Dynamics of the QR-flow for upper Hessenberg real matrices

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Abstract

We investigate the main phase space properties of the QR-flow when restricted to upper Hessenberg matrices. A complete description of the linear behavior of the equilibrium matrices is given. The main result classifies the possible α- and ω-limits of the orbits for this system. Furthermore, we characterize the set of initial matrices for which there is convergence towards an equilibrium matrix. Several numerical examples show the different limit behavior of the orbits and illustrate the theory.

Contents

1 Introduction

One of the basic problems in numerical linear algebra is the computation of eigenvalues and eigenvectors of a matrix. To this end, one uses iterative methods and, among them, the methods based on QR-iteration are usually effective [15]. As an alternative, there are methods based on the solution of a Cauchy problem of an ordinary differential equation in the space of n-dimensional matrices. We refer to the review [6] for a general description of this type of methods. Here, we study the so called QR-flow which can be seen as the continuous analogous of the QR-iteration.

The QR-flow for symmetric Jacobi (real tridiagonal) matrices X was first derived in the context of the Toda lattice [29] for a finite number of mass points. In Flaschka variables [13] the equations of motion can be rewritten in Lax form

$$X' = [X, k(X)] = Xk(X) - k(X)X, \tag{1}$$

where $k(X) = X^- - (X^-)^\top$ is a skew-symmetric projection of X (where X^- is the strictly lower triangular part of X). The Lax pair structure implies the following properties:

1. The eigenvalues of the linear operator X are integrals of motion of (1), that is, the flow $X(t)$ is isospectral, see [13]. In particular, the coefficients of the characteristic polynomial of X are also first integrals.

2. Given a real n-dimensional matrix A, the (unshifted) QR-iteration algorithm applied to $X_0 = \exp(A)$ is the evaluation at integer times of $X(t)$ (see [28, 11]). For this reason, the flow is usually referred as the QR-flow.

For the Toda lattice, the coefficients of the characteristic polynomial of X are independent first integrals in involution, see [19, 13]. Moreover, the solution $X(t)$ converges to a diagonal matrix, see [23]. The last convergence property also holds whenever the QR-flow is considered on the set of symmetric matrices, see [24]. We refer to [30, 22] for further details on the Toda lattice and its generalizations.

The goal of this paper is to analize the dynamics of the QR-flow when applied to upper Hessenberg general matrices. This flow interpolates (in a proper way) the QR-iterates of the exponential of the initial matrix. The main result gives a detailed description of the structure of the α, ω-limit sets for any initial n-dimensional upper Hessenberg matrix.

We recall (see for example [17]) that, given an ordinary differential equation on $\mathbb{R}^n$ such that defines a complete flow φ (that is, a flow defined for all $t \in \mathbb{R}$), the α, ω-limit sets of a point $p \in \mathbb{R}^n$ are

$$\alpha(p) = \{q \in \mathbb{R}^n;\ \exists (t_n) \text{ such that } t_n \to -\infty \text{ and } \varphi(t_n, p) \to q \text{ as } n \to \infty\},$$
$$\omega(p) = \{q \in \mathbb{R}^n;\ \exists (t_n) \text{ such that } t_n \to \infty \text{ and } \varphi(t_n, p) \to q \text{ as } n \to \infty\}.$$

One has $\omega(\varphi(t,p)) = \omega(p)$ and $\alpha(\varphi(t,p)) = \alpha(p)$ for all $t \in \mathbb{R}$. As the orbits of the QR-flow are bounded, see Section 2, these sets are non-empty, compact, invariant by the flow and connected. A point q is homoclinic to an equilibrium point p if $\alpha(q) = \omega(q) = \{p\}$. If, on the other hand, if there exist $p_1 \neq p_2$ equilibria such that $\alpha(q) = \{p_1\}$ and $\omega(q) = \{p_2\}$ then we say that the orbit of q is heteroclinic to p_1 and p_2.

We emphasize in this sense that the QR-flow is one of the exceptional cases in which such a description of the α, ω-limit sets can be obtained. As discussed in this work, this fact is indeed a consequence of the concrete Lax pair formulation that admits the QR-flow, see Prop.2.2. Other results presented in this work and that are required to obtain the main result or that follow from it include:

- A complete list of the equilibrium matrices is given.
- The behavior of the linearization at the equilibrium matrices is described.
- A characterization of the set of matrices for which the QR-flow converges to a limit matrix, that is, of the set matrices that are homo/heteroclinic to equilibria.
- The structure of the α- and ω-limits, including equilibrium matrices, periodic orbits and/or invariant tori.
- Estimates on the rate of convergence towards the α- and ω-limits are explicitly derived.
- Several paradigmatic examples of the different theoretical situations are given.

We believe that these results can be potentially useful to design adapted algorithms for computing the spectrum of a general matrix (we recall that any matrix can be reduced to an upper Hessenberg, see for instance [15]). In this sense we remark that

- We show that a real Schur normal form for any initial upper Hessenberg matrix can be computed (but clearly not in an efficient way!) by combining the QR-flow with the QR-iteration without any shift strategy.
- Any effective implementation of the QR-iteration requires shifts strategies. Although in the flow setting such strategies are useless, the QR-flow can be numerically integrated using a variable stepsize integrator. If the numerical integrator is chosen properly, the timesteps can be larger than one. Since the time-one map of the QR-flow corresponds to QR-iterates of the exponential of the matrix, the previous property can be though as performing more than one QR-iterate per integration step.
- In general, continuous realization methods can have some advantages in front their discrete counterpart, as was pointed out in [5]. In this direction, we would like to add that the variational equations can be used as a tool to study the smooth dependence of solutions on parameters. For the QR-flow setting, this means that one can obtain (in a straightforward way) Taylor expansion of the eigenvalues for matrices depending on parameters. This can be useful for bifurcation analysis.

The QR-flow is a particular example of an isospectral flow, that is, a flow such that all the points of an orbit are similar matrices. In the case of the QR-flow, the points of an orbit are orthogonally similar matrices. Due to the simple structure of the QR-flow equations we have used a high-order Taylor method for numerical integration of the ODE. We are aware of adapted methods for isospectral flows [2, 3] which can be very convenient in this setting. However, we stress that by keeping the local error below the machine accuracy, the Taylor integrator respects isospectrality up to numerical errors (but nothing prevents from a bias for very long times of integration). We refer to [21] for analogous comments in the symplectic setting.

The main result of this paper is inspired by the analogous Parlett's result in [26] for the QR-iteration. For reader's convenience we briefly recall the Parlett's result. Such a result describes the form to what tend the QR-iterates $(H^{(s)})_{s\in\mathbb{N}}$ of an initial unreduced upper Hessenberg matrix $H^{(1)}$. Concretely,

Theorem 1.1. *(Parlett'68) Let $w_1 > \dots w_r > 0$ the distinct modulus of the eigenvalues of $H^{(1)}$. Of the eigenvalues of modulus w_i, let $p(i)$ have even multiplicities $m_1^i \geq m_2^i \geq \cdots \geq m_{p(i)}^i > m_{p(i)+1}^i = 0$, and let $q(i)$ have odd multiplicities $n_1^i \geq n_2^i \geq \cdots \geq n_{q(i)}^i > n_{q(i)+1}^i = 0$. If $0 \notin \operatorname{Spec}(H^{(1)})$ then, as $s \to \infty$, $H^{(s)}$ becomes block triangular with blocks $H_{i,j}^{(s)}$, $i \leq j$. Moreover, $\operatorname{Spec}(H_{i,i}^{(s)})$ converges to the set of eigenvalues with modulus ω_i. Each $H_{i,i}^{(s)}$ tends to a block triangular substructure in which emerge $m_j^i - m_{j+1}^i$ unreduced diagonal blocks of dimension j for $1 \leq j \leq p(i)$, and $n_j^i - n_{j+1}^i$ unreduced diagonal blocks of dimension j for $1 \leq j \leq q(i)$. The union of the spectra of these blocks converges to the eigenvalues of even and odd multiplicity respectively. If, on the other hand, $0 \in \operatorname{Spec}(H^{(1)})$ and has multiplicity m, the last m columns and m rows are discarded from $H^{(s)}$ and the previous statement holds.*

The paper is organized as follows. In Section 2 we present the equations of the QR-flow for upper Hessenberg matrices and we state some basic properties. In Section 3 we obtain the equilibrium matrices for the QR-flow and we describe their linear behavior (that is, the behavior of the linearization of the QR-flow at the equilibrium matrix). In particular, a complete description of the spectrum of the linearised system at the equilibrium matrices is given. Some technical lemmas used to this end are given in Appendix A. Next, in Section 4, we discuss about the structure of the elements of the ω-limit set of any initial matrix. As said, the main result can be seen as the analogous of Parlett's classical result in [26] but in the flow setting. Moreover, for the QR-flow a detailed description of the ω-limit set, making explicit the order in which the diagonal blocks appear, can be obtained: given $X_0 \in \mathbf{H}_n$ which is not an equilibrium matrix, either $\omega(X_0)$ is an equilibrium matrix or it does not contain equilibrium matrices. In the latter case, the orbit of an element $Y \in \omega(X_0)$ is a (multi-)periodic orbit on a torus of suitable dimension. A complete proof is given in Section 4, with the help of some technical lemmas included in Appendices B and C. As a consequence of the previous result we provide in Sections 4.1 and 4.2 the convergence results in the Wilkinson and/or Parlett sense, useful for the numerical computation of eigenvalues using the QR-flow. Also, the velocity of convergence towards the ω-limit behavior is explicitly obtained. In Section 4.3 we characterize the set of initial matrices for which there is convergence of the QR-flow to an equilibrium matrix. In particular, the orthogonal, normal and symmetric matrices are included in this set, see Section 4.3.1.

In Section 5 we consider the restriction of the QR-flow to dimensions 2,3 and 4, for which further details on the dynamics of the ω-limit can be explicitly given. In Section 6 we illustrate the complexity of the phase space in a simple 3-dimensional example. Finally, in Section 7 we provide some conclusions and possible future research directions.

The notation in Table 1 will be used through the text. On the other hand, we shall use the Diag and diag operators through the text: if $A_i \in \mathcal{M}_{n_i,n_i}$, $1 \leq i \leq m$, then

$$\operatorname{Diag}(A_1, \dots A_m) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} \in \mathbf{BD}^n_{n_1,\dots,n_m}$$

and, if $A \in \mathbf{BUT}^n_{n_1,\dots,n_m} \cup \mathbf{BLT}^n_{n_1,\dots,n_m}$, then $\operatorname{diag}(A) = (A_{1,1}, \dots, A_{m,m}) \in \mathcal{M}_{n_1,n_1} \times \cdots \times \mathcal{M}_{n_m,n_m}$.

$\mathcal{M}_{n,m}$	Set of $n \times m$ real matrices.
I_n	Identity matrix of dimension n
Real matrices subsets:	
$\mathbf{H}_n$	Set of n-dimensional upper Hessenberg matrices
$\mathbf{H}_n^\star$	Set of n-dimensional unreduced upper Hessenberg matrices
$\mathbf{T}_n$	Set of n-dimensional upper triangular matrices
$\mathbf{D}_n$	Set of n-dimensional diagonal matrices
$\mathbf{O}_n$	Set of n-dimensional orthogonal matrices
$\mathbf{Sym}_n$	Set of n-dimensional symmetric matrices
$\mathbf{Skew}_n$	Set of n-dimensional skew-symmetric matrices
Block real matrices subsets:	
$\mathbf{BUT}^n_{n_1,\ldots,n_m}$	Set of n-dimensional block upper triangular matrices with diagonal blocks belonging to $\mathcal{M}_{n_i,n_i}$, $i = 1, \ldots, m$, $n = n_1 + \cdots + n_m$.
$\mathbf{BD}^n_{n_1,\ldots,n_m}$	Set of n-dimensional block diagonal matrices with diagonal blocks belonging to $\mathcal{M}_{n_i,n_i}$, $i = 1, \ldots, m$, $n = n_1 + \cdots + n_m$.
$\mathbf{BLT}^n_{n_1,\ldots,n_m}$	Set of n-dimensional block lower triangular matrices with diagonal blocks belonging to $\mathcal{M}_{n_i,n_i}$, $i = 1, \ldots, m$, $n = n_1 + \cdots + n_m$.

Table 1: Notation for the matrices sets used through the text.

2 The QR-flow on $\mathbf{H}_n$ as an isospectral flow

It is well-known, see for example [6] and references therein, that given a smooth map $k : \mathcal{M}_{n,n} \longrightarrow \mathcal{M}_{n,n}$ the solution of the Cauchy problem

$$X' = [X, k(X)], \qquad X(0) = X_0, \tag{2}$$

is $X(t) = G(t)^{-1}X_0G(t)$, where $G(t)$ is the solution of the Cauchy problem $G' = Gk(X(t))$, $G(0) = I_n$. Hence the flow defined by (2) is an isospectral flow. On the other hand, if we consider the Cauchy problem

$$\begin{aligned} X' &= [X, k(X)], & X(0) &= X_0, \\ G' &= Gk(X), & G(0) &= I_n, \end{aligned}$$

then, as before, $X(t) = G(t)^{-1}X_0G(t)$, and $G(t)$ is orthogonal if, and only if, $k(X) \in \mathbf{Skew}_n$ for all $X \in \mathcal{M}_{n,n}$. Note that in this case the solution of (2) satisfies $\|X(t)\|_2 = \|X_0\|_2$ and then the flow is complete (defined for all $t \in \mathbb{R}$).

The QR-flow is obtained by considering $k(X) = X^- - (X^-)^\top \in \mathbf{Skew}_n$, being X^- the strictly lower triangular part of X, in (2) above. The following proposition, see [7, 6], relates the QR-flow at integer times of $X_0 \in \mathcal{M}_{n,n}$ with the QR-iteration of $\exp(X_0)$ and associates to the isospectral flow suitable Cauchy problems with solution the Q and R matrices of the QR-factorization. Note that the QR-factorizations considered here, and from now on, have $\mathrm{diag}(R)$ with positive elements. This guarantees uniqueness of the QR-factorization provided the matrix to be factorized is non-singular.

Proposition 2.1. *Given $X_0 \in \mathcal{M}_{n,n}$ denote by $X(t)$ the solution of the Cauchy problem $X' = [X, k(X)],\ X(0) = X_0$. Define $Q(t)$ and $R(t)$ as the solutions of the Cauchy problems*

$$Q' = Q\,k(X(t)), \quad Q(0) = I, \qquad \text{and} \qquad R' = k_c(X(t))R, \quad R(0) = I, \tag{3}$$

respectively, where $k_c(X) = X - k(X)$. Then, for all $t \in \mathbb{R}$, the following four properties hold:

1. $e^{tX_0} = Q(t)R(t)$,
2. $Q(t) \in \mathbf{O}_n$ *and* $R(t) \in \mathbf{T}_n$,
3. $X(t) = Q(t)^\top X_0 Q(t) = R(t)X_0R(t)^{-1}$,
4. $e^{tX(t)} = R(t)Q(t)$.

Proof: One has $Q(t) \in \mathbf{O}_n$ and $X(t) = Q(t)^\top X_0 Q(t)$. First we will prove that R is upper triangular and $X(t) = R(t)X_0R(t)^{-1}$. The first statement is trivial, because $k_c(X)$ is upper triangular for any matrix X. To see the second one we define $C(t) = R(t)X_0R(t)^{-1}$. Then $C'(t) = [k_c(X(t)), C(t)],\ C(0) = X_0$. Moreover, $X'(t) = [X, k(X)] = [k_c(X(t)), X(t)],\ X(0) = X_0$, and by uniqueness of solutions of the Cauchy problem we get that $X(t) = C(t)$, for all $t \in \mathbb{R}$. With this we have proved 2. and 3. In order to prove 1., we define $S(t) = Q(t)^\top e^{tX_0}$. Then $S(0) = I$ and

$$S'(t) = (Q(t)^\top)' e^{tX_0} + Q^\top X_0 e^{tX_0} = (-k(X(t)) + X(t))S(t) = k_c(X(t))S(t).$$

Again by uniqueness of the solutions we have that $S(t) = R(t)$, which proves 1. Finally, 4. follows from $R(t)Q(t) = Q(t)^\top e^{tX_0} Q(t) = e^{tQ(t)^\top X_0 Q(t)} = e^{tX(t)}$. $\square$

As said, the last proposition implies that $e^{X(n)}$, $n \geq 1$, are the iterates of the QR-iteration applied to the matrix e^{X_0}. Precisely, this means that $A_1 = e^{X(0)} = Q_1R_1$, $A_2 = R_1Q_1 = e^{X(1)} = Q_2R_2$, $A_3 = R_2Q_2 = e^{X(2)} = Q_3R_3$, *etc*, hence the QR-iterates of $e^{X(0)}$ are given by A_k. This was first observed in [27]. Note that, in general, $Q(k) \neq Q_k$ and $R(k) \neq R_k$, $k \in \mathbb{N}$.

It is immediate to see that the set $\mathbf{H}_n$ is invariant by the QR-flow. Indeed, if $X_0 \in \mathbf{H}_n$, by Prop. 2.1, we know that $X(t) = R(t)X_0R(t)^{-1}$. As $R(t)$ is upper triangular, it follows that $X(t) \in \mathbf{H}_n$ for all $t \in \mathbb{R}$. From now on, we shall consider the QR-flow restricted to $\mathbf{H}_n$. In this case, the equation $X' = [X, k(X)]$ defines a dynamical system on $\mathbb{R}^{N_d}$, where $N_d = (n^2+3n-2)/2$ is the dimension of the set of upper Hessenberg matrices.

Remark 2.1. It is interesting to comment about the connection of our results and the results on integrability of the QR-flow. In [10] it is proved the integrability of the QR-flow on the open set $\mathcal{G}$ of matrices $M \in \mathcal{M}_{n,n}$ satisfying the generic assumptions

1. if $(M)_k \in \mathcal{M}_{n-k,n-k}$ denotes the matrix obtained by deleting the first k rows and the last k columns of M, then one assumes, for $1 \leq k \leq [n/2]$, that $P_k(M, \lambda) = \det(M - \lambda I_n)_k$ is a polynomial in λ of degree $n - 2k$ (that is, the leading coefficient does not vanish),
2. the matrix $(M - M^\top)/2 \in \mathbf{Skew}_n$ has a simple spectrum,

3. the roots of $P_k(M,\lambda)$ for $0 \le k \le [(n-1)/2]$ are distinct, and
4. zero is a regular value of the function $J : \mathbb{C}^2 \to \mathbb{C}$ given by $J(h,z) = \det((1-h)M + hM^\top - zI_n)$.

The authors construct action-angle variables and derive the integrability of the QR-flow on $\mathcal{G}$. As a consequence of this construction it was proved that the orbit of $M \in \mathcal{G}$ under the QR-flow is contained into an invariant connected set diffeomorphic to $\mathbb{R}^l \times \mathbb{T}^c$ for suitable dimensions l and c. Note that:

- As already pointed out in [10] matrices in $\mathbf{H}_n$ are not generic in the sense above but integrability on a suitable open set $\hat{\mathcal{G}}$ containing $\mathbf{H}_n$ could be proved by adapting the previous result, see related comments in [9]. For generic matrices $H \in \mathbf{H}_n$ the invariant set containing its orbit must be diffeomorphic to $\mathbb{R}^l \times \mathbb{T}^c$ being c the number of pairs of complex eigenvalues of H.
- For matrices $H \in \mathbf{H}_n$ (not necessarily in $\hat{\mathcal{G}}$) having at least a pair of complex eigenvalues $\omega(H)$ can be contained into $W \cong \mathbb{T}^c$. For example in Figs. 5 and 6 we consider matrices $H_0, H_1 \in \mathbf{H}_4$ such that $\omega(H_0) \cong \mathbb{T}^2$ while $\omega(H_1)$ is a periodic orbit contained in $W \cong \mathbb{T}^2$, in agreement with the statement of the previous item. But there are also matrices H having at least a pair of complex eigenvalues such that $\omega(H)$ is an equilibrium matrix, see for example cases 5, 6 and 7 in Section 5.3 for $H \in \mathbf{H}_4$. This situation happens when H is on the stable invariant manifold of an equilibrium matrix which is neither upper triangular nor skew-symmetric, see Section 4.3.

3 Equilibrium matrices

In this section we characterize which upper Hessenberg matrices are equilibria of the QR-flow. To this end, it is important to introduce for $X \in \mathcal{M}_{n,n}$ and $Y \in \mathcal{M}_{m,m}$ the operator $\mathcal{B}_{X,Y} : \mathcal{M}_{n,m} \to \mathcal{M}_{n,m}$ defined by

$$\mathcal{B}_{X,Y}(Z) = XZ - ZY. \tag{4}$$

We also define, for $n = m$ and $X = Y$, $L_X = \mathcal{B}_{X,X}$. Then $L_X(Z) = [X, Z]$. We summarize in Appendix A some properties of these linear operators.

A matrix $X \in \mathbf{H}_n$ is an equilibrium matrix of the QR-flow if $[X, k(X)] = 0$. Given $X \in \mathbf{H}_n$ there is $m \in \mathbb{N}$ and there are (unique) $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$X = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ & \ddots & \vdots \\ & & A_{m,m} \end{pmatrix} \in \mathbf{BUT}^n_{n_1,\ldots,n_m}, \tag{5}$$

where $A_{i,i} \in \mathbf{H}^\star_{n_i}$. With this notation we have the following result.

Theorem 3.1. *A matrix $X \in \mathbf{H}_n$ is an equilibrium matrix of the QR-flow if, and only if, for $i = 1, \ldots, m$, the block $A_{i,i}$ in (5) is of the form $A_{i,i} = \lambda_i I_{n_i} + k(A_{i,i})$ and $\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}(A_{i,j}) = 0$ for all $1 \le i < j \le m$.*

Proof: One has $[X, k(X)] = 0$ if, and only if, $[A_{i,i}, k(A_{i,i})] = 0$ for all i and $B_{k(A_{i,i}),k(A_{j,j})} = 0$ for all $i < j$. Since $k(A_{i,i}) \in \mathbf{Skew}_{n_i} \cap \mathbf{H}^\star_{n_i}$ has simple eigenvalues then $A_{i,i}$ can be expressed as a polynomial of degree $n_i - 1$ in $k(A_{i,i})$ (see for example Theorem 3.1 in [32]). But $A_{i,i} \in \mathbf{H}^*_n$ and one infers sequentially for $s = n_i - 1, n_i - 2, \ldots, 2$ that the coefficient associated to the monomial $k(A_{i,i})^s$ vanishes. Hence $A_{i,i} = \lambda_i I_{n_i} + \mu_i k(A_{i,i})$, for suitable coefficients $\mu_i, \lambda_i \in \mathbb{R}$. But $k(A_{i,i}) = \lambda_i k(I_{n_i}) + \mu_i k^2(A_{i,i}) = \mu_i k(A_{i,i})$ and one gets $\mu_i = 1$.

