Advanced Course on Long Time Integrations

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Outline

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Objective

Given

- a **real analytic quasi–periodic function**, $f(t)$,
- N equally–spaced values of it on an interval $[0, T]$,

$$\{f(t_l)\}_{l=0}^{N-1}, \quad t_l = l\frac{T}{N},$$

the goal is to compute a trigonometric approximation

$$Q_f(t) = A_0^c + \sum_{l=1}^{N_f} \left(A_l^c \cos(2\pi \frac{\nu_l}{T} t) + A_l^s \sin(2\pi \frac{\nu_l}{T} t)\right.$$

whose **frequencies and amplitudes** are a **good approximation** of the ones of f.

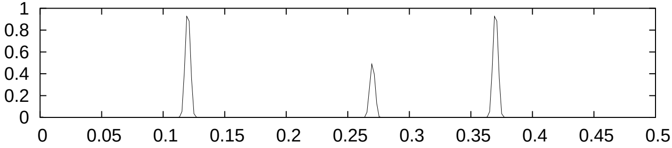
Previous methodology

- A common approach in experimental sciences:
  "perform a DFT and look for peaks"

$$\mathrm{DFT} : f, T, N \longrightarrow \{F_{f,T,N}(k)\}_{k=0}^{N-1}$$

$$F_{f,T,N}(k) := \frac{1}{N}\sum_{j=0}^{N-1} f(t_j) e^{-i2\pi k \frac{j}{N}} = \frac{1}{N}\sum_{j=0}^{N-1} f(t_j) e^{-i2\pi \frac{k}{T} t_j}$$

For instance, for $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$, the **modulus of the DFT**, $|F_{f,T,N}(k)|$, is



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The DFT can be thought as a function of **harmonics** (left) or **frequencies** (right):

$$\begin{array}{rcl} 0 & \mapsto & F_{f,T,N}(0) \\ 1 & \mapsto & F_{f,T,N}(1) \\ & \vdots & \\ N-1 & \mapsto & F_{f,T,N}(N-1) \end{array} \qquad \begin{array}{rcl} 0 & \mapsto & F_{f,T,N}(0) \\ 1/T & \mapsto & F_{f,T,N}(1) \\ & \vdots & \\ (N-1)/T & \mapsto & F_{f,T,N}(N-1) \end{array}$$

Previous methodology

- A common approach in experimental sciences:
  "perform a DFT and look for peaks"
- Laskar's procedure[4, 3]: consider the function that maps an harmonic to the corresponding Fourier coefficient of f:

$$k \longmapsto |a_k| := \left| \frac{1}{T} \int_0^T f(t) e^{-i2\pi kt} dt \right|$$

For $f(t) = e^{i2\pi\omega t}$,

  - $|a_k|$ has a **global maximum** at $k = \omega$.
  - The $|$ DFT $|$ samples approximations of $|a_k|$ at **integer values** of k.

Previous methodology

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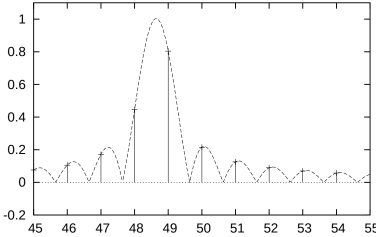

$$f(t) = e^{i2\pi\omega t},\ \omega = 0.76,\ T = 64,\ N = 256 \Longrightarrow \omega T = 48.64$$

Previous methodology

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$$k \longmapsto |a_k| := \left| \frac{1}{T} \int_0^T f(t) e^{-i2\pi kt} dt \right|$$

By using a **numerical quadrature rule**, the previous integral can be evaluated from the samples of f for **any** value of k.

