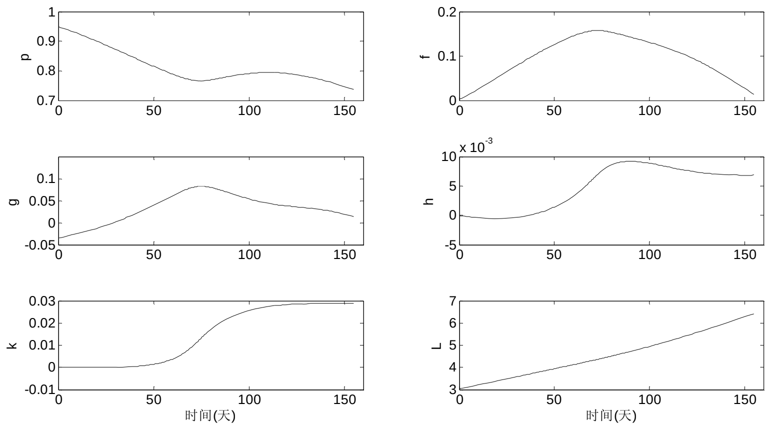
$$C=\zeta_{\mathrm{ThrustArc}}\left(t_{\mathrm{c}}\right)-\zeta_{\mathrm{ws}}\left(t_{\mathrm{c}}\right)=0,\quad \left(\zeta=p,f,g,h,k,L\right)$$

$$J=T_{\mathrm{ThrustArc}}$$









Thrust-coast-thrust IPS transfer

$$\boldsymbol{Z}=[t_{\mathrm{L}},t_{\mathrm{A}},t_{\mathrm{wu}},t_{\mathrm{ws}},\lambda_{\mathrm{p}}^{1},\lambda_{\mathrm{f}}^{1},\lambda_{\mathrm{g}}^{1},\lambda_{\mathrm{h}}^{1},\lambda_{\mathrm{k}}^{1},\lambda_{\mathrm{L}}^{1},\lambda_{\mathrm{m}}^{1},\lambda_{\mathrm{p}}^{2},\lambda_{\mathrm{f}}^{2},\lambda_{\mathrm{g}}^{2},\lambda_{\mathrm{h}}^{2},\lambda_{\mathrm{k}}^{2},\lambda_{\mathrm{L}}^{2},\lambda_{\mathrm{m}}^{2},\delta v_{\mathrm{L}},\delta v_{\mathrm{A}},t_1,t_2]$$

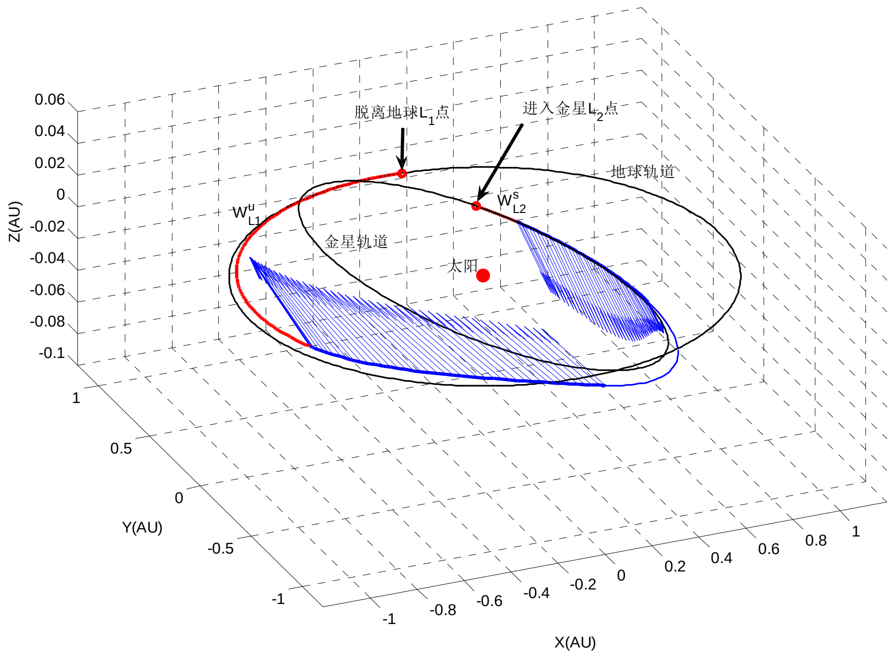
$$\delta v_{\mathrm{L}}\in\left[\delta v_{\mathrm{L}}^{\mathrm{l}},\delta v_{\mathrm{L}}^{\mathrm{u}}\right],\quad \delta v_{\mathrm{A}}\in\left[\delta v_{\mathrm{A}}^{\mathrm{l}},\delta v_{\mathrm{A}}^{\mathrm{u}}\right],\quad t_1\in[0,1),\quad t_2\in[0,1)$$