Reciprocally, if $A_{i,i} = \lambda_i I_{n_i} + k(A_{i,i})$ then $[A_{i,i}, k(A_{i,i})] = 0$ for all $1 \le i \le m$, and $X \in \mathbf{H}_n$, given by (5), is an equilibrium matrix. □

Remark 3.1. 1. Every fixed point $X \in \mathbf{H}_n$ is the sum $X = A + R$, of a matrix $A \in \mathbf{Skew}_n \cap \mathbf{H}_n$ and a matrix $R \in \mathbf{T}_n$ such that they commute. This follows since $[X, k(X)] = [R, A]$ which vanishes if they commute.

2. The set of equilibria of the QR-flow contains $\mathbf{T}_n$ and the set

$$\mathbf{Z}_n = \{A \in \mathbf{H}_n,\ A = \lambda I_n + B,\ B \in \mathbf{Skew}_n,\ \lambda \in \mathbb{R}\}.$$

More general, if one considers $X \in \mathbf{Skew}_n \cap \mathbf{H}_n$ as a matrix of $\mathbf{BUT}^n_{n_1,\ldots,n_m}$ with $A_{i,i} \in \mathbf{H}^\star_{n_i}$ and given $D = \mathrm{Diag}(D_{1,1}, \ldots, D_{m,m}) \in \mathbf{BD}^n_{n_1,\ldots,n_m}$ with $D_{i,i} = \lambda I_{n_i}$ then $X + D$ is an equilibrium matrix.

3. Under the hypotheses of the Theorem 3.1, if X is an equilibrium matrix then, for any $Q \in \mathbf{BD}^n_{n_1,\ldots,n_m} \cap \mathbf{O}_n$ the matrix $Q^\top X Q$ is an equilibrium matrix.

4. Given $1 \le i < j \le m$ such that $\mathrm{Spec}\,(k(A_{i,i})) \cap \mathrm{Spec}\,(k(A_{j,j})) = \emptyset$ then $A_{i,j} = 0$. This is a consequence of the fact that $A_{i,j} \in \mathrm{Ker}\,(\mathcal{B}_{k(A_{i,i}),k(A_{j,j})})$ and that $\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$ is an invertible operator, see Prop. A.1. Consequently, if this is true for all $1 \le i < j \le m$ then X is normal (i.e. $X^\top X = X X^\top$).

5. Only for $n = 2$ the set of equilibria is equal to $\mathbf{T}_n \cup \mathbf{Z}_n$. For $n = 3$ if an equilibrium matrix is not upper triangular then it is normal. For $n \ge 4$ there are equilibrium matrices which are neither upper triangular nor normal, see Section 5.3.

3.1 Eigenvalues of equilibria

The goal of this section is to study the linear behavior at an equilibrium matrix $X \in \mathbf{H}_n$. We denote by $\mathcal{F} : \mathbf{H}_n \to \mathbf{H}_n$ the vector field $\mathcal{F}(X) = [X, k(X)]$. One has $\mathcal{F}(X + \epsilon Y) = \mathcal{F}(X) + \epsilon([X, k(Y)] + [Y, k(X)]) + \mathcal{O}(\epsilon^2)$ because $k(X + \epsilon Y) = k(X) + \epsilon k(Y)$ by linearity of k. Hence $D\mathcal{F}(X)Y = [X, k(Y)] + [Y, k(X)]$, for all $Y \in \mathbf{H}_n$.

Consider X as given by (5). The following theorem characterizes the spectrum of $D\mathcal{F}(X)$. In particular, we prove that all the hyperbolic directions of the equilibrium matrix X are associated to real eigenvalues of $D\mathcal{F}(X)$. That is the restriction of the dynamics to the unstable/stable invariant manifolds $W^{u/s}(X)$ is of repelling/attractor node type (i.e. no focus type hyperbolic components are present). On the other hand, all equilibrium matrices X have a non-trivial center manifold $W^c(X)$ of, at least, dimension $n - m + 1$.

Theorem 3.2. *With the previous notation, if $X \in \mathbf{H}_n$ is an equilibrium matrix one has*

$$\mathrm{Spec}\,(D\mathcal{F}(X)) = \mathrm{Spec}\,(D\mathcal{F}(k(X))) \cup \mathrm{Spec}\,\mathcal{B} \cup D_\lambda,$$

where

$$\mathrm{Spec}\,(D\mathcal{F}(k(X))) = \bigcup_{1 \le i \le m} \mathrm{Spec}\,(D\mathcal{F}(k(A_{i,i})), \quad \mathrm{Spec}\,\mathcal{B} = \bigcup_{1 \le i < j \le m} \mathrm{Spec}\,(\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}),$$

and

$$D_\lambda = \bigcup_{1 \le i \le m-1} \{\lambda_{i+1} - \lambda_i\}.$$

Moreover, all the non-zero real eigenvalues are contained in D_λ. Concretely:

1. *The eigenvalues of $\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$ are $\pm\mathrm{i}\,(\mu_1 \pm \mu_2)$ where $\pm\mathrm{i}\,\mu_1 \in \mathrm{Spec}\,(k(A_{i,i}))$ and $\pm\mathrm{i}\,\mu_2 \in \mathrm{Spec}\,(k(A_{j,j}))$.*

2. *Denote by* $\pm\mathrm{i}\,\mu_j$, $1 \le j \le r$, *with* $0 \le \mu_1 < \mu_2 < \cdots < \mu_r$, *the eigenvalues of* $k(A_{i,i})$, *where* $r = (n_i+1)/2$ *if* n_i *odd and* $r = n_i/2$ *otherwise. Then the non-zero eigenvalues of* $D\mathcal{F}(k(A_{i,i}))$ *are* $\pm\mathrm{i}\,(\mu_i - \mu_j)$, $1 \le i < j \le r$, *and* $\pm\mathrm{i}\,(\mu_i + \mu_j)$, $1 \le i \le j \le r$, *where we assume* $i + j \ne 2$ *whenever* n_i *odd.*

Proof: We consider the following subspaces of $\mathbf{H}_n$. Let $\mathbf{M}_1 \subset \mathbf{BUT}^n_{n_1,\ldots,n_m}$ be the subspace of strict upper block triangular matrices, $\mathbf{M}_2 = \mathbf{BD}^n_{n_1,\ldots,n_m}$ and $\mathbf{M}_3 \subset \mathbf{H}_n$ be the subspace of matrices B of the form $B_{i,j} = A_{i,j}$ if $i < j$, where $A \in \mathbf{M}_1$, and $B_{i,j} = -B_{j,i}$ if $i > j$, and $B_{i,i} = 0$. Note that $\mathbf{H}_n = \mathbf{M}_1 \oplus \mathbf{M}_2 \oplus \mathbf{M}_3$ and $D\mathcal{F}(X)(\mathbf{M}_1) \subset \mathbf{M}_1$, $D\mathcal{F}(X)(\mathbf{M}_2) \subset \mathbf{M}_1 \oplus \mathbf{M}_2$. This implies that

$$\mathrm{Spec}(D\mathcal{F}(X)) = \mathrm{Spec}(\mathcal{A}_1) \cup \mathrm{Spec}(\mathcal{A}_2) \cup \mathrm{Spec}(\mathcal{A}_3),$$

where $\mathcal{A}_1 = D\mathcal{F}(X)|\mathbf{M}_1$, $\mathcal{A}_2 = \Pi_2 \circ D\mathcal{F}(X)|\mathbf{M}_2$, $\mathcal{A}_3 = \Pi_3 \circ D\mathcal{F}(X)|\mathbf{M}_3$, and $\Pi_i : \mathbf{H}_n \to \mathbf{M}_i$ is the projection of $\mathbf{H}_n$ over $\mathbf{M}_i$, $i = 2,3$. Moreover, we have that

$$\mathbf{M}_1 = \bigoplus_{i=1}^{m} \bigoplus_{j=i+1}^{m} \mathbf{M}_1^{(i,j)}, \qquad \mathbf{M}_2 = \bigoplus_{i=1}^{m} \mathbf{M}_2^{(i)}, \qquad \mathbf{M}_3 = \bigoplus_{i=1}^{m-1} \mathbf{M}_3^{(i)}, \qquad \text{where}$$

$$\begin{aligned}
\mathbf{M}_1^{(i,j)} &= \{Y = (Y_{k\ell})_{1\le k,\ell\le m} \in \mathbf{M}_1 \,|\, Y_{k,\ell} = 0 \text{ for } (k,\ell) \ne (i,j)\},\\
\mathbf{M}_2^{(i)} &= \{Y = (Y_{k\ell})_{1\le k,\ell\le m} \in \mathbf{M}_2 \,|\, Y_{k,\ell} = 0 \text{ for } (k,\ell) \ne (i,i)\},\\
\mathbf{M}_3^{(i)} &= \{Y = (Y_{k\ell})_{1\le k,\ell\le m} \in \mathbf{M}_3 \,|\, Y_{k,\ell} = 0 \text{ for } k > \ell \text{ s.t. } (k,\ell) \ne (i+1,i)\}.
\end{aligned}$$

In order to finish the proof of the theorem, we will use the following lemma:

Lemma 3.1. *The operators* $\mathcal{A}_i$, $i = 1,2,3$ *satisfy the following properties:*

1. $\mathcal{A}_1(\mathbf{M}_1^{(i,j)}) \subset \mathbf{M}_1^{(i,j)}$, *and* $\mathcal{A}_1|\mathbf{M}_1^{(i,j)} = -\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$,

2. $\mathcal{A}_2(\mathbf{M}_2^{(i)}) \subset \mathbf{M}_1 \oplus \mathbf{M}_2^{(i)}$, *and* $\Pi_2 \circ \mathcal{A}_2|\mathbf{M}_2^{(i)} = D\mathcal{F}(A_{i,i})$

3. $\mathcal{A}_3(\mathbf{M}_3^{(i)}) \subset \mathbf{M}_1 \oplus \mathbf{M}_2 \oplus \mathbf{M}_3^{(i)}$ *and* $\Pi_3 \circ \mathcal{A}_3|\mathbf{M}_3^{(i)}(Y) = (\lambda_{i+1} - \lambda_i)Y$, *for all* $Y \in \mathbf{M}_3^{(i)}$.

Proof of Lemma 3.1: If $Y \in \mathbf{M}_1^{(i,j)}$ then $M = D\mathcal{F}(X)Y = [Y, K(X)] \in \mathbf{M}_1^{(i,j)}$, where $M_{i,j} = Y_{i,j}k(A_{j,j}) - k(A_{i,i})Y_{i,j} = -\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$. This proves 1.

To prove 2. first we define $D = \mathrm{Diag}(A_{1,1}, \ldots, A_{m,m}) \in \mathbf{BD}^n_{n_1,\ldots,n_m}$ Then $X = D + \widehat{X}$, where $\widehat{X} \in M_1$. If $Y \in \mathbf{M}_2^{(i)}$, then $D\mathcal{F}(X)Y = [D + \widehat{X}, k(Y)] + [Y, k(D)] = [D, k(Y)] + [Y, k(D)] + [\widehat{X}, k(Y)]$, where $[D, k(Y)] + [Y, k(D)] \in \mathbf{M}_2^{(i)}$ and $[\widehat{X}, k(Y)] \in \mathbf{M}_1$. Therefore, $\Pi_2 \circ \mathcal{A}_2|\mathbf{M}_2^{(i)} = D\mathcal{F}(A_{i,i})$, and $\mathcal{A}_2(\mathbf{M}_2^{(i)}) \subset \mathbf{M}_1 \oplus \mathbf{M}_2^{(i)}$.

Finally 3. follows because, if $Y \in \mathbf{M}_3^{(i)}$,

$$\begin{aligned}
D\mathcal{F}(X)Y =&[D + \widehat{X}, k(Y)] + [Y, k(D)] = [\widehat{X}, k(Y)] + [k_c(Y), k(D)] + [k_c(D), k(Y)]\\
=&[\widehat{X}, k(Y)] + [k_c(Y), k(D)] - [k_c(D), (Y^-)^\top] + [k_c(D), Y^-].
\end{aligned}$$

Then, $[\widehat{X}, k(Y)] + [k_c(Y), k(D)] - [k_c(D), (Y^-)^\top] \in \mathbf{M}_1 \oplus \mathbf{M}_2$ and $[k_c(D), Y^-] = Z_1 + Z_2$, where $Z_1 \in \mathbf{M}_1 \oplus \mathbf{M}_2$ and $Z_2 \in \mathbf{M}_3^{(i)}$, such that $(Z_2)_{i+1,i} = (\lambda_{i+1} - \lambda_i)Y_{i+1,i}$.

□

Following with the proof of the theorem, we have

$$\mathrm{Spec}(\mathcal{A}_1) = \bigcup_{i=1}^{m} \bigcup_{j=i+1}^{m} \mathrm{Spec}(\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}),$$

$$\mathrm{Spec}(\mathcal{A}_2) = \bigcup_{i=1}^{m} \mathrm{Spec}(D\mathcal{F}(A_{i,i})), \qquad \mathrm{Spec}(\mathcal{A}_3) = \bigcup_{i=1}^{m-1} \{\lambda_{i+1} - \lambda_i\}.$$

It remains to see that the eigenvalues of $\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$ and of $D\mathcal{F}(A_{i,i})$ are of the form $\gamma = \mathrm{i}\,\mu, \mu \in \mathbb{R}$. From Prop. A.1 in Appendix A, taking into account that $k(A_{i,i})$ are skew-symmetric, it follows that the eigenvalues of $\mathcal{B}_{k(A_{i,i}),k(A_{j,j})}$ are of this form. From Prop. 3.2 it follows the result about the spectrum of $D\mathcal{F}(A_{i,i})$, see below. □

Remark 3.2. In particular, it follows from the proof of Lemma 3.1 that if $X \in \mathbf{T}_n$ is an equilibrium matrix then $D\mathcal{F}(X)$ is diagonalizable. This was proved also in [4].

As a consequence of Theorem 3.2, in order to obtain all the eigenvalues of $D\mathcal{F}(X)$, we need to determine the eigenvalues of the blocks $A_{i,i} \in \mathbf{Skew}_n \cap \mathbf{H}_n^\star$. Hence, below we consider $X \in \mathbf{H}_n^\star$. First we begin with a proposition relating the eigenvalues and the eigenvectors of the operator L_X with those of $D\mathcal{F}(X)$.

Proposition 3.1. *Let $X \in \mathbf{H}_n^\star \cap \mathbf{Skew}_n$.*

1. *If $S \in \mathrm{Ker}\, L_X \cap \mathbf{Sym}_n$, $S \neq 0$, then $R = S - k(S) \in \mathbf{T}_n$, $R \neq 0$, is such that $D\mathcal{F}(X)R \in \mathbf{Skew}_n$ and $D\mathcal{F}(X)^2(R) = 0$. Moreover, $D\mathcal{F}(X)(R) = 0$, iff $S = R = \lambda I$.*
2. *If $S \in \mathbf{Sym}_n \oplus \mathrm{i}\,\mathbf{Sym}_n$ is an eigenvector of eigenvalue $\lambda \neq 0$ of L_X and $A = k(S)$, $R = S - k(S)$ then $A + \lambda^{-1}[X, A] + R$ is an eigenvector of eigenvalue $-\lambda$ of $D\mathcal{F}(X)$. Moreover, if $X \in \mathbf{H}_n$ then $A + \lambda^{-1}[X, A] + R \in \mathbf{H}_n$.*

Proof:

1. Recall that $D\mathcal{F}(X)R = [X, k(R)] + [R, k(X)] = [R, k(X)] = [R, X]$. Moreover, by hypothesis
$$L_X(S) = [X, S] = [X, k(S)] + [X, R] = 0.$$
This means that $B = [R, X] = [X, k(S)]$ is skew-symmetric. Finally,
$$D\mathcal{F}(X)B = [X, B] + [B, X] = 0.$$

   To prove the second part, notice that if $D\mathcal{F}(R) = [R, X] = 0$ then $[X, k(S)] = [X, S] = 0$.

   First we claim that if $[X, S] = (a_{ij})_{1 \le i,j \le n}$, $[X, k(S)] = (b_{ij})_{1 \le i,j \le n}$ and $a_{11} = b_{11} = 0$ and $a_{ij} = b_{ij} = 0$, for $2 \le i, j \le n$ then S is a diagonal matrix. Indeed, if $n = 2$ is obviously true. Suppose that it is true for $n - 1 \ge 2$ then we can write
$$S = \begin{pmatrix} s_{11} & c^\top \\ c & S_1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & -b^\top \\ b & X_1 \end{pmatrix},$$
where S_1 and X_1 are $(n-1) \times (n-1)$ matrices such that $S_1 = S_1^\top$ and $X_1^\top = -X_1$, and $b, c \in \mathbb{R}^{n-1}$, such that $b^\top = (\beta, 0, \ldots, 0)$, and $\beta \neq 0$. It is immediate to see that
$$k(S) = \begin{pmatrix} 0 & -c^\top \\ c & k(S_1) \end{pmatrix}.$$

Then

$$[X,S] = \begin{pmatrix} -2b^\top c & b^\top(s_{11}I - S_1) - c^\top X_1 \\ (s_{11}I - S_1)b + X_1 c & bc^\top + cb^\top + [X_1, S_1] \end{pmatrix},$$

$$[X,k(S)] = \begin{pmatrix} 0 & c^\top X_1 - b^\top k(S_1) \\ X_1 c - k(S_1)b & cb^\top - bc^\top + [X_1, k(S_1)] \end{pmatrix}.$$

If $c = (c_1, \dots, c_{n-1})$, as, by hypothesis, $b^\top c = 0$ then $c_1 = 0$ and the matrices $cb^\top \pm bc^\top$ have all their elements equal to zero except, perhaps, the ones of the first row and first column from the second element on. Then, by the induction hypothesis, we have that S_1 is a diagonal matrix. This implies that $bc^\top = cb^\top$, and, therefore, $c = 0$, which implies that S is also a diagonal matrix.

Finally, in order to finish the proof of the first part of the Prop. 3.1, we note that if $[X,S] = 0$ and S is diagonal, (this implies that $[X,k(S)] = [X,0] = 0$) then $S = \lambda I$.

2. Suppose that $[X,S] = \lambda S$, where $\lambda \neq 0$. If $A = k(S)$ and $R = S - k(S)$ we have

$$\begin{aligned} D\mathcal{F}(X)S &= [X, k(A+R)] + [A+R, k(X)] = [X,A] + [A,X] + [R,X] = [R,X] \\ &= [S,X] - [A,X] = -\lambda S - [A,X] = -\lambda A - [A,X] - \lambda R \\ &= -\lambda(A + \lambda^{-1}[A,X] + R). \end{aligned}$$

Then, if $H = A + \lambda^{-1}[A,X] + R$ we have that $D\mathcal{F}(X)H = -\lambda H$. Finally, if X is upper Hessenberg then $[R,X] + \lambda R = -\lambda(A + \lambda^{-1}[A,X])$, and as $[R,X]$ and R are upper Hessenberg, so is H. □

Proposition 3.2. *Let $X \in \mathbf{H}_n^\star \cap \mathbf{Skew}_n$. Then*

1. *The dimension of* $\operatorname{Ker} D\mathcal{F}(X)$ *is n. In particular* $\mathbf{Skew}_n \subset \operatorname{Ker} D\mathcal{F}(X)$.
2. *All the non-zero eigenvalues of $D\mathcal{F}(X)$ are pure imaginary and simple.*
3. *The dimension of the generalized eigenspace of eigenvalue zero is* $(3n-2)/2$ *if n is even and* $(3n-1)/2$ *if n is odd.*

Proof: Consider $Y \in \mathbf{H}_n$ such that $D\mathcal{F}(X)(Y) = 0$. We express $Y = R + A$, $R \in \mathbf{T}_n$, $A \in \mathbf{Skew}_n$. One has $D\mathcal{F}(X)(Y) = D\mathcal{F}(X)(R) = 0$. By Prop. 3.1 the matrix R is of the form $R = \lambda I_n$. This proves item 1.

By Prop. A.2 all the non-zero eigenvalues of $L_X|\mathbf{Sym}_n$ correspond to eigenvalues of $D\mathcal{F}(X)$ (note that there is a change of sign). Moreover, there are n^2 simple pure imaginary eigenvalues of $L_X|\mathbf{Sym}_n$ if n is even and n^2-1 otherwise. This proves that there are, at least, n^2 (or n^2-1 if n odd) eigenvalues of $D\mathcal{F}(X)$ different from zero.

Now, consider $Y \in \mathbf{H}_n$ such that $D\mathcal{F}(X)^2(Y) = 0$. Expressing $Y = R + A$, $R \in \mathbf{T}_n$, $A \in \mathbf{Skew}_n$, then $D\mathcal{F}(X)^2(Y) = D\mathcal{F}(X)^2(R) = 0$. By Prop. A.2 the dimension of the set of matrices $R \in \mathbf{T}_n$ such that $D\mathcal{F}(X)^2(R) = 0$ equals the dimension of $\operatorname{Ker} L_X \cap \mathbf{Sym}_n$ and this dimension is $n/2$ if n is even and $(n+1)/2$ otherwise. Adding $n-1$ (the dimension of $\mathbf{Skew}_n$) we obtain that the dimension is larger or equal than $(3n-2)/2$ if n even and $(3n-1)/2$ otherwise.

Finally, since the sum of the dimensions of all the eigenspaces equals the dimension of $\mathbf{H}_n$, there are no other matrices in the generalized eigenspace of eigenvalue zero of $D\mathcal{F}(X)$ and there are no more eigenvalues of $D\mathcal{F}(X)$ different from zero. □

Remark 3.3. The previous theorems provide a systematic way to proceed to determine the equilibrium matrices and its spectra. We shall give further details for dimensions $n \leq 4$ in Section 5 where a complete description is possible and helps to clarify the general procedure.