Outline

A windowed Fourier Transform

- Fourier Fransform:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi\omega t} dt.$$

- Windowed Fourier Transform (window $\chi_{[0,T]}(t)$):

$$\int_{-\infty}^{+\infty} \chi_{[0,T]}(t) f(t) e^{-i2\pi\omega t} dt = \int_{0}^{T} f(t) e^{-i2\pi\omega t} dt.$$

- Normalized, Windowed Fourier Transform (WFT, the one we will consider):

$$\phi_{f,T}(\omega) = \frac{1}{T} \int_{0}^{T} f(t) e^{-i2\pi\omega t} dt.$$

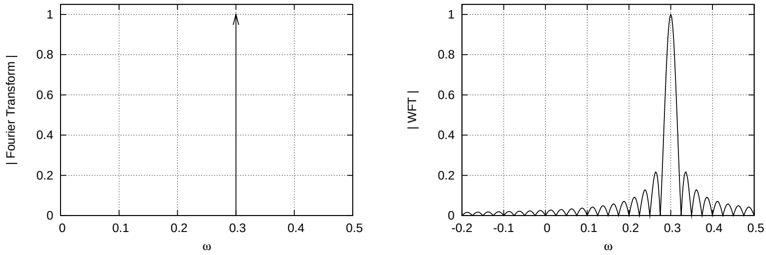
Leakage

For a **complex exponential term**, $f(t) = e^{i2\pi\nu t}$,

$$\phi_{e^{i2\pi\nu t},T}(\omega) = \frac{e^{i2\pi(\nu-\omega)T} - 1}{i2\pi(\nu-\omega)T}.$$

We have

$$\left|\frac{e^{i2\pi x} - 1}{i2\pi x}\right| = \left|\frac{\sin \pi x}{\pi x}\right| = |\operatorname{sinc} x|,$$



Leakage

For **several complex exponential terms**,

$$f(t) = a_\nu e^{i2\pi\nu t} + \sum_{\xi \in \Omega \setminus \{\nu\}} a_\xi e^{i2\pi\xi t},$$

(Ω is the set of frequencies fo the signal), so that

$$|\phi_{f,T}(\omega)| = |a_\nu| \, |\, \mathrm{sinc}\big(T(\nu - \omega)\big)| + \sum_{\xi \in \Omega \setminus \{\nu\}} O\left(\frac{1}{T(\xi - \omega)}\right),$$

The **second term** in the last equation is responsible for the **peak** of $|\phi_{f,T}(\omega)|$ near $\omega = \nu$ **not being exactly** at ν.

Leakage

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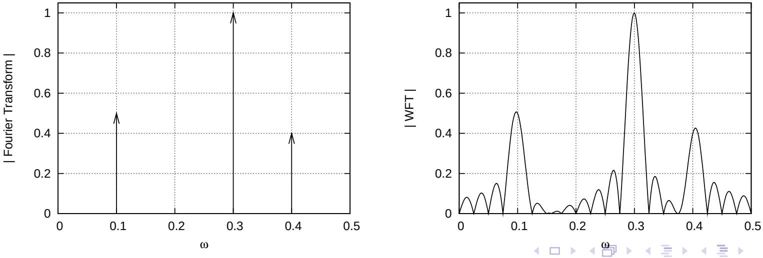
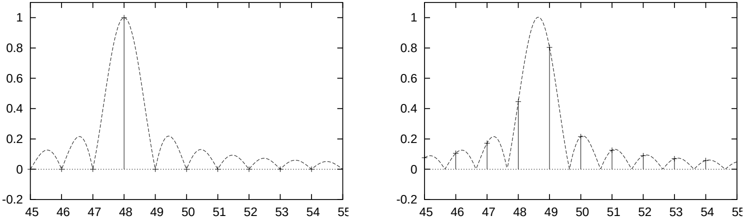


Leakage

In order to **not to have leakage**, the set of frequencies needs to satisfy

$$\Omega \subset \{k/T, k \in \mathbb{Z}\}.$$



$$f(t) = \cos(2\pi\omega t), T = 64 \text{ and } N = 256$$
$$\text{Left: } \omega = 0.75 \Longrightarrow T\omega = 48$$
$$\text{Right: } \omega = 0.76 \Longrightarrow T\omega = 48.64$$

Reducing leakage: filtering

We can substitue $\chi_T(t)$ in the WFT by the **Hanning window** (of filter) of order n_h:

$$H_T^{n_h}(t) = q_{n_h}\left(1 - \cos\frac{2\pi t}{T}\right)^{n_h}.$$

being $q_{n_h} = n_h!/\big((2n_h - 1)!!\big)$.

