On exponentially accurate approximation of a near the identity map by an autonomous flow

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October 18, 2024

Abstract

This paper contains a proof of a refined version of Neishtadt's theorem which states that an analytic near-identity map can be approximated by the time-one map of an autonomous flow with exponential accuracy. We provide explicit expressions for the vector fields and give explicit bounds for the error terms.

Keywords— near-the-identity maps, discrete averaging, embedding of a map into flow

The classical result of Neishtadt [3] states that an analytic near-the-identity family of maps can be approximated by time-one maps of autonomous vector fields, with approximation error decaying exponentially fast as the parameter vanishes. The rate of decay is controlled by the ratio of two parameters δ/ε, where ε characterises the distance to the identity in a complex δ-neighbourhood of the domain of the map. Neishtad's theorem provides a useful tool for studying dynamics of close-to-the identity maps. Its proof is based on the classical averaging for rapidly oscillating time-periodic flows and does not provide explicit expressions for the vector fields in terms of the original map. Therefore, checking the accuracy of an approximation for an individual map becomes difficult and finding an expression for the vector field impossible from the practical point of view.

In this note we use discrete averaging to provide explicit expressions for the vector fields which approximate a near-identity map f and give explicit expressions for the approximation errors.

We consider an analytic (or real-analytic) map $f : D_0 \to \mathbb{C}^n$ defined on a subset $D_0 \subset \mathbb{C}^n$ (or $\mathbb{R}^n$). We suppose that there is $\delta > 0$ such that the analytic continuation of f onto D_δ, a complex δ-neighbourhood of D_0, is close to the identity map ξ and define

$$\varepsilon = \|f - \xi\|_{D_\delta}. \tag{1}$$

In this paper we use the infinity norm for vectors and supremum norms for functions. Let $m \in \mathbb{N}$ and define an interpolating vector field of order m,

$$X_m(x) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \Delta_k(x), \tag{2}$$

where the finite differences are defined recursively

$$\Delta_0(x) = x, \qquad \Delta_k(x) = \Delta_{k-1}(f(x)) - \Delta_{k-1}(x) \quad \text{for } k \geq 1. \tag{3}$$

We say that X_m is obtained with the help of *discrete averaging* as X_m is a weighted sum of $f^k(x)$ for $0 \leq k \leq m$. Indeed, it is not too difficult to check that

$$\Delta_k(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f^i(x).$$

Theorem 1. *If a map f is analytic in D_δ and $\varepsilon/\delta \leq 1/6e$, then the interpolating vector field X_m of order $2 \leq m \leq M_\varepsilon + 1$, where $M_\varepsilon = \frac{\delta}{6e\varepsilon}$, is analytic in $D_{\delta/3}$, $\|X_m\|_{D_{\delta/3}} \leq 2\varepsilon$ and*

$$\|\Phi_{X_m} - f\|_{D_0} \leq 3\varepsilon \left(\frac{6(m-1)\varepsilon}{\delta} \right)^m. \tag{4}$$

Moreover, for $m = \lfloor M_\varepsilon \rfloor + 1$

$$\|\Phi_{X_\mathrm{m}} - f\|_{D_0} \leq 3\,\varepsilon \exp\left(-\delta/6\mathrm{e}\varepsilon\right). \tag{5}$$

Proof. We consider the map f as a member of the family

$$f_\mu = (1 - \mu)\xi + \mu f$$

where μ is a complex parameter. Obviously the function f_μ is analytic in the same domain D_δ as the function f. Then $|\Delta_1(x)| = |f_\mu(x) - x| \leq |\mu|\varepsilon$ for any $x \in D_\delta$ and any μ. Let $\mu_1 = \delta/\varepsilon$ and

$$\mu_m = \frac{2\delta}{3\varepsilon(m-1)} \qquad \text{for } m \geq 1\ .$$

If $|\mu| \le \mu_m$, then for every $x_0 \in D_{\delta/3}$ the first iterates $x_k := f_\mu^k(x_0) \in D_\delta$ and $|x_{k+1} - x_k| \le |\mu|\varepsilon$ provided $0 \le k \le m-1$. The definition (3) implies that

$$\Delta_k(x_0) = \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{k-1}{j} \Delta_1(x_j).$$