Example 3.1. We consider

$$X = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 3 & 0 & -4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \in \mathbf{H}_4^{\star} \cap \mathbf{Skew}_4. \tag{6}$$

The eigenvalues of X are

$$\lambda_{\pm,\pm} = \pm \mathrm{i}\sqrt{(29 \pm 3\sqrt{65})/2} \in \mathrm{i}\,\mathbb{R}.$$

Then $D\mathcal{F}(X) \in \mathcal{M}_{13,13}$ and, by Theorem 3.2,

$$\mathrm{Spec}\,(D\mathcal{F}(X)) = \left\{0^5, \pm 3\sqrt{5}\,\mathrm{i}\,, \pm\sqrt{13}\,\mathrm{i}\,, \pm\sqrt{58 \pm 6\sqrt{65}}\,\mathrm{i}\right\}.$$

On the other hand, $\mathrm{Ker}\,(D\mathcal{F}(X)) = \langle I_4\rangle \oplus (\mathbf{Skew}_4 \cap \mathbf{H}_4)$. From Prop. 3.2 one has $\dim(\mathrm{Ker}\,(D\mathcal{F}(X))) = 4$, and $D\mathcal{F}(X)$ posseses a generalized eigenvector of eigenvalue 0 associated to a Jordan block.

As follows from the theoretical results, this is, indeed, the general situation: given $X = (x_{i,j})_{1\le i,j\le n} \in \mathbf{Skew}_4 \cap \mathbf{H}_4$ one has $\mathrm{Ker}\,(D\mathcal{F}(X)) = \langle I_4\rangle \oplus (\mathbf{Skew_4} \cap \mathbf{H}_4)$ and a generalized eigenvector is given by

$$T = \begin{pmatrix} 0 & 0 & x_{2,1} & 0 \\ 0 & -\frac{x_{3,2}}{2} & 0 & x_{4,3} \\ & 0 & \frac{x_{2,1}^2 - x_{4,3}^2 - x_{3,2}^2}{2x_{3,2}} & 0 \\ & & 0 & \frac{x_{2,1}^2 - x_{4,3}^2}{2x_{3,2}} \end{pmatrix},$$

as one can check that $D\mathcal{F}(X)(T) \in \mathbf{Skew}_4 \cap \mathbf{H}_4$ (i.e., it belongs to $\mathrm{Ker}\,(D\mathcal{F}(X))$).

If $x_{3,2} = 0$ (i.e. if X is reduced), the linear dynamics around X changes. Given $X \in \mathbf{Skew}_4 \cap \mathbf{H}_4^{\star}$ there exists $Q \in \mathbf{O}_4$ such that $X_R := Q^{\top} X Q = X_R \in \mathbf{Skew}_4 \cap (\mathbf{H}_4 \setminus \mathbf{H}_4^{\star})$. Obviously $\mathrm{Spec} DF(X_R) = \mathrm{Spec} DF(X)$. However, $\mathrm{Ker}\,(D\mathcal{F}(X_R)) = \langle I_2^s\rangle \oplus \langle I_2^i\rangle \oplus (\mathbf{Skew}_4 \cap \mathbf{H}_4)$ being

$$I_2^s = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad I_2^i = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix},$$

and, consequently, $\dim(\mathrm{Ker}\,(D\mathcal{F}(X_R))) = 5 > \dim(\mathrm{Ker}\,(D\mathcal{F}(X)))$.

Remark 3.4. Let $X, R \in \mathbf{T}_n$. Then $D\mathcal{F}(X)R = 0$ and, therefore, zero is an eigenvalue. The other eigenvalues of $D\mathcal{F}(X)$ are $x_{i+1,i+1} - x_{i,i}$, $i = 1, \ldots, n-1$, where $X = (x_{ij})_{1\le i,j\le n}$.

4 Asymptotic behavior of the orbits of the QR-flow

In this section we prove that, given $X_0 \in \mathbf{H}_n$ the set $\omega(X_0)$ is an equilibrium matrix or it does not contain equilibrium matrices. In the latter case, the orbit of an element $Y \in \omega(X_0)$ is a (multi-)periodic orbit on a torus of suitable dimension. Concretely, we characterize the block diagonal part (for suitable blocks) of the elements of the ω-limit of an initial matrix $X_0 \in \mathbf{H}_n$. In the general case, given $Y \in \omega(X_0)$, the orbit of Y is a (multi-)periodic function defined over a torus of dimension the number of eigenvalues with non-vanishing imaginary part divided by two. Hence, this torus has dimension $\le n/2$ and it is embedded into a phase space of dimension

$N_d = (n^2 + 3n - 2)/2$. Note that resonances between the frequencies can decrease the number of fundamental periods to describe the orbit and, in such a case, the torus can be of lower dimension.

The ordering of the diagonal blocks of Y gives us a decomposition of the torus into a product of lower dimensional tori, obtained as a projection of the torus into suitable invariant subspaces (under the QR-flow) of lower dimension related to the number of non-real eigenvalues of Y with the same real part.

Remark 4.1.
1. Note that the equation (1) is invariant with respect to $t \mapsto -t$, $X \mapsto -X$. In particular, the reversing involution implies that the ω- and α-limit sets have the same properties. Consequently, we just consider the ω-limit set in the following.
2. Let $X_0 \in \mathbf{H}_n$ and assume that $\omega(X_0)$ contains an equilibrium matrix Y. Then, $X_0 = A + B$ where $A \in \mathbf{Skew}_n$, $\mathrm{Spec}(B) \subset \mathbb{R}$ and $AB = BA$. Indeed, by Remark 3.1 item 1, $Y = \tilde{A} + R$ with $\tilde{A} \in \mathbf{Skew}_n$ and $R \in \mathbf{T}_n$. The property follows because there exists $Q \in \mathbf{O}_n$ such that $X_0 = Q^\top Y Q = Q^\top \tilde{A} Q + Q^\top R Q = A + B$.

Remark 4.2. We will suppose that the matrix X_0 is unreduced. Otherwise, we divide X_0 into blocks with unreduced diagonal blocks. The form of the system $X' = [X, k(X)]$ implies that if $X_0 = (x_{ij})_{1\le i,j\le n}$ the QR-flow is invariant in any hypersurface $x_{j+1,j} = 0$. Then, $X(t) = (X_{i,j}(t))_{1\le i,j\le m} \in \mathbf{BUT}^n_{n_1,\dots,n_m}$ being $X_{i,i}(t)$, $1 \le i \le n$, unreduced. One has $e^{tX(0)} = Q(t)R(t)$, where $\mathrm{diag}(e^{tX(0)}) = (e^{tX_{1,1}(0)}, \dots, e^{tX_{m,m}(0)})$. Moreover, $Q(t) = \mathrm{Diag}(Q_1(t), \dots, Q_m(t)) \in \mathbf{BD}^n_{n_1,\dots,n_m}$ and $X_{i,j}(t) = Q_i^\top(t) X_{i,j}(0) Q_j(t)$, $1 \le i \le j \le m$. This implies that $Y \in \omega(X_0)$ admits the same block partition defined by the diagonal blocks of X_0, that we call $Y_{1,1}, \dots, Y_{k,k}$, and $Y_{j,j} \in \omega(X_{j,j})$, $j = 1, \dots, k$.

The Theorem 4.1 below characterizes the structure of the matrices $Y \in \omega(X_0)$. A basic tool to prove it is to reduce $X_0 \in \mathbf{H}^\star_n$ to a suitable reordered Jordan normal form. The existence of such a Jordan normal form follows from by Prop. 4.1 below. Let us first introduce some notations. Given $\lambda \in \mathrm{Spec}(X_0)$ we denote by $\mathrm{mult}(\lambda)$ the (algebraic) multiplicity of λ. Assume that $\alpha_1 > \alpha_2 > \dots > \alpha_m$, $m \le n$, are the real parts of the eigenvalues of X_0. One has

$$\mathrm{Spec}(X_0) = \bigcup_{j=1}^{m} \mathcal{R}_j = \bigcup_{j=1}^{m} (\mathcal{R}_j^0 \cup \mathcal{R}_j^1),$$

where $\mathcal{R}_j = \{\lambda \in \mathrm{Spec}(X_0), \mathrm{Re}\,\lambda = \alpha_j\}$, $\mathcal{R}_j^0$ is the set of $\lambda \in \mathcal{R}_j$ of even multiplicity and $\mathcal{R}_j^1$ is the set of $\lambda \in \mathcal{R}_j$ of odd multiplicity. Denote by $2m_j^i - i$ the maximum multiplicity of the eigenvalues of $\mathcal{R}_j^i$, $i = 0, 1$. Note that either $m_j^0 = m_j^1$ or $m_j^0 = m_j^1 - 1$ assuming that all blocks (associated to eigenvalues of different multiplicity) are present, see Remark 4.3 for other cases. We shall denote by $d_j = 2s_j - 1$ where $s_j = m_j^0 + m_j^1$. Furthermore,

$$\mathcal{R}_j^i = \bigcup_{k=1}^{m_j^i} \mathcal{R}_{j,k}^i, i = 0, 1, \text{ where } \mathcal{R}_{j,k}^i = \{\lambda \in \mathcal{R}_j^i, \text{ with multiplicity } 2k - i\}.$$

Let $c_{j,k}^i := \#\mathcal{R}_{j,k}^i$, $i = 0, 1$, and assume that

$$\mathcal{R}_{j,k}^i = \{\alpha_j, \alpha_j \pm \mathrm{i}\,\beta_{j,k,i}^{(1)}, \dots, \alpha_j \pm \mathrm{i}\,\beta_{j,k,i}^{((c_{j,k}^i-1)/2)}\}, \qquad \text{if } c_{j,k}^i \text{ odd, and}$$
$$\mathcal{R}_{j,k}^i = \{\alpha_j \pm \mathrm{i}\,\beta_{j,k,i}^{(1)}, \dots, \alpha_j \pm \mathrm{i}\,\beta_{j,k,i}^{(c_{j,k}^i/2)}\}, \qquad \text{if } c_{j,k}^i \text{ even.}$$

If $c_{j,k}^i$ is odd we consider the vector of block-matrices $\Lambda_{j,k}^i = (\Lambda_{j,k,1}^i, \dots, \Lambda_{j,k,(c_{j,k}^i+1)/2})$, where

$$\Lambda_{j,k,1}^i = \alpha_j, \quad \Lambda_{j,k,\ell}^i = \begin{pmatrix} \alpha_j & -\beta_{j,k,i}^{(\ell-1)} \\ \beta_{j,k,i}^{(\ell-1)} & \alpha_j \end{pmatrix}, \qquad \ell = 2, \dots, (c_{j,k}^i + 1)/2,$$

Analogously, for $c^i_{j,k}$ even, we consider $\Lambda^i_{j,k} = (\Lambda^i_{j,k,1}, \ldots, \Lambda_{j,k,c^i_{j,k}/2})$, where

$$\Lambda^i_{j,k,\ell} = \begin{pmatrix} \alpha_j & -\beta^{(\ell)}_{j,k,i} \\ \beta^{(\ell)}_{j,k,i} & \alpha_j \end{pmatrix}, \qquad \ell = 1, \ldots, c^i_{j,k}/2.$$

Proposition 4.1. *Let $X_0 \in \mathbf{H}^\star_n$ and assume the previous described structure and the notation introduced above. There exists $T \in GL(n, \mathbb{R})$ with the following properties:*

1. *$T^{-1}X_0T = D$ where $D \in \mathbf{BD}^n_{n_1,\ldots,n_m}$ is such that*

$$D_{j,j} = \begin{pmatrix} D^{(j)}_{1,1} & & \\ \vdots & \ddots & \\ D^{(j)}_{d_j,1} & \cdots & D^{(j)}_{d_j,d_j} \end{pmatrix} \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}},$$

being $D^{(j)}_{k,k} \in \mathbf{BLT}^{r_{k,j}}_\nu$, where $\nu = \nu(j,k)$ is equal to $(2, 2, \ldots, 2)$ if $r_{k,j}$ is even, and $\nu = (1, 2, \ldots, 2)$ otherwise. Moreover,

$$\begin{aligned} \operatorname{diag}\left(D^{(j)}_{s_j \pm 2k, s_j \pm 2k}\right) &= \left(\Lambda^1_{j,m_j}, \Lambda^1_{j,m_j-1}, \ldots, \Lambda^1_{j,k+1}\right), \\ \operatorname{diag}\left(D^{(j)}_{s_j \pm (2k+1), s_j \pm (2k+1)}\right) &= \left(\Lambda^0_{j,m_j}, \Lambda^0_{j,m_j-1}, \ldots, \Lambda^0_{j,k+1}\right), \end{aligned}$$

and

$$\begin{aligned} \operatorname{Spec}(D^{(j)}_{s_j-2k,s_j-2k}) &= \operatorname{Spec}(D^{(j)}_{s_j+2k,s_j+2k}) &&= R^1_j \setminus \cup_{\ell=1}^k R^1_{j,\ell}, \\ \operatorname{Spec}(D^{(j)}_{s_j-2k-1,s_j-2k-1}) &= \operatorname{Spec}(D^{(j)}_{s_j+2k+1,s_j+2k+1}) &&= R^0_j \setminus \cup_{\ell=1}^k R^0_{j,\ell}, \end{aligned}$$

2. *$T^{-1} = LU$, where $L \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ and $U \in \mathbf{BUT}^n_{n_1,\ldots,n_m}$. The matrix U is such that*

   (a) *The matrices $U_{j,j}$ are of the form*

$$U_{j,j} = \begin{pmatrix} U^{(j)}_{1,1} & \cdots & U^{(j)}_{1,d_j} \\ & \ddots & \vdots \\ & & U^{(j)}_{d_j,d_j} \end{pmatrix}, \qquad U^{(j)}_{k,k} \in \mathbf{BUT}^{r_{k,j}}_\nu.$$

   (b) *$UX_0U^{-1} = D + N$, where $N \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ is such that $N_{i,i} = 0$ for $1 \le i \le m$.*

See Appendix B for a proof of Prop. 4.1. The reordered Jordan normal form that we will use is derived as follows. For $m \ge n$, we denote by $E(m,n)$ the matrix

$$E(m,n) = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \in \mathcal{M}_{m,n}.$$

For each block $D_{j,j} \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$ there exists a matrix $\hat{L}_{j,j} \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$, such that $\hat{L} = \operatorname{Diag}(\hat{L}_{1,1}, \ldots, \hat{L}_{m,m})$ reduces $D = \operatorname{Diag}(D_{1,1}, \ldots, D_{m,m})$ to the following form

$$\hat{L}D\hat{L}^{-1} =: \tilde{D} = \operatorname{Diag}(\tilde{D}_{1,1}, \ldots, \tilde{D}_{m,m}), \quad \tilde{D}_{j,j} = \hat{L}_{j,j}D_{j,j}\hat{L}^{-1}_{j,j} \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}, \tag{7}$$

where

$$\tilde{D}^{(j)}_{s_j\pm 2k,s_j\pm 2k} = \mathrm{Diag}(\Lambda^1_{j,m_j}, \Lambda^1_{j,m_j-1}, \ldots, \Lambda^1_{j,k+1}) = \mathrm{Diag}(\tilde{D}^{(j)}_{s_j\pm 2k+2,s_j\pm 2k+2}, \Lambda^1_{j,k+1}),$$

$$\tilde{D}^{(j)}_{s_j\pm(2k+1),s_j\pm(2k+1)}=\mathrm{Diag}(\Lambda^0_{j,m_j}, \Lambda^0_{j,m_j-1}, \ldots, \Lambda^0_{j,k+1})=\mathrm{Diag}(\tilde{D}^{(j)}_{s_j\pm(2k+3),s_j\pm(2k+3)}, \Lambda^0_{j,k+1}),$$

and the non-zero blocks of $\tilde{D}$ outside the diagonal are:

$$\tilde{D}_{s_j+k+2,s_j+k}=E(r_{s_j+k,j}, r_{s_j+k+2,j})^\top,\ \tilde{D}_{s_j-k+1,s_j-k-1}=E(r_{s_j+k-1,j}, r_{s_j+k+1,j}),\ k\geq 0.$$

The existence of the lower triangular $\hat{L}$ introduced above follows from the fact that $X_0 \in \mathbf{H}^\star_n$ and hence it is nonderogatory (i.e. such that all the eigenvalues have geometric multiplicity one). The matrix $\tilde{D}$ is the *reordered Jordan normal form* of X_0 that will be used to prove Theorem 4.1. See Fig. 1 right for an illustration of the form of a typical block $\tilde{D}_{j,j}$ of $\tilde{D}$. Similar canonical normal forms were used in [25, 26] to study the convergence of the QR-iteration in the complex setting.

Theorem 4.1. *Let $X_0 \in \mathbf{H}^\star_n$ with the structure and notation introduced in Prop 4.1. The structure of $Y \in \omega(X_0)$ has the following properties:*

1. *Y is of the form*

$$Y = \begin{pmatrix} Y_{1,1} & \cdots & Y_{1,m} \\ & \ddots & \vdots \\ & & Y_{m,m} \end{pmatrix} \in \mathbf{BUT}^n_{n_1,\ldots,n_m},$$

   where $\mathrm{Spec}(Y_{j,j}) = \mathcal{R}_j$ *and* $n_j = \sum_{\lambda\in\mathcal{R}_j} \mathrm{mult}(\lambda)$.

   The subdiagonal elements of $X(t)$ that lead to the block structure of Y above are $o(\exp((\alpha_{j+1}-\alpha_j+\eta)t))$, as $t\to\infty$ and for any $\eta>0$, see Lemma 4.1.

2. *The matrices $Y_{j,j}$ are of the form*

$$Y_{j,j} = \begin{pmatrix} Y^{(j)}_{1,1} & \cdots & Y^{(j)}_{1,d_j} \\ & \ddots & \vdots \\ & & Y^{(j)}_{d_j,d_j} \end{pmatrix} \in \mathbf{BUT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}},$$

   being $Y^{(j)}_{k,k} \in \mathbf{H}^\star_{r_{k,j}}$, for $1\leq k\leq d_j = 2s_j-1$, $s_j = m^0_j + m^1_j$, $r_{k,j} = \#\mathrm{Spec}(Y^{(j)}_{k,k})$, therefore all the eigenvalues of $Y^{(j)}_{k,k}$ are simple. Moreover,

$$\mathrm{Spec}(Y^{(j)}_{s_j-2k,s_j-2k})=\mathrm{Spec}(Y^{(j)}_{s_j+2k,s_j+2k})=\mathcal{R}^1_j \setminus \bigcup_{\ell=1}^{k} \mathcal{R}^1_{j,\ell}, \qquad 0\leq k\leq m^1_j-1,$$

$$\mathrm{Spec}(Y^{(j)}_{s_j-2k-1,s_j-2k-1})=\mathrm{Spec}(Y^{(j)}_{s_j+2k+1,s_j+2k+1})=\mathcal{R}^0_j \setminus \bigcup_{\ell=1}^{k} \mathcal{R}^0_{j,\ell}, \qquad 0\leq k\leq m^0_j-1.$$

   The subdiagonal elements of $(X(t))_{j,j}$ that lead to the block structure of $Y_{j,j}$ tend to zero (at least, see Remark 4.3 below) as t^{-1}, see Lemma 4.2.

Remark 4.3. An example of the form of a block $Y_{j,j}$ is shown in Fig. 1 left. We remark that, in a general situation, some of the blocks $Y_{j,j}$ might be missing depending on the multiplicity of the eigenvalues of X_0. However, one can consider zero-dimensional blocks to apply the general theorem. This will provide zero-dimensional blocks in the ω-limit that have no meaning and should be removed from the final structure. Accepting this convection, Theorem 4.1 provides the structure of the elements of the ω-limit in all situations. See the following Example 4.1.

Example 4.1. Let $X_0 \in \mathbf{H}_7^\star$ such that $\mathrm{Spec}\,(X_0) = \{1+\mathrm{i}\,, 1-\mathrm{i}\,, 1+\mathrm{i}\,, 1-\mathrm{i}\,, 2+\mathrm{i}\,, 2-\mathrm{i}\,, 2\}$, that is $\mathrm{mult}(1+\mathrm{i}\,) = \mathrm{mult}(1-\mathrm{i}\,) = 2$, $\mathrm{mult}(2+\mathrm{i}\,) = \mathrm{mult}(2-\mathrm{i}\,) = \mathrm{mult}(2) = 1$. One has $R_1 = \{2+\mathrm{i}\,, 2-\mathrm{i}\,, 2\}$ and $R_2 = \{1+\mathrm{i}\,, 1-\mathrm{i}\,\}$. Let us examine the block associated to R_1 and R_2 separately.

The structure of the block $Y_{1,1}$ associated to R_1 is compatible with the general structure of Theorem 4.1. The eigenvalues of $Y_{1,1}$ are those of X_0 with real part equal to two. One has $R_1^0 = \emptyset$, $R_1^1 = R_1$ and $m_1^0 = 0$, $m_1^1 = 1$. Moreover, $s_1 = d_1 = 1$ and $n_1 = r_{1,1} = 3$, which gives only one three-dimensional sub-block $Y_{1,1} = Y_{1,1}^{(1)} \in \mathbf{H}_3^\star$.

The block associated to R_2 does not have the general structure of the Theorem 4.1 since all the eigenvalues of R_2 have multiplicity two and the block associated to multiplicity one is missed. One has $R_2^0 = R_2$, $R_2^1 = \emptyset$. Hence, we add a "ficticious" zero-dimensional block so that $m_2^0 = m_2^1 = 1$ and we consider the block $Y_{2,2}$ associated to this modified structure of R_2. Then, $s_2 = 2$, $d_2 = 3$ and $n_2 = 4$. This gives three diagonal sub-blocks $Y_{i,i}^{(2)} \in \mathbf{H}_{r_{i,i}}^\star$, $i = 1, 2, 3$, of $Y_{2,2}$ with dimensions $r_{1,1} = 2$, $r_{2,2} = 0$ and $r_{3,3} = 2$. This holds since $R_{2,1}^0 = \{1+i, 1-i\}$ and $R_{2,1}^1 = \emptyset$, hence $\mathrm{Spec}\,(Y_{2,2}^{(2)}) = \emptyset$ and $\mathrm{Spec}\,(Y_{1,1}^{(2)}) = \mathrm{Spec}\,(Y_{3,3}^{(2)}) = R_2^0$. The block $Y_{2,2}$ contains the eigenvalues of X_0 with real part equals one. The block $Y_{2,2}^{(2)}$ is zero-dimensional and we remove it.