The corresponding WFT is denoted by

$$\phi_{f,T}^{n_h}(\omega) := \frac{1}{T}\int_0^T H_T^{n_h}(t) f(t) e^{-i2\pi\omega t}dt,$$

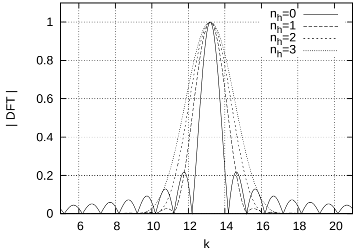
Reducing leakage: filtering

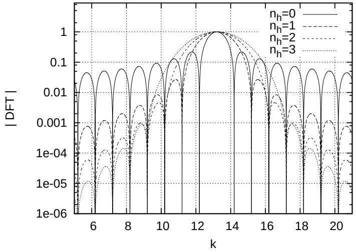
For $f(t) = e^{i2\pi\nu t}$,

$$\phi_{e^{i2\pi\nu t},T}(\omega) = \frac{e^{i2\pi(\nu-\omega)T} - 1}{i2\pi(\nu-\omega)T} = O\left(\frac{1}{(\nu-\omega)T}\right),$$

vs

$$\phi^{n_h}_{e^{i2\pi\nu t},T}(\omega) = \frac{(-1)^{n_h}(n_h!)^2\left(e^{i2\pi(\nu-\omega)T} - 1\right)}{i2\pi\prod_{j=-n_h}^{n_h}\left((\nu-\omega)T + j\right)} = O\left(\frac{1}{\left((\nu-\omega)T\right)^{1+2n_h}}\right)$$





Reducing leakage: filtering

The DFT can also be filtered:

$$F_{f,T,N}^{n_h}(k) := \frac{1}{N}\sum_{j=0}^{N-1} H_N^{n_h}(j) f(j\frac{T}{N}) e^{-i2\pi \frac{k}{N} j}.$$

Aliasing

The DFT is N–periodic:

$$F^{n_h}_{f,T,N}(k) := \frac{1}{N}\sum_{j=0}^{N-1} H^{n_h}_N(j) f(j\tfrac{T}{N}) e^{-i2\pi \frac{k}{N} j}.$$

It is actually the N–periodification of the WFT:

$$F^{n_h}_{f,T,N}(k) = \phi^{n_h}_{f,T}\left(\frac{k}{T}\right) + \sum_{l=1}^{\infty}\left(\phi^{n_h}_{f,T}\left(\frac{k+lN}{T}\right) + \phi^{n_h}_{f,T}\left(\frac{k-lN}{T}\right)\right).$$

Moreover, for f real (always in practice), we have the symmetry

$$F^{n_h}_{f,T,N}(k) = \overline{F^{n_h}_{f,T,N}(N-k)}.$$

Therefore, a **fundamental domain** for the DFT is

$$[0, N/2] \quad \text{(harmonics)}, \qquad [0, N/(2T)] \quad \text{(frequencies)}.$$

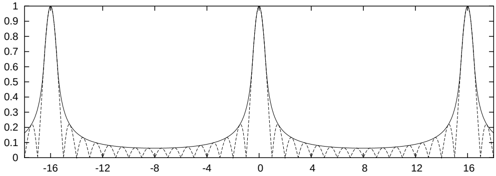
Aliasing

The **aliasing effect** consists on:

- In **frequency domain**: due to N–periodicity and the symmetry, any frequency of the signal outside $[0, N/(2T)]$ gives a peak in $[0, N/(2T)]$.
  Such a peak is called an **alias** or an **aliased frequency**.
- In **time domain**: two periodic terms corresponding to aliased frequencies

$$e^{i2\pi\nu t}, \quad e^{i2\pi(\nu+kN/T)t},$$

  have the same values at $\{j\frac{T}{N}\}_{j=0}^{N-1}$ (stagecoach wheel effect).