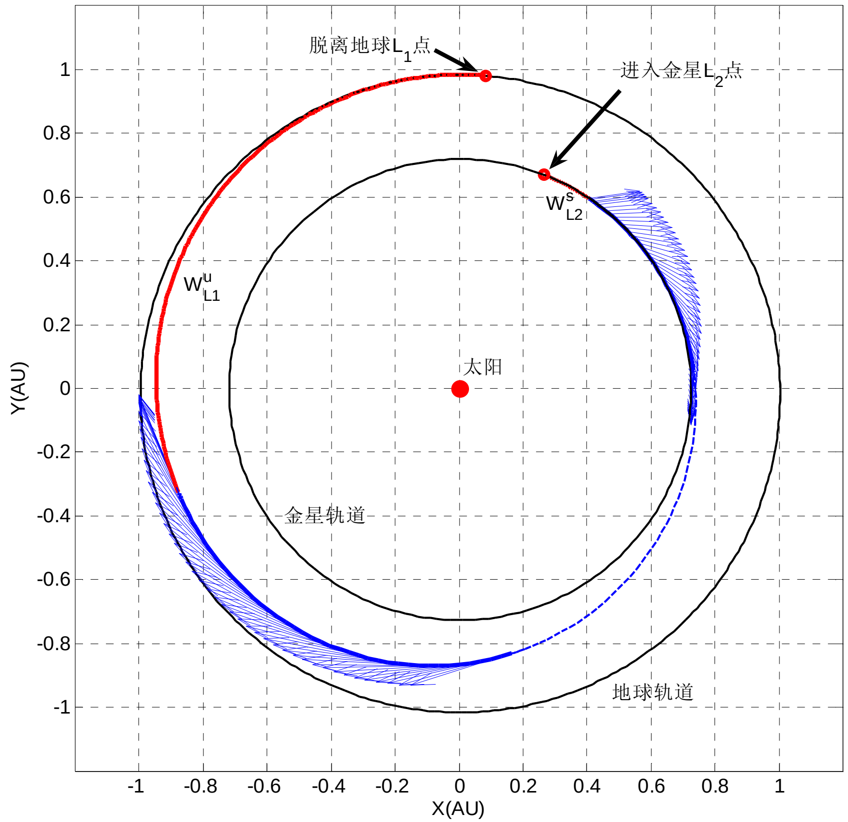
$$T_{\mathrm{TCT}}=(t_{\mathrm{A}}-t_{\mathrm{L}})\times 24\times 3600/\mathrm{TU}-t_{\mathrm{wu}}-t_{\mathrm{ws}}\geq 0$$

$$t_1+t_2\leq 1$$

$$C=\zeta_{\mathrm{TCT}}(t_{\mathrm{c}})-\zeta_{\mathrm{ws}}(t_{\mathrm{c}})=0,\quad (\zeta=p,f,g,h,k,L)$$

$$J=\left[(t_{\mathrm{A}}-t_{\mathrm{L}})\times 24\times 3600/\mathrm{TU}-t_{\mathrm{wu}}-t_{\mathrm{ws}}\right]\times(t_1+t_2)$$





Comparison between the different scenarios

	Low-thrust	One thrust arc IPS	T-C-T IPS
Initial mass (kg)	1200	1200	1200
Mass at the end of the escape segment (kg)	938.358	913.285	913.285
Mass at the end of the cruise segment (kg)	645.974	686.076	762.602
Arrive mass (kg)	543.749	576.507	638.019
Payload percentage	45.31%	48.042%	53.168%
TOF of the cruise segment (days)	173.396	259.016	280.168

Thank you！