Finally, we conclude from the previous considerations that $Y \in \omega(X_0)$ is such that $\mathrm{diag}(Y) = (Y_{1,1}, Y_{2,2})$, $Y_{1,1} \in \mathbf{H}_3^\star$, $\mathrm{diag}(Y_{2,2}) = (Y_{1,1}^{(2)}, Y_{3,3}^{(2)})$, $Y_{i,i}^{(2)} \in \mathbf{H}_2^\star$, $i = 1, 3$.

Proof: Let $T \in \mathcal{M}_{n,n}$ be the matrix that reduces $X_0 \in \mathbf{H}_n^\star$ to the form given by Prop. 4.1. In particular, $X_0 = TDT^{-1}$ and $T = U^{-1}L^{-1}$. One has

$$e^{tD}L = e^{tD}Le^{-tD}e^{tD} = \tilde{L}(t)e^{tD}, \tag{8}$$

where $\tilde{L}(t) \in \mathbf{BLT}_{n_1,\ldots,n_m}^n$ with $\tilde{L}(t)_{i,i} = I_{n_i}$. Indeed, $\lim_{t\to\infty} \tilde{L}(t) = I_n$. This follows from the following fact. Consider $i > j$, then $\alpha_i - \alpha_j < 0$ and

$$\tilde{L}_{i,j}(t) = e^{tD_{i,i}}L_{i,j}e^{-tD_{j,j}} = e^{(\alpha_i-\alpha_j)t}e^{t(D_{i,i}-\alpha_i I_{n_i})}L_{i,j}e^{-t(D_{j,j}-\alpha_j I_{n_j})},$$

which tends to 0 as $t \to \infty$ because $e^{t(D_{i,i}-\alpha_i I_{n_i})}$ grows, at most, as a polynomial in t.

Denote by $T = Q_1R_1$ the QR-factorization of T. Using (8) it follows that

$$e^{tX_0} = Te^{tD}T^{-1} = Q_1R_1e^{tD}LU = Q_1R_1\tilde{L}(t)e^{tD}U. \tag{9}$$

Now, consider the QR-factorization $R_1\tilde{L}(t) = \widehat{Q}(t)\widehat{R}(t)$. Then $\lim_{t\to\infty} \widehat{Q}(t)\widehat{R}(t) = R_1$ and, by uniqueness of the QR factorization, it follows that $\lim_{t\to\infty} \widehat{Q}(t) = I_n$.

By Prop. 2.1 we have that e^{tX_0} has a QR factorization $e^{tX_0} = Q(t)R(t)$, and the solution of the QR-flow with initial condition X_0 is $X(t) = Q(t)^{-1}X_0Q(t)$.

On the other hand, $\widehat{R}(t)e^{tD}U \in \mathbf{BUT}^n_{n_1,\ldots,n_m}$. Therefore, the orthogonal matrix of its QR-factorization is $\mathrm{diag}(\overline{Q}_1(t),\ldots,\overline{Q}_m(t))$. By the uniqueness of the QR-factorization, we obtain

$$Q(t) = Q_1\widehat{Q}(t)\mathrm{Diag}(\overline{Q}_1(t),\ldots,\overline{Q}_m(t)), \tag{10}$$

and, since $Q_1^\top X_0 Q_1 = R_1 D R_1^{-1}$,

$$X(t) = \mathrm{Diag}(\overline{Q}_1(t)^\top,\ldots,\overline{Q}_m(t)^\top)\widehat{Q}(t)^\top R_1 D R_1^{-1}\widehat{Q}(t)\mathrm{Diag}(\overline{Q}_1(t),\ldots,\overline{Q}_m(t)).$$

If $Y \in \omega(X_0)$ then let $(t_k)_{k\geq 0} \to \infty$ be a sequence such that $\lim_{k\to\infty} X(t_k) = Y$ and $\overline{Q}_j^0 = \lim_{k\to\infty} \overline{Q}_j(t_k)$, $1 \leq j \leq m$. Then,

$$Y = \mathrm{Diag}((\overline{Q}_1^0)^\top,\ldots,(\overline{Q}_m^0)^\top)R_1 D R_1^{-1}\mathrm{Diag}(\overline{Q}_1^0,\ldots,\overline{Q}_m^0), \tag{11}$$

which proves the first part of the theorem.

The second part follows from a similar argument but using the reordered Jordan normal form (7). Hence, the analogous of (9) in this case is

$$e^{tX_0} = T\hat{L}^{-1}e^{t\tilde{D}}\hat{L}T^{-1}, \tag{12}$$

which, by Lemma C.1, reduces to

$$e^{tX_0} = T\hat{L}^{-1}PPe^{t\tilde{D}}\hat{L}T^{-1} = Q_2R_2M(t)U = Q_2R_2\bar{L}(t)\bar{R}(t)U, \tag{13}$$

where $M(t) = Pe^{t\tilde{D}}\hat{L}L = \bar{L}(t)\bar{R}(t)$, $\bar{L}(t) \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ is such that $\mathrm{diag}(\bar{L}(t)) = (\bar{L}_{1,1}(t),\ldots,\bar{L}_{m,m}(t))$ where $\bar{L}_{j,j}(t) \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$, $\lim_{t\to\infty}\bar{L}(t) = I_n$, and Q_2R_2 is the QR-factorization of $T\hat{L}^{-1}P$. Note also that $\mathrm{diag}(\bar{R}(t)) = (\bar{R}_{1,1}(t),\ldots,\bar{R}_{m,m}(t))$ where $\bar{R}_{j,j}(t) \in \mathbf{BUT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$, $1 \leq j \leq m$. The last expression has the same structure than (9) but with $\bar{R}(t)$ instead of e^{tD}. The same reasoning as in the first part gives

$$Y = \mathrm{Diag}((\overline{Q}_1^0)^\top,\ldots,(\overline{Q}_s^0)^\top)R_2P\tilde{D}PR_2^{-1}\mathrm{Diag}(\overline{Q}_1^0,\ldots,\overline{Q}_s^0), \tag{14}$$

where $s = d_1 + \cdots + d_m$. This gives the structure of the ω-limit of the second part of the theorem. It remains to prove that the blocks are unreduced, the corresponding proof will given in Lemma 4.3. □

Next, we discuss about the decay of the subdiagonal elements of $X(t) = (x_{i,j})_{i,j} \in \mathbf{H}_n^\star$ between the m blocks of its ω-limit structure described by (11).

Lemma 4.1. *For $1 \leq j \leq m-1$, let $\ell(j) = \sum_{i=1}^j n_i$. One has*

$$|x_{\ell(j)+1,\ell(j)}| \leq M_j\, e^{(\alpha_{j+1}-\alpha_j)t}\, t^{k_j}, \quad M_j > 0,$$

for $|t|$ large enough, where $0 \leq k_j \leq \max(2m_{j+1}^0, 2m_{j+1}^1 - 1) + \max(2m_j^0, 2m_j^1 - 1) - 2$.

Proof: From Prop. 2.1, $X(t) = R(t)X_0R(t)^{-1}$ and $e^{tX_0} = Q(t)R(t)$. From (9), $e^{tX_0} = Q_1\hat{Q}(t)\hat{R}(t)e^{tD}U$ and, using (10), $R(t) = \mathrm{Diag}(\overline{Q}_1(t)^\top,\ldots,\overline{Q}_m(t)^\top)\hat{R}(t)e^{tD}U$. We consider the block partition of $X(t)$ induced by $\mathrm{Diag}(\overline{Q}_1(t)^\top,\dots,\overline{Q}_m(t)^\top)$ and we denote by $(X(t))_{i,j}$ the corresponding blocks. If $H_{i+1,i}{:=}U_{i+1,i+1}(X(0))_{i+1,i}U_{i,i}^{-1}$, then

$$\begin{aligned}(X(t))_{i+1,i} &= \overline{Q}_{i+1}(t)^\top \hat{R}_{i+1,i+1}(t)e^{tD_{i+1,i+1}}H_{i+1,i}e^{-tD_{i,i}}\hat{R}_{i,i}(t)^{-1}\overline{Q}_i(t)\\ &= e^{(\alpha_{i+1}-\alpha_i)t}Z_i(t),\end{aligned}$$

where

$$Z_i(t){=}\overline{Q}_{i+1}(t)^\top \hat{R}_{i+1,i+1}(t)e^{t(D_{i+1,i+1}-\alpha_{i+1}I_{n_{i+1}})}H_{i+1,i}e^{-t(D_{i,i}-\alpha_i I_{n_i})}\hat{R}_{i,i}(t)^{-1}\overline{Q}_i(t).$$

Note that $Z_i(t)\in\mathcal{M}_{n_{i+1},n_i}$ and, since $X(t)\in\mathbf{H}_n^\star$, only $(Z_i(t))_{1,n_i}\neq 0$. Finally, since $\overline{Q}_i(t)$ are bounded, $\hat{R}(t)$ has limit when $t\to\infty$ and the eigenvalues of $D_{i,i}-\alpha_i I_{n_i}$ are purely imaginary, the growth of $Z_i(t)$ is polynomial with degree bounded by the sum of maximum of the multiplicities of the eigenvalues of $D_{i,i}$ and $D_{i+1,i+1}$ minus two. This implies the result. □

Next lemma discusses the decay of the subdiagonal elements of a block $(X(t))_{j,j}$ that lead to the blocks $Y_{k,l}^{(j)}$, that is, sub-blocks inside a block $Y_{j,j}$. For simplicity, we denote by $X(t)=(x_{i,j})_{i,j}\in\mathbf{H}_n^\star$ one of these blocks.

Lemma 4.2. *For $1\le j\le r-1$, let $\hat{\ell}(j)=\sum_{i=1}^{j} r_i$. One has, for $|t|$ large enough,*

$$|x_{\hat{\ell}(j)+1,\hat{\ell}(j)}|\le M_j\,t^{-1},\quad M_j>0.$$

Proof: Using (12) one obtains

$$R(t)=\mathrm{Diag}(\overline{Q}_1(t)^\top,\dots,\overline{Q}_d(t)^\top)\hat{R}(t)\bar{R}(t)U. \tag{15}$$

Consider the block partition so that $\mathrm{diag}(\Xi(t))=(\Xi_{1,1}(t),\dots,\Xi_{d,d}(t))$ for the matrices $\Xi=R,\hat{R},\bar{R},U$ and $X(t)$. If we denote by $H_{i+1,i}=U_{i+1,i+1}(X(0))_{i+1,i}U_{i,i}^{-1}$, $1\le i\le d-1$, then

$$(X(t))_{i+1,i}=\overline{Q}_{i+1}(t)^\top\hat{R}_{i+1,i+1}(t)\overline{R}_{i+1,i+1}(t)H_{i+1,i}\overline{R}_{i,i}(t)^{-1}\hat{R}_{i,i}(t)^{-1}\overline{Q}_i(t), \tag{16}$$

where $\overline{R}(t)=e^{t\alpha}Pe^{tA}P\overline{\overline{R}}(t)$ according to Lemma C.3. Moreover, from Lemma C.4

$$\overline{\overline{R}}(t)=W(t)R_1(t)\widehat{W}(t), \tag{17}$$

where $L_1(t)R_1(t)$ is the LU-block factorization of $R_0N(t)$, $N(t)=\widehat{W}(t)\hat{L}\widehat{W}^{-1}(t)$. Using (24),

$$R_0N(t)=\begin{pmatrix}\sum_{i=1}^d t^{p(i,1)}(R_0)_{1,i}\hat{L}_{i,1} & \sum_{i=2}^d t^{p(i,2)}(R_0)_{1,i}\hat{L}_{i,2} & \dots & (R_0)_{1,d}\hat{L}_{d,d}\\ \vdots & \vdots & & \vdots\\ t^{p(d,1)}(R_0)_{d,d}\hat{L}_{d,1} & t^{p(d,2)}(R_0)_{d,d}\hat{L}_{d,2} & \dots & (R_0)_{d,d}\hat{L}_{d,d}\end{pmatrix},$$

and, since $p(i,j-1)\le p(i,j)\le p(i-1,j)$, one has $(R_0N(t))_{i,j}=\mathcal{O}(1)$, $i\ge j$, and $(R_0N(t))_{i,j}=\mathcal{O}(t^{p(i,j)})$ otherwise. Note that, since $(R_0)_{i,i+1}=0$, it follows that $(R_0N(t))_{i,i}=(R_0)_{i,i}\hat{L}_{i,i}(I+\mathcal{O}(t^{-1}))$ and, moreover, that $(R_0N(t))_{i,i+1}=\mathcal{O}(t^{-1})$. The LU-block factorization of $(R_0N(t))$ verifies

$$\begin{aligned}&(L_1(t))_{i,i}=I_{r_i},\quad (L_1(t))_{i,j}=\mathcal{O}(t^{p(i,j)}),\ i>j,\\ &(R_1(t))_{i,i}=(R_0)_{i,i}\hat{L}_{i,i}+\mathcal{O}(t^{-1}),\quad (R_1(t))_{i,i+1}=\mathcal{O}(t^{p(i,i-1)}),\\ &(R_1(t))_{i,j}=\mathcal{O}(1),\ j\ge i+2.\end{aligned}$$

From (17) and the expressions in Lemma C.4, one obtains

$$(\overline{\overline{R}}(t))_{i,i} = t^{s-i}\,(R_0)_{i,i}\hat{L}_{i,i}(I + \mathcal{O}(t^{-1})),\ \text{where}\ s = (d+1)/2. \tag{18}$$

We conclude from (16) that $X(t)_{i+1,i} = t^{-1}B(t)$ where $B(t)$ is a bounded matrix. This proves the result. $\square$

Remark 4.4. We recall that some blocks of the $\omega(X_0)$ might be missing because the multiplicities of the eigenvalues. In case there is a missing block the expected decay is $\mathcal{O}(t^{-2})$, see Example 4.2.

Finally, we prove that the blocks $Y_{k,k}^{(j)}$, $1 \leq k \leq d_j$, in Theorem 4.1 are unreduced. We consider a block $(X(t))_{k,k}^{(j)}$ that corresponds to the block $Y_{k,k}^{(j)}$. As before, for simplicity, we denote by $X(t) = (x_{i,j})_{i,j} \in \mathbf{H}_r^\star$ one of these blocks.

Lemma 4.3. *The blocks $Y_{k,k}^{(j)}$ are unreduced, that is, there exists a constant $M > 0$ such that*

$$|x_{i+1,i}| \geq M, \quad 1 \leq i \leq r-1, \qquad \textit{for all } t \in \mathbb{R}.$$

Proof: From (15), taking into account that $Pe^{tA}P \in \mathbf{BD}^n_{r_1,\ldots,r_d}$, it follows that

$$\mathcal{R}(t) = e^{-t\alpha}\,t^{k-s}\,R_{k,k}(t) = Q_k(t)^\top \hat{R}_{k,k}(t)B(t)U_{k,k},$$

where $B(t) = \tilde{B}(t)(R_0)_{k,k}\hat{L}_{k,k}(I + \mathcal{O}(t^{-1}))$, being $\tilde{B}(t) = (Pe^{tA}P)_{k,k} \in \mathbf{O}_r$. In particular, $B(t)$ and $B(t)^{-1}$ are bounded. Then, $\mathcal{R}(t)$ and $\mathcal{R}(t)^{-1}$ are bounded (because $\lim_{t\to\infty} \hat{R}(t) = R_2$ is invertible). It follows that $(\mathcal{R}(t))_{i,i} > 0$ are upper bounded and lower bounded. Since $X(t) = \mathcal{R}(t)X_0\mathcal{R}(t)^{-1}$, if we denote $X_0 = (x^0_{i,j})_{1\leq i,j\leq n}$, then, for some constant $M > 0$, one has

$$|x_{i+1,i}| = (\mathcal{R}(t))_{i+1,i+1}\;|x^0_{i+1,i}|\;(\mathcal{R}(t))^{-1}_{i,i} \geq M.$$

$\square$

Example 4.2. We illustrate the decay of the corresponding subdiagonal elements for a concrete matrix of the form of Example 4.1. Concretely, we consider $X_0 \in \mathbf{H}_7^\star$ similar to the matrix $\mathrm{Diag}(A_1, A_2, 2)$ by a random similarity matrix with coefficients in $[0, 1]$, where

$$A_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad A_2 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

As discussed in Example 4.1 one expects $(X(t))_{4,3} \to 0$ and $(X(t))_{6,5} \to 0$ as $t \to \infty$. Concretely, in Fig. 2 we observe that $(X(t))_{4,3} \sim e^{-t}$ while $(X(t))_{6,5} \sim t^{-2}$, as expected from Lemma 4.2. The coefficient -2 of the power-law decay of $(X(t))_{6,5}$ is due to the fact that R_2^1 is empty (the corresponding block is missed), see details in Example 4.1.

4.1 Convergence in Wilkinson's sense

Let us comment about Wilkinson (essential) convergence in the QR-iteration setting. Given $X_0 \in \mathbf{H}_n$ we consider the sequence $\{X_k\}_{k\geq 0}$ of QR-iterates.

Definition 4.1. The QR-iteration algorithm applied to X_0 essentially converges if the sequence $\{X_k\}_{k\geq 0}$ tends to an upper triangular matrix as $k \to \infty$.

In [14] it was proved that if the eigenvalues of X_0 are assumed to be of different modulus then the QR-iteration converges essentially to an upper triangular matrix. We can consider the same idea of Wilkinson convergence but within the QR-flow setting. Indeed, from Theorem 4.1, it immediately follows the following result. Note that in this setting we have convergence instead of essential convergence. By remark 4.2 we can extend the convergence property to $X_0 \in \mathbf{H}_n$.

Corollary 4.1 (Wilkinson convergence)**.** *If $X_0 \in \mathbf{H}_n$ has real eigenvalues then $\omega(X_0)$ is an upper triangular equilibrium matrix.*

Example 4.3. For a fixed prime number p, we consider the matrix $A \in \mathcal{M}_{n,n}$, $n = p-1$, defined by $a_{i,j} = \left(\frac{i+j}{p}\right)$, where $\left(\frac{x}{y}\right)$ denotes the Legendre symbol. The eigenvalues are $1, -1, \sqrt{p}$ and $\sqrt{-p}$ with multiplicities $1, 1, n/2-1, n/2-1$ respectively, see [16]. Since A has multiple real eigenvalues, the upper Hessenberg reductions of A are reduced. For $p = 7$ we consider

$$X_0 = \begin{pmatrix} 1 & 4 & -2 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 1 \\ 0 & -2 & -1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbf{H}_6 \setminus \mathbf{H}_6^\star. \tag{19}$$

The subspaces $\{x_{4,3} = 0\}$ and $\{x_{6,5} = 0\}$ are invariant by the flow. Each diagonal block of X_0 contains simple eigenvalues with different real parts. Hence Lemma 4.1 guarantees exponential convergence towards $Y \in \mathbf{T}_6$. For example, for $t = 10^5$ $X(t)$ has trace $\mathcal{O}(10^{-15})$. The eigenvalues within the diagonal blocks are sorted according to Theorem 4.1.

For $p = 13$, $X_0 \in \mathbf{H}_{12} \setminus \mathbf{H}_{12}^*$ and the subspaces $\{x_{4,3} = 0\}$, $\{x_{7,6} = 0\}$, $\{x_{9,8} = 0\}$ and $\{x_{11,10} = 0\}$ are invariant by the QR-flow. The diagonal entries of $X(t)$ for $t = 10^5$ are ordered as follows: $\lambda, -1, -\lambda, \lambda, 1, -\lambda, \lambda, -\lambda, \lambda, -\lambda, \lambda, -\lambda$, and $\lambda = \sqrt{13}$ is computed with error $\mathcal{O}(10^{-14})$.

4.2 Convergence in Parlett's sense

Before stating the results concerning Parlett convergence for the QR-flow, let us briefly describe the Parlett's results for the QR-iteration algorithm.

Definition 4.2. Given $X_0 \in \mathbf{H}_n$ we denote by $X_n = \left(x_{i,j}^{(n)}\right)$ the nth iterate of the QR-iteration applied to X_0. We say that the QR-iteration converges in the Parlett sense if $x_{j+1,j}^{(n)} x_{j,j-1}^{(n)} \to 0$ as $n \to \infty$.

Note that Parlett convergence is enough to numerically obtain an approximation of the spectrum of X_0. It was stated in [26] that the QR-iteration applied to X_0 converges (in Parlett's sense) if, and only if, 1) the number of eigenvalues of X_0 of equal modulus with even algebraic multiplicity is less or equal than two and 2) the same holds for the number of eigenvalues of X_0 of equal modulus with odd algebraic multiplicity. For the QR-flow we have the following analogous result.

Corollary 4.2 (Parlett convergence)**.** *Let $X_0 \in \mathbf{H}_n^\star$. Denote by N_{even} (resp. N_{odd}) the number of eigenvalues of X_0 of equal real part with even (resp. odd) algebraic multiplicity. The QR-flow with initial condition X_0 converges (in the Parlett sense) if, and only if, both N_{even} and N_{odd} are less or equal than two.*

We want to emphasize that criteria for the QR-iteration and QR-flow are not equivalent. It turns out that either the QR-iteration and/or the QR-flow applied to a matrix $X_0 \in \mathbf{H}_n$ with $n \leq 3$ converges in the Parlett sense. The following is an example with $n = 4$ for which the two algorithms fail in the convergence.

Example 4.4 (Non-convergence of QR-flow nor of QR-iteration)**.** The matrix

$$X_0 = \begin{pmatrix} -251/214 & 448843/100259 & -4447435/656338 & 459/214 \\ -937/428 & 965997/200518 & -6834801/1312676 & 519/428 \\ 0 & 1312676/877969 & -9216351/2873779 & 1188/937 \\ 0 & 0 & -3762055/9406489 & 1725/3067 \end{pmatrix},$$

has $\operatorname{Spec} DX_0 = \{0, 1, i, -i\}$. According to [26] the QR-iteration does not converge because X_0 has 3 eigenvalues of algebraic multiplicity 1 with modulus 1. On the other hand, from Corollary 4.2, we conclude that the QR-flow does not converge because X_0 has 3 eigenvalues of multiplicity 1 with zero real part.