Since $\sum_{j=0}^{k} \binom{k}{j} = 2^k$ we get

$$\|\Delta_k\|_{D_{\delta/3}} \le 2^{k-1} |\mu| \varepsilon \, .$$

Since $|\Delta_k(x)| \le \|\Delta'_{k-1}\| \, |\mu|\varepsilon$ where the supremum norm is taken over $|\mu| < \mu_k$ and $|x - x_0| < \mu_k \varepsilon$, we can check that $\Delta_k(x_0) = O(\mu^k)$. Applying the maximum modulus principle (MMP)[1] in μ to each component of $\Delta_k(x_0)$, we get

$$\|\Delta_k\|_{D_{\delta/3}} \le 2^{k-1} \mu_k \varepsilon \left(\frac{|\mu|}{\mu_k} \right)^k \, .$$

Let $X_{m,\mu}$ be defined by (2) with f replaced by f_μ. Then $X_{m,\mu}$ is analytic in $D_{\delta/3}$ for $|\mu| \le \mu_m$ and admits the following upper bound

$$\|X_{m,\mu}\|_{D_{\delta/3}} \le \sum_{k=1}^{m} \frac{1}{k} \|\Delta_k\|_{D_{\delta/2}} \le \frac{\varepsilon \mu_m}{2} \sum_{k=1}^{m} \left(\frac{2|\mu|}{\mu_m} \right)^k \le \frac{\varepsilon |\mu|}{1 - \frac{2|\mu|}{\mu_m}} \, .$$

Then we get that for $|\mu| \le \mu_m/4$

$$\|X_{m,\mu}\|_{D_{\delta/3}} \le 2\varepsilon |\mu| \, .$$

For our range of m we have $\mu_m \ge 4$. Then the domain of validity of the upper bound includes $\mu = 1$ and we get

$$\|X_m\|_{D_{\delta/2}} \le 2\varepsilon.$$

We also get that for $|\mu| \le \mu_m/4$ and $m \ge 2$

$$\|X_{m,\mu}\|_{D_{\delta/3}} \le \frac{\varepsilon \mu_m}{2} = \frac{\delta}{3(m-1)} \le \frac{\delta}{3} \, .$$

Then the orbit of the vector field $X_{m,\mu}$ with an initial condition in D_0 remains in $D_{\delta/3}$ during one unit of time and

$$\|\Phi_{X_{m,\mu}} - \xi\|_{D_0} \le \|X_{m,\mu}\|_{D_{\delta/3}} \le 2\varepsilon |\mu| \, .$$

[1] We use the following simple statement of Complex Analysis: if a function g is an analytic function of μ bounded in an open disk $|\mu| < r$ and $g^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m$, then the maximum modulus principle implies that $|g(\mu)| \le (|\mu|/r)^m \sup_{|\mu|<r} |g(\mu)|$. Of course, if the function extends continuously onto the boundary of the disk, the supremum can be replaced by the maximum over $|\mu| = r$.

In order to apply arguments based on the MMP, we need to check that $\Phi_{X_{m,\mu}}$ has the same Taylor polynomial of degree m in μ as the map f_μ. Proofs of similar claims can be found in [1, 2]. First we define an auxiliary vector field

$$Y_{m,\mu}(x) = \sum_{k=1}^{m} \mu^k a_k(x)$$

where $a_1 = f - \xi$ and a_k with $k \geq 2$ are defined recursively by

$$a_k = -\sum_{j=2}^{k} \frac{1}{j!} \sum_{i_1+\dots+i_j=k} L_{a_{i_1}} \dots L_{a_{i_j}} \xi$$

where differential operators $L_a g = a \cdot \nabla g$ act on a vector valued function g component-wise. Expanding the time-t map $\Phi^t_{Y_{m,\mu}}$ in Taylor series in t we get

$$\Phi^t_{Y_{m,\mu}} = \xi + tY_{m,\mu} + \sum_{k=2}^{m} \frac{t^k}{k!} L^k_{Y_{m,\mu}} \xi + O((t\mu)^{m+1}). \tag{6}$$

Our choice of a_k implies that the terms of order μ^k cancel each other for $k = 2, \dots, m$ when $t = 1$:

$$\Phi^1_{Y_{m,\mu}} = \xi + Y_{m,\mu} + \sum_{k=2}^{m} \frac{1}{k!} L^k_{Y_{m,\mu}} \xi + O(\mu^{m+1}) = \xi + \mu a_1 + O(\mu^{m+1}) = f_\mu + O(\mu^{m+1}).$$