Outline

Algorithm[1]

1. Set an starting **thresold** for collecting **peaks of the modulus of the DFT** of f(t).
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

Computing amplitudes from known frequencies

We ask $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$, being

$$Q_f(t) = A_0^c + \sum_{l=1}^{N_f} (A_l^c \cos(2\pi \frac{\nu_l}{T} t) + A_l^s \sin(2\pi \frac{\nu_l}{T} t).$$

Since we work with **real** signals, we use the sine and cosine transforms:

$$\begin{aligned} c_{f,T,N}^{n_h}(k) &= \frac{2}{N} \sum_{j=0}^{N-1} f(j\tfrac{T}{N}) H_N^{n_h}(j) \cos\left(2\pi \tfrac{k}{N} j\right), \quad k = 0, \ldots, \tfrac{N}{2}, \\ s_{f,T,N}^{n_h}(k) &= \frac{2}{N} \sum_{j=0}^{N-1} f(j\tfrac{T}{N}) H_N^{n_h}(j) \sin\left(2\pi \tfrac{k}{N} j\right), \quad k = 1, \ldots, \tfrac{N}{2} - 1. \end{aligned}$$

They are realted to the DFT in complex form by

$$F_{f,T,N}^{n_h}(k) = \frac{1}{2}\left(c_{f,T,N}^{n_h}(k) - i s_{f,T,N}^{n_h}(k)\right), \qquad k = 0, \ldots, N/2.$$

Computing amplitudes from known frequencies

The system of equations to be solved is **linear** and $(1+2N_f)\times(1+2N_f)$:

$$\begin{aligned}
A_0^c c^{n_h}_{1,T,N}(0) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(0) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(0)\right) &= c^{n_h}_{f,T,N}(0) \\
A_0^c c^{n_h}_{1,T,N}(j) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(j) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(j)\right) &= c^{n_h}_{f,T,N}(j) \\
\sum_{l=1}^{N_f}\left(A_l^c \overline{s}^{n_h}_{\nu_l,T}(j) + A_l^c \widetilde{s}^{n_h}_{\nu_l,T}(j)\right) &= s^{n_h}_{f,T,N}(j)
\end{aligned}$$

where $j=[\nu_l+0.5]$, $l=1\div N_f$, and

$$\begin{aligned}
c_1^{n_h}(j) &= c^{n_h}_{1,T,N}(j), \\
\overline{c}^{n_h}_{\nu_l,N}(j) &= c^{n_h}_{\cos(\frac{2\pi\nu_l}{T}),T,N}(j), \quad & \overline{s}^{n_h}_{\nu_l,N}(j) &= s^{n_h}_{\cos(\frac{2\pi\nu_l}{T}),T,N}(j), \\
\widetilde{c}^{n_h}_{\nu_l,N}(j) &= c^{n_h}_{\sin(\frac{2\pi\nu_l}{T}),T,N}(j), \quad & \widetilde{s}^{n_h}_{\nu_l,N}(j) &= s^{n_h}_{\sin(\frac{2\pi\nu_l}{T}),T,N}(j).
\end{aligned}$$

Simultaneous improvement of frequencies and amplitudes

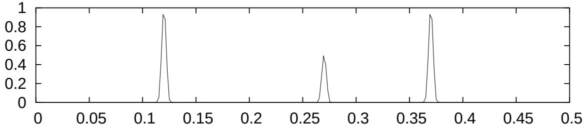
We solve by **Newton's method** the following $(1+3N_f)\times(1+3N_f)$ **non–linear system**:

$$
\begin{array}{rcl}
A_0^c c_{1,T,N}^{n_h}(0) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}_{\nu_l,N}^{n_h}(0) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(0)\right) & = & c_{f,T,N}^{n_h}(0) \\
A_0^c c_{1,T,N}^{n_h}(j_i) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(j_i)\right) & = & c_{f,T,N}^{n_h}(j_i) \\
\sum_{l=1}^{N_f}\left(A_l^c \overline{s}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{s}_{\nu_l,N}^{n_h}(j_i)\right) & = & s_{f,T,N}^{n_h}(j_i) \\
A_0^c cs_{1,T,N}^{n_h}(j_i^+) + \sum_{l=1}^{N_f}\left(A_l^c \overline{cs}_{\nu_l,N}^{n_h}(j_i^+) + A_l^s \widetilde{cs}_{\nu_l,N}^{n_h}(j_i^+)\right) & = & cs_{f,T,N}^{n_h}(j_i^+)
\end{array}
$$

being $j_i = [\nu_i + 0.5], j_i^+ \neq j_i, |j_i^+ - j_i| = 1.$

A (toy) example

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
1. Starting thresold: 0.8
modulus of the DFT of the input data:



$\Rightarrow$ peaks $j = 61, j = 189$.

A (toy) example

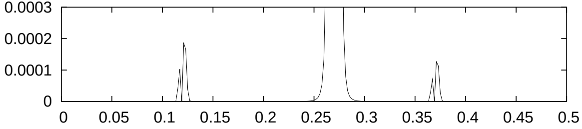
For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
2. Approximation of frequencies (Laskar's method):

$$\begin{aligned} \text{peak } 61 \quad &\Rightarrow \text{frequency } 0.11999948789 \\ \text{peak } 189 \quad &\Rightarrow \text{frequency } 0.36999965075 \end{aligned}$$

3. Computation of amplitudes from known frequencies:

Frequency	Cosine amplitude	Sine amplitude
0.119999487888	0.999907666367	-0.000823654552
0.369999650752	0.000561727398	0.999937098420

modulus of the DFT of the residual

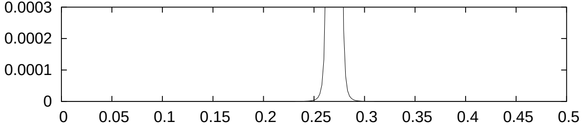


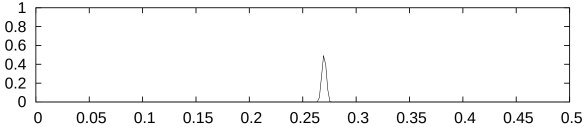
A (toy) example

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.120000000003	1.000000000686	0.000000005106
0.369999999995	0.000000006660	1.000000000297

5. modulus of the DFT of input signal minus step 4:





New threshold: 0.2

A (toy) example

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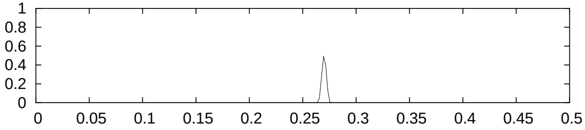
$T = N = 512$, $n_h = 0$.
5. modulus of the DFT of input signal minus step 4:



New threshold: 0.2
2. Approximation of frequencies (Laskar's method):

$$\text{peak } 138 \quad \Rightarrow \text{frequency } 0.26999999999988849$$

3. Amplitudes from known frequencies:

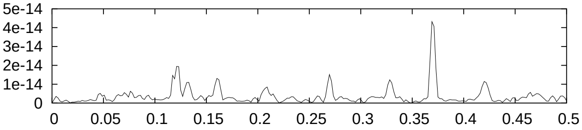
Frequency	Cosine amplitude	Sine amplitude
0.120000000003	1.000000000586	0.000000005245
0.369999999995	0.000000007480	0.999999999164
0.269999999999	-0.000000000897	-0.499999999946

A (toy) example

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.12000000000000	0.99999999999999	-0.00000000000008
0.37000000000000	-0.00000000000001	1.00000000000000
0.27000000000000	0.00000000000000	-0.50000000000000

modulus of the DFT of the residual:



Outline

Assumptions

- f is **real analytic** and **quasi–periodic**,

$$f(t) = A_0^c + \sum_{\substack{k\in\mathbb{Z}^m \\ k\omega>0}} \left(A_k^c \cos(2\pi\langle k,\omega\rangle t) + A_k^s \sin(2\pi\langle k,\omega\rangle t)\right) = \sum_{k\in\mathbb{Z}^m} A_m e^{i2\pi\langle k,\omega\rangle t},$$