Denote by $X(t) = (x_{i,j})_{i,j}(t)$ the solution of the Cauchy problem with initial condition X_0. The numerical integration of the QR-flow shows that $x_{2,1}$ tends to zero and the eigenvalue 1 (i.e. the one with largest real part) is isolated in the diagonal. However, the subdiagonal elements $x_{3,2}$ and $x_{4,3}$ do not tend to zero (they behave 2π periodically in t, remaining away from zero). That is, $\omega(X_0)$ has a 3×3 block in the diagonal which, in particular, does not allow to obtain the remaining eigenvalues. The elements $x_{1,2}$, $x_{1,3}$ and $x_{1,4}$ are 2π-periodic functions of t. The elements in the 3×3 diagonal block describe a 2π-periodic orbit. See also example 5.1.

It follows from the previous results that the computation of the eigenvalues of any (upper Hessenberg) matrix can be performed by combination of the QR-iteration and QR-flow strategies. For example, given $A_0 \in \mathcal{M}_{n,n}$ and $\epsilon > 0$ one can compute QR-iterates of A_0 until one obtains A_k such that the absolute value of some of the components of the subdiagonal are below ϵ. Then, when no other decays of the subdiagonal elements size is observed, one fills up with zeroes the subdiagonal components with size less than ϵ and switches to integrate the QR-flow starting from A_k. The zero components of the subdiagonal define a block partition of A_k. In each of the blocks the QR-flow converges (in Parlett sense).

Example 4.5. Consider $X_0 \in \mathcal{M}_{7,7}$ with eigenvalues $1 \pm \mathrm{i}$, $e^{\pm \mathrm{i}}$, $e^{\pm \mathrm{i}\sqrt{2}}$ and 1. Neither QR-iteration nor the QR-flow converges. After some number of QR-iterates a 2×2 diagonal block "separates" (for example, when $x_{3,2} = \mathcal{O}(10^{-8})$)). We integrate the QR-flow starting with the 5×5 remaining block up to large enough time to numerically observe (Parlett) convergence towards $T \in \mathbf{T}_5$. Note also that one can proceed in the inverse order, that is, by first integrating the QR-flow and then perfoming iterates of the QR-iteration. This will also converge (in Parlett sense).

4.3 A characterization of the homo/heteroclinic connections

In this section we characterize the homo/heteroclinic orbits and we prove the non-existence of homo/heteroclinic connections in the α, ω-limit sets. We will need the following proposition. We use the notation in Prop. 4.1 and we also consider the matrices

$$\check{\Lambda}^i_{j,k,1} = \alpha_j, \quad \check{\Lambda}^i_{j,k,\ell} = \begin{pmatrix} \alpha_j & 0 \\ 0 & \alpha_j \end{pmatrix}, \qquad \ell = 2, \ldots, (c^i_{j,k} + 1)/2,$$

if $c^i_{j,k}$ odd. For the case $c^i_{j,k}$ even we consider

$$\check{\Lambda}^i_{j,k,\ell} = \begin{pmatrix} \alpha_j & 0 \\ 0 & \alpha_j \end{pmatrix}, \qquad \ell = 1, \dots, c^i_{j,k}/2.$$

Also we consider $\hat{\Lambda}^i_{j,k,\ell} = \Lambda^i_{j,k,\ell} - \check{\Lambda}^i_{j,k,\ell}$.

Proposition 4.2. *Let $X_0 = A+B \in \mathbf{H}^\star_n$ such that $A, B \in \mathcal{M}_{n,n}$, $A \in \mathbf{Skew}_n$, all the eigenvalues of B are real and $AB = BA$. Then, one can choose $T \in \mathcal{M}_{n,n}$ in Prop. 4.1 such that $T^{-1}BT = \check{D}$ where $\check{D} \in \mathbf{BD}^n_{n_1,\dots,n_m}$ is such that*

$$\check{D}_{j,j} = \begin{pmatrix} \check{D}^{(j)}_{1,1} & & \\ \vdots & \ddots & \\ \check{D}^{(j)}_{d_j,1} & \cdots & \check{D}^{(j)}_{d_j,d_j} \end{pmatrix} \in \mathbf{BLT}^{n_j}_{r_{1,j},\dots,r_{d_j,j}},$$

being $\check{D}^{(j)}_{k,k} \in \mathbf{BLT}^{r_{k,j}}_{\nu}$, where $\nu = \nu(j,k)$ is equal to $(2,2,\dots,2)$ if $r_{k,j}$ is even, and $\nu = (1,2,\dots,2)$ otherwise. Moreover,

$$\operatorname{diag}\left(\check{D}^{(j)}_{s_j\pm 2k, s_j\pm 2k}\right) = \left(\check{\Lambda}^1_{j,m_j}, \check{\Lambda}^1_{j,m_j-1}, \dots, \check{\Lambda}^1_{j,k+1}\right),$$
$$\operatorname{diag}\left(\check{D}^{(j)}_{s_j\pm(2k+1), s_j\pm(2k+1)}\right) = \left(\check{\Lambda}^0_{j,m_j}, \check{\Lambda}^0_{j,m_j-1}, \dots, \check{\Lambda}^0_{j,k+1}\right),$$

and

$$\begin{aligned} \operatorname{Spec}(\check{D}^{(j)}_{s_j-2k,s_j-2k}) &= \operatorname{Spec}(\check{D}^{(j)}_{s_j+2k,s_j+2k}) &&= \{\alpha_j\}, \\ \operatorname{Spec}(\check{D}^{(j)}_{s_j-2k-1,s_j-2k-1}) &= \operatorname{Spec}(\check{D}^{(j)}_{s_j+2k+1,s_j+2k+1}) &&= \{\alpha_j\}. \end{aligned}$$

See Appendix B for a proof of this proposition.

Remark 4.5. The previous result holds if A has purely imaginary eigenvalues (including zero) even if $A \notin \mathbf{Skew}_n$.

From the previous result one can obtain a reordered Jordan normal form with the same nilpotent part as in the case of Proposition 4.1. Denote by $\hat{D} = D - \check{D}$. Then $\hat{D}$ and $\check{D}$ are block diagonal and commute. Since $A \in \mathbf{Skew}_n$, $\hat{D}$ is semisimple. The change $\hat{L}$ that reduces D to the reordered Jordan normal form, see 7, indeed reduces $\check{D}$ to the same form changing the diagonal blocks $\Lambda_{k,\ell}$ by $\check{\Lambda}_{k,\ell}$. This is because $D_{i,i} = (\hat{D}_{i,i} + \alpha_i I_{n_i}) + (\check{D}_{i,i} - \alpha_i I_{n_i})$ is the Jordan-Chevalley decomposition [20] of $D_{i,i}$, and hence $(\hat{L}_{i,i}\hat{D}_{i,i}\hat{L}^{-1}_{i,i} + \alpha_i I_{n_i}) + (\hat{L}_{i,i}\check{D}_{i,i}\hat{L}^{-1}_{i,i} - \alpha_i I_{n_i})$ is the Jordan-Chevalley decomposition of reordered Jordan normal form $\hat{L}_{i,i}D_{i,i}\hat{L}^{-1}_{i,i}$. By uniqueness of such a decomposition, the nilpotent part of the Jordan-Chevalley decomposition of the reordered Jordan normal form is $\hat{L}_{i,i}\check{D}_{i,i}\hat{L}^{-1}_{i,i} - \alpha_i I_{n_i}$.

The main result of this section is the following theorem that characterizes the homo/heteroclinic matrices to equilibria.

Theorem 4.2. *Let $X_0 \in \mathbf{H}^\star_n$. There exist matrices $A, B \in \mathcal{M}_{n,n}$ such that $A \in \mathbf{Skew}_n$, $Spec(B) \subset \mathbb{R}$, $AB = BA$ and $X_0 = A + B$ if, and only if, there exist equilibrium matrices $X_1, X_2 \in \mathbf{H}_n$ such that $\alpha(X_0) = \{X_1\}$ and $\omega(X_0) = \{X_2\}$.*

Proof: Suppose that $X_0 = A + B$, with $A \in \mathbf{Skew}_n$, $\operatorname{Spec}(B) \subset \mathbb{R}$, $AB = BA$. Then $e^{tX_0} = e^{tA}e^{tB}$ and, by Prop. 4.2, $e^{tX_0} = e^{tA}T\hat{L}^{-1}e^{t\check{D}}\hat{L}T^{-1}$. If $\check{Q}(t)\check{R}(t)$ is the QR factorization of e^{tB} then

$$X(t) = Q(t)^\top(A+B)Q(t) = \check{Q}(t)^\top e^{-tA}(A+B)e^{tA}\check{Q}(t) = \check{Q}(t)^\top(A+B)\check{Q}(t).$$

By Appendix C (see also the final remark), and proceeding again as in Theorem 4.1, we have that

$$e^{tB} = Q_2R_2\bar{L}(t)\bar{R}(t)U,$$

where $\bar{L}(t) \in \mathbf{BLT}^n_{n_1,\dots,n_m}$ is such that $\operatorname{diag}(\bar{L}(t)) = (\bar{L}_{1,1}(t),\dots,\bar{L}_{m,m}(t))$ where $\bar{L}_{j,j}(t) \in \mathbf{BLT}^{n_j}_{r_{1,j},\dots,r_{d_j,j}}$, $\lim_{t\to\infty}\bar{L}(t) = I_n$, and Q_2R_2 is the QR-factorization of $T\hat{L}^{-1}P$. By equation (18) and using the expressions in Lemma C.3 relating $\overline{R}_{j,j}(t)$ and $\overline{\overline{R}}_{j,j}(t)$, we have that

$$(\overline{R}_{i,i}(t))_{j,j} = e^{t\alpha_i}t^{s_i-j}(R_0)_{j,j}\hat{L}_{j,j}(I_{r_{i,j}} + \mathcal{O}(t^{-1})), \qquad 1 \le j \le d_i.$$

Now, we have that the QR factorization $R_2\overline{L}(t) = Q_3(t)R_3(t)$ satisfies that $Q_3(t) \to I_n$ and $R_3(t) \to R_3^\infty \in \mathbf{T}_n$, when $t \to \infty$. Then

$$e^{tB} = Q_2Q_3(t)R_3(t)\bar{R}(t)U.$$

Finally, the orthogonal matrix of the QR factorization of $R_3(t)\bar{R}(t)U$ is simply $\operatorname{Diag}(\overline{Q}_1(t),\dots,\overline{Q}_s(t))$, because $R_3(t)\bar{R}(t)U$ is block upper triangular. If we write

$$\begin{aligned}\operatorname{diag}(R_3(t)) &= ((R_3)_{1,1}(t),\dots,(R_3)_{m,m}(t)), \qquad \text{and}\\ \operatorname{diag}(R_3)_{i,i}(t) &= (((R_3)_{i,i})_{1,1}(t),\dots,((R_3)_{i,i})_{d_i,d_i},\end{aligned}$$

to get one of these orthogonal matrices we have to compute the QR factorization of

$$((R_3)_{i,i})_{j,j}(t)(R_0)_{j,j}\hat{L}_{j,j}(I_{r_{i,j}} + \mathcal{O}(t^{-1}))(U_{i,i})_{j,j}.$$

As this matrix has a limit when $t \to \infty$, we have that $\operatorname{Diag}(\overline{Q}_1(t),\dots,\overline{Q}_s(t))$ has a limit and, therefore, $\check{Q}(t)$ has a limit when $t \to \infty$. Hence, $X(t)$ has limit when $t \to \infty$.

The other implication follows directly from Remark 4.1 item 2. $\square$

Corollary 4.3. *Let $X_0 \in \mathbf{H}_n$. Then, either both ω and α-limit sets of X_0 are singletons (hence formed by an equilibrium matrix) or they do not contain equilibrium matrices.*

Proof: Suppose that there exists an equilibrium matrix Y such that $Y \in \omega(X_0)$. Then, there exists $Q \in \mathbf{O}_n$ such that $X_0 = Q^TYQ$. By Remark 3.1 item 1, $Y = A + R$ where $A \in \mathbf{Skew}_n$ and $R \in \mathbf{T}_n$ commute. Then, $X_0 = Q^TAQ + Q^TRQ$.

If $X_0 \in \mathbf{H}_n^\star$, Theorem 4.2 implies that $\omega(X_0) = \{Y\}$ and there exists an equilibrium matrix Z such that $\alpha(X_0) = Z$.

If $X_0 \in \mathbf{H}_n \setminus \mathbf{H}_n^\star$, by Remark 4.2, one has $Y_{j,j} \in \omega((X_0)_{j,j})$ and $X_{i,j}(t) = Q_i^\top(t)(X_0)_{i,j}Q_j(t)$, $1 \le i \le j \le m$. One has:

- By definition of ω-limit and because $Q_i(t)$ are bounded, there exists a sequence $(t_k)_{k\ge0} \to \infty$ as $k \to \infty$ such that $X_{i,j}(t_k) \to Y_{i,j}$ and $Q_i(t_k) \to \tilde{Q}_i$. Hence, $(X_0)_{i,j} = \tilde{Q}_iY_{i,j}\tilde{Q}_j^\top$.
- Since Y is an equilibrium matrix, so it is $Y_{j,j}$. Hence, we write $Y_{j,j} = A_{j,j} + R_{j,j}$, where $Y_{j,j} = A_{j,j} + R_{j,j}$, $A_{j,j} \in \mathbf{Skew}_{n_j}$ and $R_{j,j} \in \mathbf{T}_{n_j}$. Moreover, $Y_{i,j}A_{j,j} = A_{i,i}Y_{i,j}$ for $i \ne j$.

- Let $\tilde{A}_{j,j} = \tilde{Q}_j A_{j,j} \tilde{Q}_j^\top$. One can check that $(X_0)_{i,j}\tilde{A}_{j,j} = \tilde{A}_{i,i}(X_0)_{i,j}$ for $i \neq j$.

- From the proof of Theorem 4.2 one has $Q_j(t) = e^{t\tilde{A}_{j,j}}\check{Q}_j(t)$ where $\check{Q}(t)$ has limit when $t \to \infty$.

Then, for $i \neq j$, $X_{i,j}(t) = Q_i^\top(t)X_{i,j}(0)Q_j(t) = \check{Q}_i^\top(t)(X_0)_{i,j}\check{Q}_j(t)$ also has limit when $t \to \infty$. This proves the statement. □

4.3.1 The case of normal, orthogonal and symmetric matrices

If $X_0 \in \mathbf{H}_n$ is a normal matrix then $X_0 = A + S$, where $A \in \mathbf{Skew}_n$, $S \in \mathbf{Sym}_n$ and $AS = SA$. Hence, if $X_0 \in \mathbf{H}_n^\star$, by Theorem 4.2, both $\alpha(X_0)$ and $\omega(X_0)$ only contain a single equilibrium matrix. Moreover, as $X(t) = Q(t)^\top X_0 Q(t)$, then $\alpha(X_0)$ and $\omega(X_0)$ are normal matrices. In this case, we can say more about the sets $\alpha(X_0)$ and $\omega(X_0)$.

Note that if $X_0 \in \mathbf{H}_n^\star$ is a normal matrix, and we follow the notation of Section 4, then $\operatorname{Spec} X_0 = \bigcup_{j=1}^m \mathcal{R}_j$, where $\mathcal{R}_j = \{\lambda \in \operatorname{Spec}(X_0), \operatorname{Re}\lambda = \alpha_j\}$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_m$, and all the eigenvalues of X_0 are simple.

Theorem 4.3. *Let $X_0 \in \mathbf{H}_n^\star$ a normal matrix. Then $\omega(X_0) = \{Y\}$, where $Y = \operatorname{Diag}(Y_{1,1}, \ldots, Y_{m,m}) \in \mathbf{BD}^n_{n_1,\ldots,n_m}$, $Y_{jj} - \alpha_j I_{n_j} \in \mathbf{Skew}_{n_j} \cap \mathbf{H}^\star_{n_j}$, $\operatorname{Spec} Y_{j,j} = \mathcal{R}_j$ and $n_j = \#\mathcal{R}_j$.*

Proof: We know that $\omega(X_0) = \{Y\}$ where Y is an equilibrium matrix. By Theorems 3.1 and 4.1, we have that $Y \in \mathbf{BUT}^n_{n_1,\ldots,n_m}$ and $Y_{jj} - \alpha_j I_{n_j} \in \mathbf{Skew}_{n_j} \cap \mathbf{H}^\star_{n_j}$. As Y is normal, then $Y \in \mathbf{BD}^n_{n_1,\ldots,n_m}$, see [32] for example. □

Remark 4.6. If $X_0 \in \mathbf{H}_n \setminus \mathbf{H}_n^\star$ is normal, then $X_0 \in \mathbf{BD}^n_{n_1,\ldots,n_m}$ and one can apply Theorem 4.3 to each block. Then $\omega(X_0) = \{Y\}$ where Y is also an equilibrium matrix.

In particular, from the previous theorem it follows that the QR-flow converges for orthogonal matrices.

Corollary 4.4. *Let $X_0 \in \mathbf{O}_n \cap \mathbf{H}_n^\star$ (hence $|\alpha_i| \leq 1$). Then $\omega(X_0) = \{Y\}$, where $Y = \operatorname{Diag}(Y_{1,1}, \ldots, Y_{m,m}) \in \mathbf{BD}^n_{n_1,\ldots,n_m}$. Moreover, if $\alpha_1 = 1$ then $n_1 = 1$ and $Y_{1,1} = (1)$, if $\alpha_m = -1$ then $n_m = 1$ and $Y_{m,m} = (-1)$, and if $|\alpha_j| < 1$ then $n_j = 2$ and*

$$Y_{j,j} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}, \qquad \alpha_j^2 + \beta_j^2 = 1.$$

Also, from Theorem 4.3 we recover the well-known result concerning the convergence for symmetric matrices.

Corollary 4.5. *Let $X_0 \in \mathbf{Sym}_n \cap \mathbf{H}_n$. Then $\omega(X_0) = \{Y\}$, where Y is a diagonal matrix.*

Remark 4.7. The convergence of the QR-flow for normal matrices can be extended to the general (not necessarily upper Hessenberg) case. Indeed, it can be seen as a gradient flow on suitable invariant manifold of normal matrices with respect to some adapted Riemannian metric [31]. The convergence for normal matrices also holds in the case of the QR iteration [12]. Note that in our setting, we have given also explicit estimates on the speed of convergence.

5 Low dimensional cases

In this section we consider the QR-flow restricted to $\mathbf{H}_k$ for $k = 2, 3$ and 4, and we illustrate the previous results concerning the equilibria, their linear stability and the asymptotic behavior of the QR-flow described in Sections 3 and 4.

5.1 The QR-flow restricted to $\mathbf{H}_2$

The equations $X' = [X, k(X)]$ are simply

$$x'_{11} = x_{21}\left(x_{12} + x_{21}\right), \qquad x'_{12} = x_{21}\left(x_{22} - x_{11}\right), \qquad x'_{21} = x'_{12}, \qquad x'_{22} = -x'_{11}. \tag{20}$$

As follows from Theorem 3.1, the set of equilibria is $\mathbf{T}_2 \cup \{A + \lambda I, A \in \mathbf{Skew_2}\}$. If $X \in \mathbf{T}_2$ is an equilibrium matrix, the eigenvalues of $D\mathcal{F}(X)$ are 0 (multiplicity three) and $x_{22} - x_{11}$. If $x_{11} = x_{22}$ then we have a unique eigenvalue 0 with geometric multiplicity equals to 3 if $x_{12} \neq 0$ and equals to 4 if $x_{12} = 0$. Therefore, X has a one-dimensional stable invariant manifold if $x_{11} > x_{22}$. Concretely, the matrix of $D\mathcal{F}(X)$ is

$$\begin{pmatrix} x_{22} - x_{11} & 0 & 0 & 0 \\ x_{12} & 0 & 0 & 0 \\ 2(x_{22} - x_{11}) & 0 & 0 & 0 \\ -x_{12} & 0 & 0 & 0 \end{pmatrix},$$

if expressed in the basis $\{e_1 e_2^\top - e_2 e_1^\top,\ e_1 e_2^\top,\ e_1 e_1^\top,\ e_2 e_2^\top\}$ of $\mathbb{R}^4$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. On the other hand, if

$$X = \begin{pmatrix} x_{11} & -x_{21} \\ x_{21} & x_{11} \end{pmatrix}, \qquad x_{21} \neq 0,$$

the matrix of $D\mathcal{F}(X)$ is

$$\begin{pmatrix} 0 & 0 & -2x_{21} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{21} \\ 0 & 0 & -4x_{21} & 0 \end{pmatrix},$$

if expressed in the basis $\{e_1 e_2^\top - e_2 e_1^\top,\ e_1 e_1^\top + e_2 e_2^\top,\ e_1 e_1^\top - e_2 e_2^\top,\ e_1 e_2^\top\}$ of $\mathbb{R}^4$. In particular, $D\mathcal{F}(X)$ has a two-dimensional kernel and eigenvalues $\pm 2\mathrm{i}\, x_{21}$.

The system (20) has three functionally independent first integrals

$$I_1 = x_{11} + x_{22}, \qquad I_2 = x_{12} - x_{21}, \qquad I_3 = (x_{11} - x_{22})^2 + (x_{12} + x_{21})^2,$$

hence it is integrable. We can use these first integrals to reduce the dimension of the phase space. If we fix $I_1 = d$ and $I_2 = c$ the reduced system is

$$x' = (y - c)(2y - c), \qquad y' = (y - c)(d - 2x), \tag{21}$$

where $x = x_{11}$ and $y = x_{12}$. It has the line $y = c$ of fixed points and the fixed point $(d/2, c/2)$, if $c \neq 0$. The restriction of I_3 to $I_1 = d, I_2 = c$ provides $J_3 = (x - d/2)^2 + (y - c/2)^2$ as a first integral of the reduced system. This was also obtained in [4]. The fixed points on $y = c$

correspond to the triangular equilibrium matrices A_x and, if $c \neq 0$, the system has also the equilibrium matrix C, where

$$A_x = \begin{pmatrix} x & c \\ 0 & d-x \end{pmatrix}, \qquad C = \begin{pmatrix} d/2 & c/2 \\ -c/2 & d/2 \end{pmatrix}.$$

We note that $A_{d/2}$ has a double eigenvalue with geometric multiplicity 1, if $c \neq 0$ and multiplicity two if $c = 0$. In Fig. 3 we sketch the phase portrait of the reduced system (21) for $c \neq 0$.