Iterating the map we get that $\Phi^k_{Y_{m,\mu}} = f^k_\mu + O(\mu^{m+1})$. Using the equation (2) with f replaced by $\Phi^1_{Y_{m,\mu}}$ we obtain a vector field $\hat{X}_{m,\mu} = X_{m,\mu} + O(\mu^{m+1})$. We note that $\hat{X}_{m,\mu}$ is the derivative at $t = 0$ of the Newton interpolating polynomial of degree m defined by the points $\Phi^t_{Y_{m,\mu}}$ with $t = 0, 1, \dots, m$. Since the interpolation is exact on polynomials of degree m, the equation (6) implies that $\hat{X}_{m,\mu} = Y_{m,\mu} + O(\mu^{m+1})$. Combining these two estimates we get that $X_{m,\mu} = Y_{m,\mu} + O(\mu^{m+1})$, i.e., $Y_{m,\mu}$ is the Taylor polynomial of degree m in μ for the vector field $X_{m,\mu}$. Since the time-one map of a vector field depends smoothly on the vector field we conclude that

$$\Phi^1_{X_{m,\mu}} = \Phi^1_{Y_{m,\mu}} + O(\mu^{m+1}) = f_\mu + O(\mu^{m+1}).$$

Therefore the Taylor expansion in μ of $\Phi_{X_{m,\mu}}$ matches the Taylor expansion of f_μ up to the order m.

Since $\|\xi - f_\mu\|_{D_0} = |\mu| \, \|\xi - f\|_{D_0} \leq |\mu|\varepsilon$, we get that

$$\|\Phi_{X_{m,\mu}} - f_\mu\|_{D_0} \leq \|\Phi_{X_{m,\mu}} - \xi\|_{D_0} + \|\xi - f_\mu\|_{D_0} \leq 3\varepsilon|\mu|.$$

The MMP based on the bound in the disk $|\mu| \leq \mu_m/4$ can be applied with $\mu = 1$ to get the desired estimate

$$\|\Phi_{X_m} - f\|_{D_0} \leq 3\varepsilon \left(\frac{4}{\mu_m}\right)^m = 3\varepsilon \left(\frac{6\varepsilon(m-1)}{\delta}\right)^m.$$

The right-hand side depends on m and takes the smallest values near M_ε. There is a unique integer $m \in [M_\varepsilon, M_\varepsilon + 1)$. Then, for this m,

$$\frac{\mu_m}{4} = \frac{\delta}{6\varepsilon(m-1)} \geq \frac{\delta}{6\varepsilon M_\varepsilon} = \mathrm{e} > 1.$$

In particular, it satisfies the assumption used in the proof, and we can conclude that

$$\|\Phi_{X_m} - f\|_{D_0} \leq 3\varepsilon\, \mathrm{e}^{-M_\varepsilon} = 3\,\varepsilon \exp\left(-\frac{\delta}{6\mathrm{e}\varepsilon}\right).$$

Theorem is proved. □

Remark 2. *For the sake of completeness we present the bounds for the case of $m = 1$ separately. The interpolating vector field is given by $X_1(x) = f(x) - x$ and*

$$\|\Phi_{X_1} - f\|_{D_0} \leq \frac{2\varepsilon^2}{\delta}.$$

Proof. In order to check this bound we can consider $X_{1,\mu} = f_\mu - \xi = \mu(f - \xi)$. Obviously,

$$\|X_{1,\mu}\|_{D_\delta} = \|\mu(f_\mu - \xi)\|_{D_\delta} = |\mu|\varepsilon.$$

Then $\|\Phi_{X_{1,\mu}} - f_\mu\|_{D_0} \leq 2|\mu|\varepsilon$ provided $|\mu|\varepsilon < \delta$. Since $\Phi_{X_{1,\mu}} - f_\mu = O(\mu^2)$, the MMP implies the desired bound

$$\|\Phi_{X_1} - f\|_{D_0} \leq \frac{2\varepsilon\mu_0}{\mu_0^2} = \frac{2\varepsilon^2}{\delta}$$

where $\mu_0 = \delta/\varepsilon$. □

The error bounds of Theorem 1 can be improved by implementing a symmetric interpolation scheme instead of the Newton one, in a way similar to [1]. We also note that in the case of a symplectic map f, the interpolating vector field (2) is typically not Hamiltonian. On the other hand, it can be shown to be a small perturbation of a Hamiltonian vector field [1, 2], with the size of the perturbation being comparable with the approximation errors of Theorem 1.

Acknowledgements

A.V. is supported by the Spanish grant PID2021-125535NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU. He also acknowledges the Catalan grant 2021-SGR-01072 and the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-0010 84-M). VG's research is supported by the Mathematics Institute of the University of Warwick.

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