  $\omega = (\omega_1, \ldots, \omega_m)$ rationally independent,
  and its Fourier coefficients a_k satisfy the **Cauchy estimates**,

$$\sqrt{(A_k^c)^2 + (A_k^s)^2} \leq Ce^{-\delta|k|} \quad \forall k \in \mathbb{Z}^m.$$

- The frequency vector $\omega = (\omega_1, \ldots \omega_m)$ satisfies a **Diophantine condition**: for $D, \tau > 0$,

$$|k\omega| \geq \frac{D}{|k|^\tau},$$

- We determine the frequencies $\{\nu_l\}_{l=1}^{N_f}$ of order $\leq r_0 - 1$ ($\nu_l \approx Tk\omega,\ 1 \leq |k| \leq r_0 - 1$).
  The **order** of the freq. $\langle k, \omega\rangle$ is $|k| = |k_1| + \cdots + |k_m|$.

Strategy[2]

Let us denote

- f_{r_0}: the truncation of f to the frequencies we want to determine:

$$f_{r_0}(t) = A_0^c + \sum_{\substack{|k|\leq r_0-1 \\ k\omega>0}} \left(A_k^c \cos(2\pi k\omega t) + A_k^s \sin(2\pi k\omega t)\right).$$

- $y = (A_0, \nu_1, A_1^c, A_1^s, \ldots, \nu_{N_f}, A_{N_f}^c, A_{N_f}^s)$: the **exact** frequencies and amplitudes.
- $y + \Delta y$: the **computed** frequencies and amplitudes.

The **system we solve** for iterative improvement of frequencies and amplitudes is

$$\underbrace{\mathrm{DFT}(Q_f)}_{g(y+\Delta y)} = \underbrace{\mathrm{DFT}(f_{r_0})}_{b} + \underbrace{\mathrm{DFT}(f - f_{r_0})}_{\Delta b}$$

We would get the **exact** frequencies and amplitudes **if** $\Delta b = 0$.

Strategy[2]

- System for iterative improvement of frequencies and amplitudes:

$$\begin{aligned}
A_0^c + \sum_{l=1}^{N_f} \left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(0) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(0) \right) &= c^{n_h}_{f_{r_0},T,N}(0) + c^{n_h}_{f-f_{r_0},T,N}(0) \\
A_0^c c_1^{n_h}(j_i) + \sum_{l=1}^{N_f} \left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(j_i) \right) &= c^{n_h}_{f_{r_0},T,N}(j_i) + c^{n_h}_{f-f_{r_0},T,N}(j_i) \\
\sum_{l=1}^{N_f} \left(A_l^c \overline{s}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{s}^{n_h}_{\nu_l,N}(j_i) \right) &= s^{n_h}_{f_{r_0},T,N}(j_i) + s^{n_h}_{f-f_{r_0},T,N}(j_i) \\
A_0^c cs_1^{n_h}(j_i^+) + \sum_{l=1}^{N_f} \left(A_l^c \overline{cs}^{n_h}_{\nu_l,N}(j_i^+) + A_l^s \widetilde{cs}^{n_h}_{\nu_l,N}(j_i^+) \right) &= cs^{n_h}_{f_{r_0},T,N}(j_i^+) + cs^{n_h}_{f-f_{r_0},T,N}(j_i^+).
\end{aligned}$$

  where $f - f_{r_0} = \sum_{|k| \geq r_0} A_k e^{i2\pi kt}$.

- The error term Δb consists of DFT
  - of **periodic terms** with **frequencies not being computed**,
  - evaluated in **harmonics** corresponding to **frequencies being computed**.

  Therefore, the error term Δb can be considered **leakage of the remainder**, $f - f_{r_0}$.