As follows from Corollary 4.1, if the matrix X has real eigenvalues then $X(t)$ converges to an upper triangular matrix. When the eigenvalues are not real then $(X - \lambda I)^2 + \mu^2 I = 0$, being $\mu \neq 0$, and

$$e^{tX} = e^{\lambda t} e^{t(X-\lambda I)} = e^{\lambda t} \left[\cos(\mu t) I + \sin(\mu t) \left(\frac{1}{\mu} X - \frac{\lambda}{\mu} I \right) \right].$$

From Prop. 2.1 one has $e^{tX} = Q(t)R(t)$ and $X(t) = Q(t)^\top X Q(t)$. If we denote $(X - \lambda I)/\mu = (a_{ij})_{1 \leq i,j \leq 2}$ then

$$Q(t) = \frac{1}{\sqrt{v(t)^\top A_1 v(t)}} \left(\cos(\mu t) I_2 + \sin(\mu t) A_2 \right),$$

where $v(t) = (\sin(\mu t), \cos(\mu t))^\top$,

$$A_1 = \begin{pmatrix} 1 & a_{11} \\ a_{11} & a_{11}^2 + a_{21}^2 \end{pmatrix}, \qquad \text{and} \qquad A_2 = \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{pmatrix}.$$

Hence,

$$X(t) = Q(t)^\top X Q(t) = \frac{1}{v(t)^\top A_1 v(t)} (\cos(\mu t) I_2 + \sin(\mu t) A_2^\top) X (\cos(\mu t) I_2 + \sin(\mu t) A_2),$$

and we see that $X(t)$ is a rational function of $\cos^2(\mu t)$, $\sin^2(\mu t)$ and $\cos(\mu t)\sin(\mu t)$. In particular, if X is not an equilibrium matrix, then $X(t)$ is a periodic function of period π/μ.

Remark 5.1. A similar argument to obtain the period will be use for $\mathbf{H}_3$. However, for $\mathbf{H}_2$, the period of $X(t)$ can be obtained directly from (21) just introducing polar coordinates (r, θ) centered at $p_C = (d/2, c/2)$. Trivially, r becomes a first integral and $\theta' = -2r \sin\theta + c$ gives period $2\pi/\sqrt{c^2 - 4r^2}$. A point at a distance r from p_C corresponds to a matrix with eigenvalues $d/2 \pm \mathrm{i}\,\mu$, $\mu = \sqrt{c^2 - 4r^2}/2$, that are complex for $r < c/2$.

5.2 The QR-flow restricted to $\mathbf{H}_3$

From the results in Section 3 it follows that there are four types of equilibrium matrices of $X' = [X, k(X)]$, $X \in \mathbf{H}_3$,

$$X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & -x_{32} \\ 0 & x_{32} & x_{22} \end{pmatrix}, \ x_{32} \neq 0,$$

$$X_3 = \begin{pmatrix} x_{11} & -x_{21} & 0 \\ x_{21} & x_{11} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}, \ x_{21} \neq 0, \qquad X_4 = \begin{pmatrix} x_{11} & -x_{21} & 0 \\ x_{21} & x_{11} & -x_{32} \\ 0 & x_{32} & x_{11} \end{pmatrix}, \ x_{21} x_{32} \neq 0.$$

By Theorem 3.2 the eigenvalues of $D\mathcal{F}(X_1)$ are 0 (multiplicity six), $x_{22}-x_{11}$ and $x_{33}-x_{22}$. On the other hand, if $\{e_1,e_2,e_3\}$ denotes the canonical basis of $\mathbb{R}^3$, the matrix of $D\mathcal{F}(X_2)$ expressed in the basis

$$\{e_3e_2^\top - e_2e_3^\top,\ e_1e_1^\top,\ e_2e_2^\top + e_3e_3^\top,\ e_2e_2^\top - e_3e_3^\top,\ e_2e_3^\top,\ e_1e_2^\top,\ e_1e_3^\top,\ e_2e_1^\top - e_1e_2^\top\}$$

of $\mathbb{R}^8$ is

$$\begin{pmatrix} 0 & 0 & 0 & -2x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & -4x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{32} & 2(x_{22}-x_{11}) \\ 0 & 0 & 0 & 0 & 0 & -x_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{22}-x_{11} \end{pmatrix}.$$

Therefore, it has a zero eigenvalue of multiplicity three, and the other five eigenvalues are $\pm 2\mathrm{i}\, x_{32}$, $\pm\mathrm{i}\, x_{32}$ and $x_{22}-x_{11}$. The linear stability properties of the equilibrium matrix X_3 are similar to those of X_2. Finally, $D\mathcal{F}(X_4)$ has non-zero double non-defective[1] eigenvalues $\pm\mathrm{i}\sqrt{x_{21}^2+x_{32}^2}$, see Theorem 3.2, and the zero eigenvalue has multiplicity 4 while the kernel of $D\mathcal{F}(X_4)$ is three-dimensional, see Prop. 3.2.

Concerning the asymptotic behavior of $X(t)$, Corollary 4.1 guarantees that it converges to a matrix of $\mathbf{T}_3$ whenever X has real eigenvalues. If not, we have that X has one real eigenvalue λ and two conjugate complex eigenvalues $\alpha \pm i\beta$, with $\beta \neq 0$. By Corollary 4.2, if $\lambda \neq \alpha$, the matrix $X(t)$ converges (in Parlett sense) to a reduced matrix when $t \to \infty$. If $\omega(X)$ is not an equilibrium matrix then $X(t)$ tends to a matrix in $\mathbf{BUT}^3_{1,2}$ with diagonal blocks λ and a 2×2 block of period π/β.

Finally, we consider the case $\lambda=\alpha$. We have that $e^{tX} = e^{\alpha t}A_1 + e^{\alpha t}[\cos(\beta t)A_2 + \sin(\beta t)A_3]$, where

$$A_1 = \frac{1}{\beta^2}[X^2 - 2\alpha X + (\alpha^2+\beta^2)I], \quad A_2 = I - A_1, \quad \text{and} \quad A_3 = \frac{1}{\beta}[X-\alpha I].$$

It is clear that A_1, A_2 and A_3 are non-zero matrices. This implies, by a similar argument to the one used in Section 5.1, that $X(t) = Q(t)^\top X Q(t)$ and it is $2\pi/\beta$ periodic unless X is an equilibrium matrix. In particular, this means that, if the matrix X is unreduced, all the coeficients of the subdiagonal of $X(t)$ do not converge to zero, because they are periodic.

Example 5.1. Fix $\epsilon_0 > 0$ and, for $|\epsilon| < \epsilon_0$, let A_ϵ and B be the matrices

$$A_\epsilon = \begin{pmatrix} 2+\epsilon & 0 & 0 \\ 0 & -9 & 15.25 \\ 0 & -8 & 13 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 5 & 9 \\ 8 & 8 & 9 \\ 5 & 1 & 0 \end{pmatrix}.$$

Denote by $X_{0,\epsilon}$ the matrix obtained by the reduction to $\mathbf{H}_3$ of $BA_\epsilon B^{-1}$ (using Householder's algorithm). The theoretical discussion of this section implies that:

- For $\epsilon = 0$ one has $\mathrm{Spec}\,(X_{0,0}) = \{2, 2+\mathrm{i}\,, 2-\mathrm{i}\,\}$ and the QR-flow does not converge, see Corollary 4.2. Since $\lambda=\alpha=2$ and $\beta=1$, the ω-limit of X_0 is a 2π-periodic orbit (and the α-limit of $X_{0,0}$ coincides with the ω-limit of $X_{0,0}$).

[1] The (algebraic) multiplicity of a defective eigenvalue is larger than its geometric multiplicity.

- For $\epsilon \neq 0$, since $\lambda = 2 + \epsilon \neq \alpha = 2$ and $\beta = 1$, the ω-limit of $X_{0,\epsilon}$ is a periodic orbit with period π (and so is the α-limit of $X_{0,\epsilon}$, $\epsilon \neq 0$). In this case, one of the subdiagonal elements decays to zero, see Lemma 4.1.

In Fig. 4 left we show the behavior of the coefficient $x_{1,1}$ as a function of t for values of $\epsilon = -10^{-2}, 0$ and 10^{-2}. One clearly sees the period 2π of the $\epsilon = 0$ in contrast with the period π shown for $\epsilon < 0$ (for $\epsilon > 0$ this coefficient tends to 2). A 3-dimensional projections of the periodic orbit obtained as ω-limit in each of the previous cases considered is shown in Fig. 4 right. The behavior of the coefficients $x_{i,j}$ as a function of t is the following:

1. For $\epsilon = 0$, all the coefficients $x_{i,j}$ are 2π-periodic in t.

2. The coefficient $x_{1,3}$ is 2π-periodic for all values of ϵ.

3. For $\epsilon > 0$: $x_{1,1} \to$ *ctant*, $x_{2,1} \to 0$, $x_{1,2}$ asymptotically becomes 2π-periodic, and the coefficients $x_{2,2}$, $x_{2,3}$, $x_{3,2}$ and $x_{3,3}$ asymptotically become π-periodic. Accordingly, the periodic orbit obtained as ω-limit for $\epsilon > 0$ (in green) is π-periodic.

4. For $\epsilon < 0$: $x_{3,2} \to 0$, $x_{3,3} \to$ *ctant*, $x_{2,3}$ asymptotically becomes 2π-periodic, and the coefficients $x_{1,1}$, $x_{1,2}$, $x_{2,1}$ and $x_{2,2}$ asymptotically become π-periodic. Accordingly, the periodic orbit obtained as ω-limit for $\epsilon < 0$ (in red) is π-periodic.

We remark that the description of the 8-dimensional phase space of the QR-flow on $\mathbf{H}_3$ is far from trivial. We refer to Section 6 for an illustration.

5.3 The QR-flow restricted to $\mathbf{H}_4$

Theorem 3.1 implies that the structure of an equilibrium matrix $X = (x_{ij}) \in \mathbf{H}_4$ is one of the following eight:

1. If $X \in \mathbf{H}_4^\star$ then $X = \lambda I_4 + A$ where $\lambda \in \mathbb{R}$ and $A \in \mathbf{Skew}_4 \cap \mathbf{H}_4$.

2. If $x_{32} = 0$ and $x_{21}x_{43} \neq 0$,

$$X = \begin{pmatrix} \lambda_1 I_2 + A & \tilde{B} \\ 0 & \lambda_2 I_2 + C \end{pmatrix}, \ A, C \in \mathbf{Skew}_2, A, C \neq 0,\ \lambda_i \in \mathbb{R}, i = 1, 2, 3.$$

   By Remark 3.1, item 4., if the eigenvalues of A and C are different then $\tilde{B} = 0$. Otherwise, we have two possibilities:

$$X = \begin{pmatrix} \lambda_1 I_2 + A & \mu Q_1 \\ 0 & \lambda_2 I_2 + A \end{pmatrix}, \quad \text{or} \quad X = \begin{pmatrix} \lambda_1 I_2 + A & \mu Q_2 \\ 0 & \lambda_2 I_2 - A \end{pmatrix},$$

   where $Q_1 \in \mathcal{M}_{2,2}$ is a rotation and $Q_2 \in \mathcal{M}_{2,2}$ is a reflection.

3. If $x_{21} = 0$ and $x_{32}x_{43} \neq 0$ then

$$X = \begin{pmatrix} \lambda_1 & v^\top \\ 0 & \lambda_2 I_3 + A \end{pmatrix}, \ A \in \mathbf{Skew}_3 \cap \mathbf{H}_3,\ \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } Av = 0.$$

4. If $x_{43} = 0$ and $x_{21}x_{32} \neq 0$ then

$$X = \begin{pmatrix} \lambda_1 I_3 + A & v \\ 0 & \lambda_2 \end{pmatrix}, A \in \mathbf{Skew}_3 \cap \mathbf{H}_3,\ \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } Av = 0.$$

5. If $x_{43} \neq 0$ and $x_{21} = x_{32} = 0$ then

$$X = \begin{pmatrix} R & 0 \\ 0 & \lambda I_2 + A \end{pmatrix}, \quad R \in \mathbf{T}_2, \ \lambda \in \mathbb{R}, A \in \mathbf{Skew}_2.$$

6. If $x_{21} \neq 0$ and $x_{32} = x_{43} = 0$ then

$$X = \begin{pmatrix} \lambda I_2 + A & 0 \\ 0 & R \end{pmatrix}, R \in \mathbf{T}_2, \ \lambda \in \mathbb{R}, \ A \in \mathbf{Skew}_2.$$

7. If $x_{32} \neq 0$ and $x_{21} = x_{43} = 0$ then

$$X = \begin{pmatrix} r_{11} & 0^\top & r_{12} \\ 0 & \lambda I_2 + A & 0 \\ 0 & 0^\top & r_{22} \end{pmatrix}, \ \lambda, r_{ij} \in \mathbb{R}, 1 \leq i \leq j \leq 2, \ A \in \mathbf{Skew}_2.$$

8. If $x_{21} = x_{32} = x_{43} = 0$ then $X \in \mathbf{T}_4$.

Now, we want to find the eigenvalues of the equilibrium matrices. We will consider the previous cases. Note that the linear operator $D\mathcal{F}$ acts on $\mathbf{H}_4$ which has dimension 13.

The eigenvalues of $D\mathcal{F}(X)$ for an equilibrium matrix of the first case are obtained as a consequence of Prop. 3.1, Prop. 3.2 and Prop. A.2. In particular, in this case, one has $\dim(\operatorname{Ker} D\mathcal{F}(X)) = 4$ and multiplicity of the zero eigenvalue equals five. The other eigenvalues are simple and given by $\pm(\mu_1 - \mu_2)\mathrm{i}$, $\pm(\mu_1 - \mu_2)\mathrm{i}$, $\pm 2\mu_1\mathrm{i}$ and $\pm 2\mu_2\mathrm{i}$, where $\pm\mu_i\mathrm{i}$, $i = 1, 2$, are the eigenvalues of A.

For an equilibrium matrix of the second case, denote by $\pm\mu_A\mathrm{i}$ and $\pm\mu_C\mathrm{i}$ the eigenvalues of A and C, respectively. By Theorem 3.2 we have that

$$\operatorname{Spec}(D\mathcal{F}(X)) = \{\pm 2\mu_A\mathrm{i}, \pm 2\mu_C\mathrm{i}, 0^4\} \cup \{(\pm\mu_A \pm \mu_C)\mathrm{i}\} \cup \{\lambda_2 - \lambda_1\}.$$

If $B = 0$, then $\dim(\operatorname{Ker} D\mathcal{F}(X)) \geq 4$. See Example 3.1 for a case where one has $\dim(\operatorname{Ker} D\mathcal{F}(X)) = 5$.

Consider X an equilibrium matrix of case 3. The matrix A has eigenvalues 0 and $\pm\mu_A\mathrm{i}$. Following the notation of Theorem 3.2 one has $H_1 = 0$ and $H_2 = A$. Hence,

$$\operatorname{Spec}(D\mathcal{F}(X)) = \{\pm\mu_A\mathrm{i}, \pm 2\mu_A\mathrm{i}, 0^5\} \cup \{0, \pm\mu_A\mathrm{i}\} \cup \{\lambda_2 - \lambda_1\}.$$

Note that the eigenvalue 0 has multiplicity at least six, and $\pm\mu_A\mathrm{i}$ has multiplicity two.

The eigenvalues for an equilibrium matrix of case 4 are the same of the previous case. Consider now an equilibrium matrix X of case 5. According to Theorem 3.2 one has $H_1 = 0$, $H_2 = 0$ and $H_3 = A$. Let $\operatorname{Spec} R = \{r_{11}, r_{22}\}$. Then,

$$\operatorname{Spec}(D\mathcal{F}(X)) = \{\pm\mu_A\mathrm{i}, 0^4\} \cup \{0, \pm\mu_A\mathrm{i}, \pm\mu_A\mathrm{i}\} \cup \{r_{22} - r_{11}, \lambda - r_{22}\}. \tag{22}$$

Cases 6 and 7 are similar to case 5. Concretely, for case 6, $\operatorname{Spec}(D\mathcal{F}(X))$ is the same as in (22) but replacing $\lambda - r_{22}$ by $\lambda - r_{11}$. In case 7, just replace $\{r_{22} - r_{11}, \lambda - r_{22}\}$ by $\{\lambda - r_{11}, r_{22} - \lambda\}$ in (22). Note that in the cases 5,6 and 7, the multiplicity of the eigenvalue $\pm\mu_A\mathrm{i}$ is three, and the multiplicity of 0 is larger or equal than five.

Finally, for an equilibrium matrix X of case 8, one has $k(A_{i,i}) = 0$, $\lambda_i = x_{ii}$, $1 \leq i \leq 4$. Theorem 3.2 implies

$$\operatorname{Spec}(D\mathcal{F}(X)) = \{0^4\} \cup \{0^6\} \cup \{\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \lambda_4 - \lambda_3\},$$

and hence the multiplicity of the eigenvalue 0 is larger or equal than 10.

Concerning the ω-limit of an arbitrary $X_0 \in \mathbf{H}_4^\star$ new possibilities appear. First, the ω-limit can be a non-normal equilibrium matrix which is not triangular. Second, $\omega(X_0)$ can be contained in a 2-dimensional torus with two different fundamental frequencies. Indeed, the frequencies are integer multiples of the imaginary parts, β_1 and β_2, of the eigenvalues of X_0. This follows since $e^{tX_0} = Q(t)R(t)$ where $Q(t)$ is a function of $\cos(\beta_i t)$, $\sin(\beta_i t)$, $i = 1, 2$. Hence, also $X(t) = Q(t)^\top X_0 Q(t)$ is a function of $\cos(\beta_i t)$, $\sin(\beta_i t)$, $i = 1, 2$, and the fundamental frequencies are multiples of β_1 and β_2. If $\beta_1/\beta_2 \in \mathbb{Q}$ then $\omega(X_0)$ is a periodic orbit. On the other hand, if $\beta_1/\beta_2 \in \mathbb{R} \setminus \mathbb{Q}$ then $\omega(X_0)$ is a 2-dimensional torus. We illustrate these facts in the following example.

Example 5.2. Let R_φ denote the rotation by angle $\varphi \in [0, 2\pi)$ and

$$X_0 = \begin{pmatrix} R_\alpha & B \\ 0 & R_\beta \end{pmatrix} \in \mathbf{BUT}^4_{2,2} \cap (\mathbf{H}_4 \setminus \mathbf{H}_4^\star), \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}.$$

We consider three cases, the values of the angles α and β determining X_0 in each case are: (i) $\alpha = \pi/2$, $\beta = \pi/4$, (ii) $\alpha = \pi/2$, $\beta = \pi/2$ and (iii) $\alpha = \pi/4$, β such that $\sin(\beta) = \sin(\pi/4)/2$ (i.e. $\beta \approx 0.361367$). The ω-limit sets are embedded in a 4D ambient space because $\operatorname{diag}(X_0) = (R_\alpha, R_\beta)$ is fixed by the flow. In case (i) the ratio of the imaginary part of the eigenvalues is $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, hence the orbit lies on a 2D torus. In cases (ii) and (iii) the ratio of the imaginary part of the eigenvalues is 1 and 1/2, respectively, and the orbit is periodic (therefore coincides with its ω-limit). The three cases are illustrated in Fig. 5.

For $X_0 \in \mathbf{H}_4^\star$ the situation is similar (but geometrically more involved). For example, consider the matrices

$$X_0^{\mathbb{Q}} = \begin{pmatrix} \frac{2409}{214} & \frac{-4538113}{761091} & \frac{16512245}{421794} & \frac{3575}{214} \\ \frac{7113}{428} & \frac{-14282597}{1522182} & \frac{643711321}{10966644} & \frac{9949}{428} \\ 0 & \frac{-1218516}{5621641} & \frac{-49518619}{20250711} & \frac{-1704}{2371} \\ 0 & 0 & \frac{1131595315}{218846043} & \frac{4879}{8541} \end{pmatrix}, \quad X_0^{\mathbb{R}\setminus\mathbb{Q}} = \begin{pmatrix} \frac{1497}{214} & \frac{-1593017}{465771} & \frac{39100385}{2394018} & \frac{2359}{214} \\ \frac{4353}{428} & \frac{-5047117}{931542} & \frac{121304017}{4788036} & \frac{6269}{428} \\ 0 & \frac{-532004}{2105401} & \frac{-9676375}{5410779} & \frac{-1104}{1451} \\ 0 & 0 & \frac{141421715}{41716323} & \frac{787}{3729} \end{pmatrix}$$

One has $\operatorname{Spec} X_0^{\mathbb{Q}} = \{\pm\mathrm{i}, \pm 2\mathrm{i}\}$ and $\operatorname{Spec} X_0^{\mathbb{R}\setminus\mathbb{Q}} = \{\pm\mathrm{i}, \pm\sqrt{2}\mathrm{i}\}$. Accordingly, the orbit of $X_0^{\mathbb{Q}}$ is a periodic orbit on a 2-dimensional torus, while the orbit of $X_0^{\mathbb{R}\setminus\mathbb{Q}}$ densely fills up a 2-dimensional torus. In both cases the 2-dimensional torus is embedded in a 13-dimensional phase space. See Fig. 6 for some projections of the corresponding orbits.

6 On the phase space complexity: an illustration

The description of the phase space of the QR-flow is, in general, quite involved. Consider the QR-flow on $\mathbf{H}_3$, which is the lowest dimension with non-trivial dynamics. Even in this case a global description of the phase space seems to be difficult. Below, we numerically investigate semi-global aspects of the phase space around some concrete equilibrium matrices and their homo/heteroclinic connections. Concretely, we consider the four equilibrium matrices

$$X_{\pm,\pm} = \begin{pmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which are orthogonally similar. We want to compute the homoclinic and heteroclinic orbits to them. To this end, we consider the orthogonal matrices

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ Q_4 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$Q_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ Q_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ Q_7 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ Q_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the initial matrices $X_i = Q_i^\top X_{+,+} Q_i \in \mathbf{H}_n$, $1 \le i \le 8$. Note that since the eigenvalues of X_i are real then $\omega(X_0) \in \mathbf{T}_3$. One can check that $\{R \in \mathbf{T}_3,\ \exists Q \in \mathbf{O}_3,\ R = Q^\top X_{+,+} Q\} = \{X_{\pm,\pm}\}$ and hence the orbits of X_i tend to one of the matrices $X_{\pm,\pm}$. Indeed, each of the initial matrices X_i give a different homoclinic or heteroclinic orbit to $X_{\pm,\pm}$. See Fig. 7.

The matrices $X_i \in$, $1 \le i \le 4$, are reduced. Their orbits are heteroclinic between different equilibrium matrices. On the other hand, the homoclinic orbits correspond to the orbits of the unreduced matrices X_i, $5 \le i \le 8$.

We remark that the points $X_{\pm,\pm}$ are complete parabolic points with 8-dimensional center manifold carrying on a non-trivial dynamics only within a 2-dimensional subspace. On the other hand, one has $\dim(\mathrm{Ker}\,(D\mathcal{F}(X_{+,+}))) = 3$ and

$$K_1 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad K_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad K_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

form a basis of $\mathrm{Ker}\,(D\mathcal{F}(X_{+,+}))$. We consider an unfolding of the form $X = X_{+,+} + \eta_1 K_1 + \eta_2 K_2$, $\eta_1, \eta_2 \ge 0$. For $\eta_1, \eta_2 > 0$, the eigenvalues λ_1, λ_2 and λ_3 of X are different and real. Assume $\lambda_1 > \lambda_2 > \lambda_3$. Near each one of the equilibrium matrices $X_{\pm,\pm}$ there are, for $\eta_1, \eta_2 > 0$, six equilibrium upper triangular matrices according to the possible orders of the eigenvalues displayed in the diagonal. Let us introduce the following notation. We denote by $X_{ijk}^{\pm,\pm}$, where $i, j, k \in \{1,2,3\}$ are distinc indices, the equilibrium matrix with diagonal equal to $(\lambda_i, \lambda_j, \lambda_k)$ located near $X_{\pm,\pm}$.

From Theorem 3.2 we conclude that, beyond having a 6-dimensional center manifold, the linear behavior at the 24 equilibria restricted to the hyperbolic directions is the following:

- the matrices $X_{123}^{\pm,\pm}$ are stable node equilibrium matrices,
- the matrices $X_{321}^{\pm,\pm}$ are unstable node equilibrium matrices, and
- the other 16 equilibrium matrices are of saddle type (with one dimensional stable and unstable manifolds).

For the illustrations we consider $\eta_1 = 0.05$ and $\eta_2 = 0.01$. For these values $\lambda_1 = 0.05$, $\lambda_2 = 0.01$ and $\lambda_3 = -0.06$. In Fig. 8 we see the equilibrium matrices $X_{ijk}^{+;+}$ that are close to the equilibrium matrix $X_{+,+}$. We also display the heteroclinic orbits between them.

In the top right plot of the figure we sketch the local heteroclinic structure. We note that the 1-dimensional stable and unstable invariant manifolds of the saddle points as well as the weakly and strong 1-stable invariant manifolds of the nodes correspond to orbits $X(t)$ associated to reduced upper Hessenberg matrices. Hence, the reduced orbits define the skeleton of the phase space and organize the dynamics.

We remark that all the branches of the invariant manifolds in Fig. 8 which have not been continued correspond to heteroclinic orbits to equilibrium matrices that are located in a neighbourhood of either $X_{+,-}$ or $X_{-,+}$. They play a role in the global structure of the phase space.

7 Conclusions and outlook

We have studied the QR-flow for upper Hessenberg matrices. The linear behavior at equilibria, including the central components, has been determined. We have also provided a complete description of the elements of the α- and ω-limit of any initial matrix. This has been used for characterize the set of matrices for which there is convergence of the QR-flow towards an equilibrium matrix. Also properties of the velocity of convergence towards the limit behavior of the orbit were explicitly derived.

There are many related questions to be investigated, including theoretical aspects but also concrete applications. As a theoretical developments we mention the use of isospectral flows in other settings like, for example, the symplectic setting (where one looks for symplectic isospectral deformations of the matrix) or the most important setting of infinite dimensional linear operators where a dynamical approach description seems much more involved.

Concerning potential applications, we would like to point out that the integration of the variational equations of the QR-flow could be useful for a systematic analysis of bifurcations. Also, a description of semi-global phase space properties seems to be plausible in simple cases, see Section 6. This could be useful for designing block reduction strategies in the computation of the eigenvalues/eigenvectors of matrices using the QR-flow.

Certainly, from a numerical point of view, the QR-flow does not currently present an effective alternative to the discrete analogous algorithms for the computation of the spectra of finite linear operators. However, it has been noticed that, in general, the infinite-dimensional QR-algorithm (IQR) cannot be sped up using shift strategies [31, 18, 8]. One might expect that in the continuous infinite-dimensional QR-flow the step-size adaptation of the numerical integration could be an efficient alternative. Also the results presented here on the finite dimensional case can be useful in a section-like approach to the infinite-dimensional computational problem. The ultimate goal could be to obtain numerical methods for high order approximation (or even validation) of eigenvalues/eigenvectors of finite/infinite dimensional linear operators.

A Properties of the operators $\mathcal{B}_{X,Y}$ and L_X

In this appendix we summarize some properties of the operators $\mathcal{B}_{X,Y}$ and $L_X = B_{X,X}$ introduced in Section 3, see (4). We denote by $\mathcal{M}_{n,m}(\mathbb{C})$ the set of complex matrices of dimension $n \times m$. Concerning the operator $\mathcal{B}_{X,Y}$ we have the following result.

Proposition A.1. *1. If $A \in \mathcal{M}_{n,n}(\mathbb{C})$ and $B \in \mathcal{M}_{m,m}(\mathbb{C})$ are non-singular then*

$$\mathrm{Spec}(\mathcal{B}_{X,Y}) = \mathrm{Spec}(\mathcal{B}_{A^{-1}XA,B^{-1}YB}).$$

2. *Let $A = (A_{ij})_{1\le i\le n, 1\le j\le m} \in \mathcal{M}_{n,m}(\mathbb{C})$ such that $A_{ij} \in \mathcal{M}_{n_i,m_j}(\mathbb{C})$, for all $1 \le i \le n$, $1 \le j \le m$. Consider the set $E_{ij} \subset \mathcal{M}_{n,m}(\mathbb{C})$ of matrices such that $A \in E_{ij}$ iff $A_{k\ell} = 0$ for all $(k,\ell) \neq (i,j)$. One has $\mathcal{M}_{n,m}(\mathbb{C}) = \bigoplus_{i=1}^n \bigoplus_{j=1}^m E_{ij}$. Let $X = \mathrm{Diag}(J_1,\ldots,J_r)$ and $Y = \mathrm{Diag}(\widehat{J}_1,\ldots,\widehat{J}_s)$ where $J_i \in \mathcal{M}_{n_i,n_i}(\mathbb{C})$, $i = 1,\ldots r$, and $\widehat{J}_i \in \mathcal{M}_{m_i,m_i}(\mathbb{C})$, $i = 1,\ldots,s$. Then, one has*
$$\mathcal{B}_{X,Y}(E_{ij}) \subset E_{ij} \quad \textit{and} \quad \mathcal{B}_{X,Y}|E_{ij} = \mathcal{B}_{J_i,\widehat{J}_j}.$$

3. *Let $\mathcal{I} : \mathcal{M}_{n,m}(\mathbb{C}) \to \mathcal{M}_{n,m}(\mathbb{C})$ be the identity. Then $\mathcal{B}_{X,Y} - (\lambda-\mu)\mathcal{I} = \mathcal{B}_{X-\lambda I, Y-\mu I}$ and $\mathcal{B}^k_{X,Y}(H) = \sum_{i=0}^k (-1)^i \binom{k}{i} X^{k-i} H Y^i$.*

4. *Let $X \in \mathcal{M}_{n,n}(\mathbb{C})$ and $Y \in \mathcal{M}_{m,m}(\mathbb{C})$. Then $\mathrm{Spec}(\mathcal{B}_{X,Y}) = \{\gamma \in \mathbb{C},\ \gamma = \lambda - \mu,\ \lambda,\mu \in \mathbb{C},\ \lambda \in \mathrm{Spec}(X),\ \mu \in \mathrm{Spec}(Y)\}$,*

5. *If $X \in \mathcal{M}_{n,n}(\mathbb{C})$ and $Y \in \mathcal{M}_{m,m}(\mathbb{C})$ are diagonalizable then the operator $\mathcal{B}_{X,Y}$ is diagonalizable. Concretely, if $\{v_i, 1 \le i \le n\}$ is a basis of eigenvectors of X and $\{w_j, 1 \le j \le m\}$ is a basis of left eigenvectors of Y, then $\{v_i w_j^\top, 1 \le i \le n, 1 \le j \le m\}$ is a basis of eigenvectors of $\mathcal{B}_{X,Y}$.*

Proof: Properties 2. and 3. can be checked by a direct computation. To prove 1. we note that if $\gamma \in \mathbb{C}$ is an eigenvalue of $\mathcal{B}_{X,Y}$ then there exists a non-zero matrix $H \in \mathcal{M}_{n,m}(\mathbb{C})$ such that $XH - HY = \gamma H$. If we define $\widehat{H} = A^{-1}HB \neq 0$ then $A^{-1}XA\widehat{H} - \widehat{H}B^{-1}YB = A^{-1}XHB - A^{-1}HYB = A^{-1}(XH - HY)B = \gamma\widehat{H}$. Items 4. and 5. follow from the previous ones. □

Concerning the operator L_X the following properties hold.

Proposition A.2. *1. If $X \in \mathbf{Skew}_n$, then one has $L_X(\mathbf{Skew}_n) \subset \mathbf{Skew}_n$ and $L_X(\mathbf{Sym}_n) \subset \mathbf{Sym}_n$.*

2. *Let $X \in \mathbf{H}_n^\star \cap \mathbf{Skew}_n$. Consider $r = n/2$ if n is even, and $r = (n-1)/2$ otherwise. The (simple) eigenvalues of X are of the form $\pm\lambda_j \mathrm{i}$, $1 \le j \le r$, with $0 \le \lambda_1 < \lambda_2 < \ldots < \lambda_r$ and where $\lambda_1 = 0$ only if n is odd.*

   By the previous item, one can consider the restriction of the linear operator L_X to $\mathbf{Sym}_n$. Then,

   (a) *if n is even, L_X has an eigenvalue 0 of multiplicity $n/2$, and $n^2/2$ non-zero simple eigenvalues $\pm\mathrm{i}\,(\lambda_j - \lambda_k)$, $1 \le j < k \le n/2$, $\pm\mathrm{i}\,(\lambda_j + \lambda_k)$, $1 \le j \le k \le n/2$.*

   (b) *if n is odd, L_X has an eigenvalue 0 of multiplicity $(n+1)/2$ and $(n^2-1)/2$ non-zero simple eigenvalues $\pm\mathrm{i}\,(\lambda_j - \lambda_k)$, $1 \le j < k \le (n-1)/2$, $\pm\mathrm{i}\,(\lambda_j + \lambda_k)$, $1 \le j \le k \le (n-1)/2$, $j + k \ge 2$.*

   Moreover, we have in both cases a basis of eigenvectors.

Proof: The first item follows immediately from the definition of L_X. To prove the second item we prove first that if v is an eigenvector of eigenvalue λ of X then v is a left eigenvector of eigenvalue $-\lambda$. Indeed, $v^\top X = (X^\top v)^\top = -(Xv)^\top = -\lambda v^\top$.

Now, suppose first that n is odd. Then $\lambda = 0$ is a simple eigenvalue of X. Denote by $w \in \operatorname{Ker} X$, $w \neq 0$, an eigenvector of eigenvalue $\lambda = 0$. Let v_j be an eigenvector of eigenvalue $\lambda_j \mathrm{i}$, $2 \leq j \leq (n-1)/2$ of X.

Then, applying Prop. A.1, we have that $ww^\top \in \mathbf{Sym}_n$ and $v_j\bar{v}_j^\top + \bar{v}_j v_j^\top \in \mathbf{Sym}_n$, $2 \leq j \leq (n-1)/2$, are linearly independent eigenvectors of eigenvalue zero.

On the other hand, $v_j w^\top + w v_j^\top \in \mathbf{Sym}_n \oplus \mathrm{i}\,\mathbf{Sym}_n$ are eigenvectors of eigenvalue $\lambda_j \mathrm{i}$, $2 \leq j \leq (n-1)/2$, and $v_j\bar{v}_k^\top + \bar{v}_k v_j^\top \in \mathbf{Sym}_n \oplus \mathrm{i}\,\mathbf{Sym}_n$, $1 \leq j,k \leq (n-1)/2$, are eigenvectors of eigenvalue $(\lambda_j - \lambda_k)\mathrm{i}$, $j \neq k$, $v_j v_k^\top + v_k v_j^\top \in \mathbf{Sym}_n \oplus \mathrm{i}\,\mathbf{Sym}_n$, $1 \leq j,k \leq (n-1)/2$, are eigenvectors of eigenvalue $(\lambda_j + \lambda_k)\mathrm{i}$, $j \leq k$, and $\bar{v}_j\bar{v}_k^\top + \bar{v}_k\bar{v}_j^\top \in \mathbf{Sym}_n \oplus \mathrm{i}\,\mathbf{Sym}_n$, $1 \leq j,k \leq (n-1)/2$ are eigenvectors of eigenvalue $-(\lambda_j + \lambda_k)\mathrm{i}$, $j \leq k$. Computing the number of symmetric eigenvectors we obtain the dimension of the space $\mathbf{Sym}_n$.

If n is even then $\operatorname{Ker} X = \{0\}$ and adapting the previous reasoning the result follows. $\square$

B Proof of Propositions 4.1 and 4.2

In this appendix we prove Prop. 4.1 needed in Theorem 4.1 and Proposition 4.2 needed in Theorem 4.2, to get the structure of the ω-limit set. We start with some preliminary results.

Proposition B.1. *Let $H \in \mathbf{H}^\star$ such that $H = A + B$, where the eigenvalues of A have zero real part,* $\operatorname{Spec}(B) \subset \mathbb{R}$ *and $AB = BA$. If $\lambda + \mathrm{i}\,\mu \in \operatorname{Spec}(H)$ then, there exists a non-zero vector $v \in \mathbb{C}^n$ such that $Av = (\mathrm{i}\,\mu)v$ and $Bv = \lambda v$.*

Proof: As A and B commute, then also H and B commute. This implies that there exists a non-zero vector $v \in \mathbb{C}^n$ such that $Hv = (\lambda + \mathrm{i}\,\mu)v$ and $Bv = \lambda_1 v$, for some $\lambda_1 \in \operatorname{Spec}(B)$. Then $Av = (H - B)v = [(\lambda - \lambda_1) + \mathrm{i}\,\mu]v$. As the eigenvalues of A have zero real part, then $\lambda = \lambda_1$, and we obtain the desired result. $\square$

Lemma B.1. *If $A \in \mathcal{M}_{n,n}$, $B \in \mathcal{M}_{p,p}$ and $X \in \mathcal{M}_{n,p}$ satisfy $AX = XB$, rank $(X) = p$ and $X = X_1 X_2$ where $X_1 \in \mathcal{M}_{n,n}$ is non-singular and X_2 is a $\mathcal{M}_{n,p}$ matrix such that*

$$X_2 = \begin{pmatrix} 0 \\ X_{22} \end{pmatrix},$$

being $X_{22} \in \mathcal{M}_{p,p}$ a non-singular matrix, then

$$X_1^{-1} A X_1 = T = \begin{array}{cc} \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix} & \begin{array}{c} n-p \\ p \end{array} \\ \begin{array}{cc} n-p & p \end{array} & \end{array} .$$

Moreover, T_{22} and B have the same spectrum.

Proof: We have

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 0 \\ X_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ X_{22} \end{pmatrix} B.$$

Then $T_{12} X_{22} = 0$, $T_{22} X_{22} = X_{22} B$. As X_{22} is non-singular, $T_{12} = 0$ and $\operatorname{Spec} B = \operatorname{Spec} T_{22}$. $\square$

Proposition B.2. *If $H \in \mathbf{H}_n^\star$ with* $\operatorname{Spec} H = \{\lambda_1, \dots, \lambda_n\}$, *$\lambda_i \in \mathbb{R}$, $1 \le i \le n$, then there exists $R \in \mathbf{T}_n$ such that*

$$R^{-1}HR = D + N,$$

where N is strictly lower triangular, $D = diag\,(\lambda_1, \dots, \lambda_n)$.

Proof: The proposition holds for $n = 1$. Suppose it holds for matrices in $\mathbf{H}_m^\star$, $1 \le m \le n-1$. Let $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be an eigenvector of eigenvalue λ of H. We have that $x = UL$, where

$$U = \begin{pmatrix} I & \tilde{x} \\ 0 & x_n \end{pmatrix}, \qquad L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Moreover, if we write $H = (h_{ij})_{1 \le i,j \le n}$, then $h_{i+1,i} \neq 0$, $i = 1, \dots, n-1$, and this implies that $x_n \neq 0$. Applying Lemma B.1 (with $B = (\lambda)$), we have that

$$T = U^{-1}HU = \begin{pmatrix} T_{11} & 0 \\ T_{21} & \lambda \end{pmatrix}, \qquad \text{and} \qquad T_{11} = H_{11} - \frac{1}{x_n}\tilde{x}w^\top,$$

where $H_{11} = (h_{i,j})_{1 \le i,j \le n-1}$ and $w^\top = (0, \dots, 0, h_{n,n-1})$. This implies that $T_{11} \in \mathbf{H}_{n-1}^\star$. By induction, there is $\tilde{U} \in \mathbf{T}_{n-1}$ such that $(\tilde{U}^{-1}T_{11}\tilde{U})^\top \in \mathbf{T}_{n-1}$. Thus, if $R = U\mathrm{Diag}(\tilde{U}, 1)$ then $R \in \mathbf{T}_n$ and $(R^{-1}HR)^\top \in \mathbf{T}_n$. $\square$

Corollary B.1. *Suppose that $H = A + B \in \mathbf{H}_n^\star$, where $AB = BA$ and* $\operatorname{Spec} A = \{0\}$, $\operatorname{Spec} B = \{\mu_1, \dots, \mu_n\} \subset \mathbb{R}$. *Then there exists $R \in \mathbf{T}_n$ such that*

$$R^{-1}AR = N_1, \qquad R^{-1}BR = D + N_2,$$

where N_1, N_2 are strictly lower triangular and $D = \mathrm{Diag}(\mu_1, \dots, \mu_n)$.

Proof: The result is trivial for $n = 1$. Suppose that it holds for matrices in $\mathbf{H}_m^\star$, $1 \le m \le n-1$. Let $\lambda \in \mathbb{C}$ an eigenvalue of H. By Proposition B.1, λ is an eigenvalue of B and there exists a non-zero vector $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $Hx = \lambda x$, $Ax = 0$ and $Bx = \lambda x$. Then, if we define, as in Proposition B.2,

$$U = \begin{pmatrix} I & \tilde{x} \\ 0 & x_n \end{pmatrix}, \qquad L = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have that

$$U^{-1}HU = \begin{pmatrix} T_{11} & 0 \\ T_{21} & \lambda \end{pmatrix}, \quad U^{-1}AU = \begin{pmatrix} T_{11}^{(1)} & 0 \\ T_{21}^{(1)} & 0 \end{pmatrix}, \quad U^{-1}BU = \begin{pmatrix} T_{11}^{(2)} & 0 \\ T_{21}^{(2)} & \lambda \end{pmatrix},$$

where $T_{11} = T_{11}^{(1)} + T_{11}^{(2)} \in \mathbf{H}_{n-1}^\star$, $T_{11}^{(1)}T_{11}^{(2)} = T_{11}^{(2)}T_{11}^{(1)}$. By the induction hypothesis, there is $\tilde{U} \in \mathbf{T}_{n-1}$ such that $(\tilde{U}^{-1}T_{11}^{(i)}\tilde{U})^\top \in \mathbf{T}_{n-1}$. As before, if we take $R = U\mathrm{Diag}(\tilde{U}, 1)$ then $R \in \mathbf{T}_n$ and $(R^{-1}AR)^\top, (R^{-1}BR)^\top \in \mathbf{T}_n$. $\square$

Lemma B.2. *Let $\lambda \neq \mu$ be two real or complex eigenvalues of $H \in \mathbf{H}_n^\star$ and let v (resp. w) be an eigenvector of eigenvalue λ (resp. μ). If $v = (v_1, v_2)$ and $w = (w_1, w_2)$, where $v_1, w_1 \in \mathbb{C}^{n-2}$, $v_2, w_2 \in \mathbb{C}^2$, then the vectors v_2, w_2, are linearly independent.*

Proof: Suppose that $\alpha_1 v + \alpha_2 w = (\tilde{v}_1, 0)^\top$, where $\tilde{v}_1 \in \mathbb{C}^{n-2}$. Multiplying the last equality by $(H - \lambda I) \in \mathbf{H}_n$ we obtain $\alpha_2(\mu - \lambda)w = (\tilde{w}_1, 0)^\top$, where $\tilde{w}_1 \in \mathbb{C}^{n-1}$. As w has the last component different from zero (since H is unreduced) and $\lambda \neq \mu$, we have that $\alpha_2 = 0$. Finally, $\alpha_1 = 0$ because the last component of v is different from zero. □

Corollary B.2. *Let $H \in \mathbf{H}_n^\star$ and let v and $w = \overline{v}$ be (complex) eigenvectors of eigenvalues λ and $\overline{\lambda}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively, If we write $v = (v_1, v_2)$, where $v_1 \in \mathbb{C}^{n-2}$ and $v_2 \in \mathbb{C}^2$ then the vectors Re v_2, Im v_2, are linearly independent.*

Proof: By Lemma B.2, v_2 and w_2 are linearly independent. The corollary follows because $2\mathrm{Re}\, v_2 = v_2 + \overline{v}_2$ and $2\mathrm{i}\,\mathrm{Im}\, v_2 = v_2 - \overline{v}_2$. □

Proposition B.3. *Let $H \in \mathbf{H}_n^\star$. There exists $R \in \mathbf{BUT}^n_{n_1,\ldots,n_m}$ with either $n_i = 1$ or $n_i = 2$ for $1 \leq i \leq m$, such that $R^{-1}HR = D + N$, being $N \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ with $N_{i,i} = 0$, and $D \in \mathbf{BD}^n_{n_1,\ldots,n_m}$. Moreover, if $\mathrm{diag}(D) = (D_1, \ldots D_m)$ then either $D_i \in \mathrm{Spec}\,(H)$ or*

$$D_i = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \qquad \textit{with} \;\; \lambda + i\mu \in \mathrm{Spec}\, H.$$

Proof: Suppose that $H \in \mathbf{H}_n^\star$ has $k \geq 0$ pairs of non-real eigenvalues. We will proceed by induction on k. If $k = 0$ the result follows from Prop. B.2. Suppose that it holds for matrices in $\mathbf{H}_\mathbf{n}^\star$ with $k - 1$ pairs of non-real eigenvalues. Let $\lambda = \gamma + i\delta \in \mathrm{Spec}\, H$, $\delta \neq 0$. By the Corollary B.2, there exists a matrix

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \text{with} \quad W_1 \in \mathcal{M}_{n-2,2},\; W_2 \in \mathcal{M}_{2,2} \text{ non-singular,}$$

such that $HW = WB$ where

$$B = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}.$$

Then, we can procceed as in the real case. We write $W = UL$ where

$$U = \begin{pmatrix} I_{n-2} & W_1 \\ 0 & W_2 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}.$$

By using Lemma B.1, we see that

$$T = U^{-1}HU = \begin{pmatrix} T_{11} & 0 \\ T_{21} & B \end{pmatrix}, \quad \text{and} \quad T_{11} = H_{11} - W_1 W_2^{-1} H_{21} \in \mathbf{H}_{n-2}^\star.$$

By the induction hypothesis there exists $\tilde{U} \in \mathbf{BUT}^{n-2}_{n_1,\ldots,n_{m-1}}$ such that $\tilde{U}^{-1}T_{11}\tilde{U} \in \mathbf{BLT}^{n-2}_{n_1,\ldots,n_{m-1}}$ with diagonal blocks of the required form. Defining $R = U\mathrm{Diag}(\tilde{U}, I_2)$, it follows that $R^{-1}HR$ has the required properties. □

The precise statement of Prop. 4.1 follows from the previous results because, since $D + N \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ and $\mathrm{Spec}\, D_{i,i} \cap \mathrm{Spec}\, D_{j,j} = \emptyset$, there exists $L \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ with $L_{i,i} = I_{n_i}$ such that $L(D + N)L^{-1} = D$ (this follows, for example, from Lemma 7.1.5 in [15]).