Strategy[2]

- The error term Δb can be considered **leakage of the remainder**

$$\mathrm{DFT}(f - f_{r_0}) = \sum_{|k| \geq r_0} A_k \, \mathrm{DFT}(e^{i2\pi k\omega t})$$

- The effect of the terms of the remainder on the error Δb is
  - The DFT of terms corresponding to **low–order frequencies**, $\{k\omega\}_{|k| \gtrsim r_0}$, evaluated at the harmonics $\{j_i, j_i^+\}$, will be **small** if the harmonics $Tk\omega$ are far from $\{j_i, j_i^+\}$.
    This can be achieved by increasing T **as long as there is no aliasing**.
  - The DFT of terms corresponding to **high–order frequencies may not be small** ($Tk\omega$ can be made arbitrarily close to a j_i for large enough $|k|$).
    However, the corresponding **amplitudes** will be small due to the Cauchy estimates
    $$\sqrt{(A_k^c)^2 + (A_k^s)^2} \leq C e^{-\delta |k|} \quad \forall k \in \mathbb{Z}^m,$$
    so they will be not harmful.

Bounding

- The **system we solve** for iterative improvement of frequencies and amplitudes is

$$\underbrace{\mathrm{DFT}(Q_f)}_{g(y+\Delta y)} = \underbrace{\mathrm{DFT}(f_{r_0})}_{b} + \underbrace{\mathrm{DFT}(f - f_{r_0})}_{\Delta b}$$

  We would get the **exact** frequencies and amplitudes **if** $\Delta b = 0$.
- The **error in frequencies and amplitudes** is given, at first order, by

$$\|\Delta y\|_\infty \lesssim \|Dg(y)^{-1}\|_\infty \|\Delta b\|_\infty.$$

- **Bounds can be obtained** for $\|Dg(y)^{-1}\|_\infty$ and $\|\Delta b\|$.
- **Main idea**: instead of the DFT, bound the WFT, for which
  - explicit formulae can be obtained, that
  - are more easily handled.

Bound for $\|Dg(y)^{-1}\|_\infty$

We can write

$$Dg(y) =: M = \begin{pmatrix} 2 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & B_{1,1} & \dots & B_{1,N_f} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & B_{N_f,N_f} \end{pmatrix}.$$

We **split** $M = M_D + M_O$,

$$M = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & B_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N_f,N_f} \end{pmatrix} + \begin{pmatrix} 0 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & 0 & \dots & B_{1,N_f} \\ 0 & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & 0 \end{pmatrix}.$$

M is **close to block-diagonal**, so the idea is to obtain **bounds for** $\|M_D^{-1}\|$, $\|M_O\|$ and use

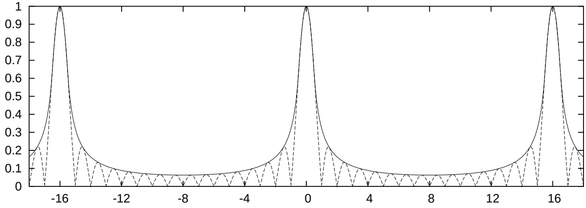
$$\|(M_D + M_O)^{-1}\| \leq \frac{\|M_D^{-1}\|}{1 - \|M_D^{-1}\|\|M_O\|}.$$

Bound for $\|\Delta b\|_\infty$

We have

$$\|\Delta b\| \leq 2C \max_{j \in J} \sum_{|k|=r_0}^{\infty} e^{-\delta|k|} |\widetilde{h}_N^{n_h}(Tk\omega - j)|$$

where $|\widetilde{h}_N^{n_h}|$ is the envelope displayed below ($N = 16$, $n_h = 0$).



Bound for $\|\Delta b\|_\infty$

We have

$$\|\Delta b\| \leq 2C \max_{j\in J} \sum_{|k|=r_0}^{\infty} e^{-\delta|k|} |\widetilde{h}_N^{n_h}(Tk\omega - j)|$$

The **Diophantine condition** gives a **lower bound for** $|Tk\omega - j|$:

$$|Tk\omega - j| \quad \geq \quad \frac{TD}{(|k| + |k_j|)^{\tau}} - 1.$$

For $|k|$ **small**, $|\widetilde{h}_N^{n_h}(Tk\omega - j)| \ll 1$.
After some order r_*, $|\widetilde{h}_N^{n_h}(Tk\omega - j)|$ may approach 1.
Therefore,

$$\|\Delta b\| \leq 2C\Big(\max_{j\in J} \sum_{|k|=r_0}^{r_*-1} e^{-\delta|k|} |\widetilde{h}_N^{n_h}(Tk\omega - j)| + \max_{j\in J} \sum_{|k|=r_*}^{\infty} e^{-\delta|k|}\Big).$$

Bound for $\|\Delta b\|_\infty$

In

$$\|\Delta b\| \leq 2C\left(\max_{j\in J}\sum_{|k|=r_0}^{r_*-1} e^{-\delta|k|}|\widetilde{h}_N^{n_h}(Tk\omega - j)| + \max_{j\in J}\sum_{|k|=r_*}^{\infty} e^{-\delta|k|}\right),$$

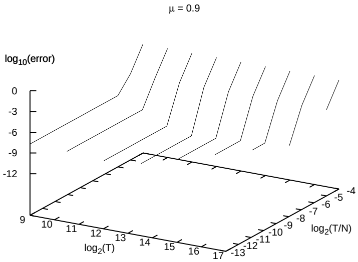
- The first term is bounded by **replacing the DFT by the WFT**. This introduces an additional error term due to this approximation.
- All the sums are reduced to **sums of the form** $\sum_j j^\alpha e^{-\delta j}$, which are bounded by **incomplete Gamma functions**.

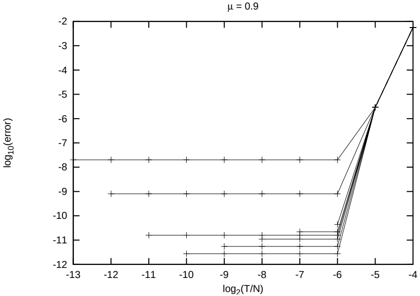
Accuracy test

We consider the **quasi–periodic function** ($\omega = (1, \sqrt{2})$, $\varphi = (0.2, 0.3)$)

$$f_{0.9}(t) = \frac{\sin(2\pi\omega_1 t + \varphi_1)}{1 - 0.9\cos(2\pi\omega_1 t + \varphi_1)} \cdot \frac{\sin(2\pi\omega_2 t + \varphi_2)}{1 - 0.9\cos(2\pi\omega_2 t + \varphi_2)}.$$

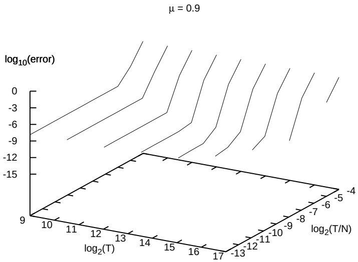
Explicit formulae for frequencies and amplitudes can be obtained, as well as the **Cauchy estimates** and the **Diophantine condition**.
We have performed **Fourier analysis** of this function for **several** T, N, computing the first 20 frequencies ($|k| \leq 5$).

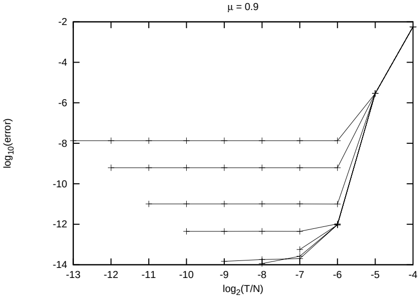




Accuracy test

Error in amplitudes only:





[1] G. Gómez, J.M. Mondelo, and C. Simó.
A collocation method for the numerical Fourier analysis of quasi–periodic functions. I: Numerical tests and examles.
Preprint, 2007.

[2] G. Gómez, J.M. Mondelo, and C. Simó.
A collocation method for the numerical Fourier analysis of quasi–periodic functions. II: Analytical error estimates.
Preprint, 2007.

[3] J. Laskar.
Introduction to Frequency Map Analysis.
In C. Simó, editor, *Hamiltonian Systems With Three or More Degrees of Freedom*, pages 134–150. Kluwer Academic Pub., 1999.

[4] J. Laskar, C. Froeschlé, and A. Celletti.
A measure of chaos by the numerical analysis of the fundamental frequencies. Application to the standard mapping.
Physica D, 56:253–269, 1992.