Corollary B.3. *Suppose that $H = A + B \in \mathbf{H}_n^\star$, where the eigenvalues of A have zero real part, $AB = BA$ and $\mathrm{Spec}\,(B) \subset \mathbb{R}$. Then there exists $R \in \mathbf{BUT}^n_{n_1,\ldots,n_m}$ with either $n_i = 1$ or $n_i = 2$ for $1 \leq i \leq m$, such that $R^{-1}AR = D^{(1)} + N^{(1)}$, $R^{-1}BR = D^{(2)} + N^{(2)}$ being $N^{(j)} \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$*

with $N^{(j)}_{i,i} = 0$, $D^{(j)} \in \mathbf{BD}^n_{n_1,\dots,n_m}$, $j = 1, 2$. *Moreover, if* $\operatorname{diag}(D^{(j)}) = (D^{(j)}_1, \dots D^{(j)}_m)$ *then either* $D^{(1)}_i = 0$ *and* $D^{(2)}_i \in \operatorname{Spec}(B)$ *or*

$$D^{(1)}_i = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \quad \textit{and } D^{(2)}_i = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \textit{with } \lambda + \mathrm{i}\,\mu \in \operatorname{Spec}(H).$$

Proof: Suppose that $H \in \mathbf{H}^\star_n$ has $k \geq 0$ pairs of non-real eigenvalues. We will proceed by induction on k. If $k = 0$ then the result follows by Proposition B.1. Suppose that it is true if H has $k-1$ pairs of non-real eigenvalues. Suppose that H has k pairs and let $\lambda + \mathrm{i}\,\mu$ an eigenvalue of H with $\mu \neq 0$. By Proposition B.1, there exists a no zero vector $v \in \mathbb{C}^n$ such that $Hv = (\lambda + \mathrm{i}\,\mu)v$, $Bv = \lambda v$ and $Av = \mathrm{i}\,\mu v$.

Note that if we write $v = x + \mathrm{i}\,y$, being $x, y \in \mathbb{R}^n$ then $y \neq 0$, and x and y are linearly independent eigenvectors of B of eigenvalue λ.

As v is an eigenvector of H of eigenvalue $\lambda + \mathrm{i}\,\mu$, by Corollary B.2, there exists a matrix

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \text{with} \quad W_1 \in \mathcal{M}_{n-2,2},\ W_2 \in \mathcal{M}_{2,2} \text{ non-singular},$$

such that $AW = WC_1$ and $BW = WC_2$, where

$$C_1 = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then, we can procceed as in the Proposition B.3. We write $W = UL$ where

$$U = \begin{pmatrix} I_{n-2} & W_1 \\ 0 & W_2 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}.$$

By using Lemma B.1, we know that

$$U^{-1}HU = \begin{pmatrix} T_{11} & 0 \\ T_{21} & B \end{pmatrix}, \quad \text{and} \quad T_{11} \in \mathbf{H}^\star_{n-2}.$$

Moreover,

$$U^{-1}AU = \begin{pmatrix} T^{(1)}_{11} & 0 \\ T^{(1)}_{21} & B_1 \end{pmatrix}, \quad U^{-1}BU = \begin{pmatrix} T^{(2)}_{11} & 0 \\ T^{(2)}_{21} & B_2 \end{pmatrix},$$

where $T^{(1)}_{11} T^{(2)}_{11} = T^{(2)}_{11} T^{(1)}_{11}$. As B_1 and B_2 commute, we can perform a block diagonal similarity that allows us to replace B_1 and B_2 by

$$\begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \text{ respectively.}$$

Then we apply the induction hypothesis to T_{11} and obtain R with the required properties. $\square$

The precise result of Prop. 4.2 follows from the following considerations. As the eigenvalues of A have real part equal to zero, we can apply Corollary B.3. This means that there exists a matrix $U \in \mathbf{BUT}_{n_1,\dots,n_m}$ such that $UX_0U^{-1} = D + N$, where $D = D^{(1)} + D^{(2)}$ and $N = N^{(1)} + N^{(2)}$. We note that since $UBU^{-1} = D^{(2)} + N^{(2)}$ the matrix $\check{D}$ in the previous statement of this Proposition denotes indeed the matrix $D^{(2)}$. As in the proof of Prop. 4.1, as $D + N \in \mathbf{BLT}^n_{n_1,\dots,n_m}$ and $\operatorname{Spec} D_{i,i} \cap \operatorname{Spec} D_{j,j} = \emptyset$, there exists $L \in \mathbf{BLT}^n_{n_1,\dots,n_m}$ with $L_{i,i} = I_{n_i}$ such that $L(D+N)L^{-1} = D \in \mathbf{BD}^n_{n_1,\dots,n_m}$.

Now, we know that $L(D + N)L^{-1} = D$ and $L(\check{D} + \check{N})L^{-1}$ commute (here $\check{N} = N^{(2)}$). Equivalently, one has that $L(\check{D} + \check{N})L^{-1}$ commutes with D. As $\operatorname{Spec}(D_{i,i}) \cap \operatorname{Spec}(D_{j,j}) = \emptyset$ then $0 \notin \operatorname{Spec}(\mathcal{B}_{D_{i,i},D_{j,j}})$, see first item of Prop. A.1. Then, $L(\check{D} + \check{N})L^{-1} = \check{D}$.

C A technical lemma

In this appendix we show how (13) in Theorem 4.1 can be derived from (12). The existence of a permutation P such that $M(t)$ has block LU-factorization with the block lower triangular matrix tending to the identity is the key to understand the block structure of the ω-limit set.

Lemma C.1. *There exists a symmetric permutation matrix $P \in \mathbf{BD}^n_{n_1,\ldots,n_m}$ (therefore $P^2 = I_n$) such that*

$$M(t) = Pe^{t\tilde{D}}\hat{L}L,$$

has block LU-factorization $M(t) = \bar{L}(t)\bar{R}(t)$ where $\bar{L}(t) \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ being

$$\bar{L}_{j,j}(t) = \begin{pmatrix} \bar{L}^{(j)}_{1,1}(t) & & \\ \vdots & \ddots & \\ \bar{L}^{(j)}_{d_j,1}(t) & \cdots & \bar{L}^{(j)}_{d_j,d_j}(t) \end{pmatrix} \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}},$$

and satisfying

1. $\bar{L}^{(j)}_{k,k}(t) = I_{r_{k,j}}$, *and*
2. $\bar{L}(t) \to I_n$ *as* $t \to \infty$.

The proof will be divided into different steps. First, we proof that we can restrict to blocks having eigenvalues with the same real part.

Lemma C.2. *Let $P = \mathrm{Diag}(P_{1,1}, \ldots, P_{m,m})$, where $P_{i,i}$, $1 \le i \le m$, is a symmetric permutation. Assume that*

$$P_{j,j}e^{t\tilde{D}_{j,j}}\hat{L}_{j,j} = \bar{L}_{j,j}(t)\bar{R}_{j,j}(t), \tag{23}$$

where $\bar{R}_{i,i}(t) \in \mathbf{BUT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$, $\bar{L}_{i,i}(t) \in \mathbf{BLT}^{n_j}_{r_{1,j},\ldots,r_{d_j,j}}$, and

1. $\bar{L}^{(j)}_{k,k} = I_{r_k}$, *and*
2. $\bar{L}_{k,k}(t) \to I_{n_j}$ *as* $t \to \infty$.

Then, there exists $\bar{L}(t)$ and $\bar{R}(t)$ such that $\mathrm{diag}(\bar{L}(t)) = (\bar{L}_{1,1}(t), \ldots, \bar{L}_{m,m}(t))$, *and* $\bar{R}(t) = \mathrm{Diag}(\bar{R}_{1,1}(t), \ldots, \bar{R}_{m,m}(t))$, *and they satisfy the thesis of Lemma C.1.*

Proof: One has $M(t) = (M_{i,j}(t))_{i,j} \in \mathbf{BLT}^n_{n_1,\ldots,n_m}$ where $M_{j,j}(t) = \bar{L}_{j,j}(t)\bar{R}_{j,j}(t)$, $1 \le j \le m$, and $M_{i,j}(t) = \bar{L}_{i,i}(t)\bar{R}_{i,i}(t)L_{i,j}$, $1 \le j < i \le m$. Then, we define

$$\bar{L}_{i,j}(t) := \bar{L}_{i,i}(t)\bar{R}_{i,i}(t)L_{i,j}\bar{R}_{j,j}(t)^{-1}, \quad 1 \le j < i \le m.$$

It just remains to check that $\bar{L}_{i,j}(t) \to 0$ whenever $i > j$. One has $\bar{L}_{i,j}(t) = P_{i,i}e^{t\tilde{D}_{i,i}}\hat{L}_{i,i}L_{i,j}\bar{R}_{j,j}(t)^{-1}$. Using (23) it follows

$$\bar{L}_{i,j}(t) = e^{t(\alpha_i-\alpha_j)}P_{i,i}e^{t(\tilde{D}_{i,i}-\alpha_i I_{n_i})}\hat{L}_{i,i}L_{i,j}\hat{L}^{-1}_{j,j}e^{-t(\tilde{D}_{j,j}-\alpha_j I_{n_j})}P_{j,j}\bar{L}_{j,j}(t)^{-1}.$$

The eigenvalues of $\tilde{D}_{i,i}$ have real part α_i, then $\|e^{\pm t(\tilde{D}_{\ell,\ell}-\alpha_\ell I_{n_\ell})}\|$ have a polynomial bound in t. On the other hand, $\bar{L}_{j,j}(t)^{-1} \to I_{n_j}$. Since $\alpha_i < \alpha_j$ because $i > j$ then $\bar{L}_{i,j}(t) \to 0$. $\square$

As a consequence of the previous Lemma C.2 it is enough to restrict the proof to blocks with eigenvalues with the same real part. We consider the jth-block which has eigenvalues with real part equal to α_j.

From the structure of $\tilde{D}_{j,j}$ described in (7) note that we can write $\tilde{D}_{j,j} = A_j + \alpha_j I_{n_j} + L_j^*$, where $A_j \in \mathbf{Skew}_{n_j}$ and $(L_j^*)^\top \in \mathbf{T}_{n_j}$ with $L_{k,k}^* = 0$.

Lemma C.3. *With the previous notation, assume that*

$$P_{j,j} e^{tL_j^*} \hat{L}_{j,j} = \overline{\overline{L}}_j(t) \overline{\overline{R}}_j(t),$$

where $\overline{\overline{R}}_j(t) \in \mathbf{BUT}^{n_j}_{r_{1,j},\dots,r_{d_j,j}}$, $\overline{\overline{L}}_j(t) \in \mathbf{BLT}^{n_j}_{r_{1,j},\dots,r_{d_j,j}}$, *and*

1. $\overline{\overline{L}}_j(t)_{k,k} = I_{r_{k,j}}$, *and*
2. $\overline{\overline{L}}_j(t) \to I_n$ *as* $t \to \infty$.

If $\bar{L}_{j,j}(t) := P_{j,j} e^{tA_j} P_{j,j} \overline{\overline{L}}_j(t) P_{j,j} e^{-tA_j} P_{j,j}$ *and* $\bar{R}_{j,j}(t) := e^{t\alpha_j} P_{j,j} e^{tA_j} P_{j,j} \overline{\overline{R}}_j(t)$ *then (23) holds.*

Proof: First, from (7) one checks that

$$\tilde{D}^{(j)}_{s_j \pm 2k, s_j \pm 2k} \tilde{D}^{(j)}_{s_j \pm 2k, s_j \pm 2k-2} = \tilde{D}^{(j)}_{s_j \pm 2k, s_j \pm 2k-2} \tilde{D}^{(j)}_{s_j \pm 2k-2, s_j \pm 2k-2},$$

$$\tilde{D}^{(j)}_{s_j \pm (2k+1), s_j \pm (2k+1)} \tilde{D}^{(j)}_{s_j \pm (2k+1), s_j \pm (2k-1)} = \tilde{D}^{(j)}_{s_j \pm (2k+1), s_j \pm (2k-1)} \tilde{D}^{(j)}_{s_j \pm (2k-1), s_j \pm (2k-1)},$$

and hence it follows that $L_j^* A_j = A_j L_j^*$. Then,

$$P_{j,j} e^{t\tilde{D}_{j,j}} \hat{L}_{j,j} = e^{t\alpha_j} P_{j,j} e^{tA_j} e^{tL_j^*} \hat{L}_{j,j} = e^{t\alpha_j} P_{j,j} e^{tA_j} P_{j,j} \overline{\overline{L}}_j(t) \overline{\overline{R}}_j(t) = \bar{L}_{j,j}(t) \bar{R}_{j,j}(t).$$

Note that the diagonal blocks of $\bar{L}_{j,j}(t)$ are identity matrices and that $\bar{L}_{j,j}(t) \to 0$ (because $e^{tA_j} \in \mathbf{O}_{n_j}$ and, hence, has bounded norm). □

It remains to prove the assumptions of the previous lemma concerning the LU factorizations of $P_{j,j} e^{tL_j^*} \hat{L}_{j,j}$. For simplicity, we shall remove the subindex j in the notation. We note that the matrix $e^{tL^*} \in \mathbf{BLT}^n_{r_1,\dots,r_d}$ has a block structure that satisfies $r_{(d+1)/2+k} = r_{(d+1)/2-k}$, $1 \le k \le (d-1)/2$ (recall that d is odd). This allows to consider P the block anti-diagonal permutation matrix with identity matrices and perform the product Pe^{tL^*} by blocks of size $r_1, \dots, r_d$.

Lemma C.4. *Let P denote the block anti-diagonal permutation matrix with identity blocks of sizes $r_1, \dots, r_d$.*

1. *Consider the matrices $W(t)$ and $\widehat{W}(t)$ where $W(t) = \mathrm{Diag}(W_{1,1}(t), \dots, W_{d,d}(t))$, $W_{i,i}(t) \in \mathbf{D}_{r_i}$,*

   $$W_{i,i}(t) = t^{-\lfloor \frac{d+1}{4} \rfloor + \lfloor \frac{d-1-2i}{4} \rfloor + 1} I_{r_i}, \quad \widehat{W}_{i,i}(t) = t^{(d+1)/2-i} [W_{i,i}(t)]^{-1}, \ 1 \le i \le d.$$

   They satisfy
   $$Pe^{tL^*} = W(t) F \widehat{W}(t),$$
   where $F \in \mathcal{M}_{n,n}$ is a constant matrix.

2. *The matrix F has block LU factorization which will be denoted by $F = L_0 R_0$.*

3. *The matrix* $R_0\widehat{W}(t)\hat{L}\widehat{W}^{-1}(t)$, *for* $|t|$ *large enough, has block LU factorization, that will be denoted by* $L_1(t)R_1(t)$.

4. *Finally,* $Pe^{tL^*}\hat{L} = \overline{\overline{L}}(t)\overline{\overline{R}}(t)$ *where* $\overline{\overline{L}}(t) = W(t)L_0L_1(t)W(t)^{-1}$, *and* $\overline{\overline{R}}(t) = W(t)R_1(t)\widehat{W}(t)$. *Moreover,* $\overline{\overline{L}}(t) \to I_n$ *as* $|t| \to \infty$.

Proof: The block structure of L^* is the same as the structure of a block of D (see Fig. 1 right) but with zero blocks in the diagonal. Let us denote $B = e^{tL*}$. Then, all blocks are 0 except $B_{i,i} = I_{r_i}$, and, for $0 \le j \le (d-1)/2 = s-1$,

$$B_{s+k+2+j,s+k-j} = \frac{t^{j+1}}{(j+1)!}E(r_{s+k+2+j}, r_{s+k-j})^\top, \quad 0 \le k \le s-j-3,$$
$$B_{s-k+1+j,s-k-1-j} = \frac{t^{j+1}}{(j+1)!}E(r_{s-k+1+j}, r_{s-k-1-j}), \quad 0 \le k \le s-j-2.$$

To prove 1. we note that the elements of $F = W(t)^{-1}PB\widehat{W}(t)^{-1}$ are given by

$$F_{i,l} = W_{i,i}(t)^{-1}B_{d-i+1,l}\widehat{W}_{l,l}(t)^{-1}.$$

If $l-i$ is odd, then $F_{i,l} = 0$. Hence we consider $l-i$ even. If $l-i > 0$ then

$$B_{d-i+1,l} = \frac{t^{j+1}}{(j+1)!}E(r_{s+k+2+j}, r_{s+k-j})^\top,$$

where $k = \frac{d+l-i-1-2s}{2}$ and $j = \frac{d-l-i-1}{2}$. From the expression of $W_{i,i}(t)$ and $\widehat{W}_{l,l}(t)$, one gets that $F_{i,l}$ has a factor t^q where

$$q = \frac{l-i}{2} + \lfloor\frac{d-1-2l}{4}\rfloor - \lfloor\frac{d-1-2i}{4}\rfloor,$$

and, since $l-i$ is even, $q = 0$. If $l-i \le 0$ then

$$B_{d-i+1,l} = \frac{t^{j+1}}{(j+1)!}E(r_{s-k+1+j}, r_{s-k-1-j}),$$

where $k = \frac{d+l-i+1-2s}{2}$ and $j = \frac{d-l-i-1}{2}$. The same computation of the power q gives $q = 0$ as before. This proves, in particular, that F is a constant matrix. Moreover, by construction, F is such that $F_{i,l} = 0$ if $i + l > d + 1$.

To prove 2. we first consider a particular case. Assume that $d = 1(\bmod 4)$. The case $d = 3(\bmod 4)$ can be handled similarly. Denote by d_+ (resp. by d_-) the number of blocks having eigenvalues of odd (resp. even) multiplicity. Since $d = 1(\bmod 4)$ one has $d_+ = (d+1)/2$ and $d_- = (d-1)/2$. Moreover, we assume that all the blocks of odd multiplicity have size r_1 and all the blocks with even multiplicity have size r_2. The general case will follow from this particular situation.

Below we use the following notation. For a fixed $j > 0$, we denote by 0_l the null matrix of dimension $l \times j$ and $\mathcal{E}(i,j,k) = (0_i, I_j, 0_k)^\top$.

For the considered case, L^* is of the form

$$L^* = \begin{pmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ I_{r_1} & 0 & 0 & & & & \\ & I_{r_2} & 0 & 0 & & & \\ & & I_{r_1} & 0 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & I_{r_1} & 0 & 0 \end{pmatrix},$$

then $F = (W(t))^{-1} P e^{tL^*} (\widehat{W}(t))^{-1}$ has block LU-factorization. This follows since $F = X_1(S_{d_+} \otimes I_{r_1}) X_1^\top + X_2(S_{d_-} \otimes I_{r_2}) X_2^\top$, where $\otimes$ denotes the Kronecker product, the matrix $X_1 \in \mathcal{M}_{n, r_1 d_+}$ is

$$X_1 = (\mathcal{E}(0, r_1, n-r_1), \mathcal{E}(r, r_1, n-2r_1-r_2), \mathcal{E}(2r, r_1, n-3r_1-2r_2), \ldots, \mathcal{E}(rd_-, r_1, 0)),$$

where $r = r_1 + r_2$; the matrix $X_2 \in \mathcal{M}_{n, r_2 d_-}$ is

$$X_2 = (\mathcal{E}(r_1, r_2, n-r), \mathcal{E}(2r_1+r_2, r_2, n-2r), \mathcal{E}(3r_1+2r_2, r_2, n-3r), \ldots, \mathcal{E}(rd_- - r_2, r_2, r_1)),$$

and, for a positive integer σ, the matrix

$$S_\sigma = \begin{pmatrix} \frac{1}{(\sigma-1)!} & \frac{1}{(\sigma-2)!} & \cdots & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{(\sigma-2)!} & \frac{1}{(\sigma-3)!} & \cdots & \frac{1}{0!} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{1!} & \frac{1}{0!} & \cdots & 0 & 0 \\ \frac{1}{0!} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

has LU-factorization $S_\sigma = L_\sigma R_\sigma$, $L_\sigma = (m_{ij})_{1 \le i,j \le \sigma}$, $R_\sigma = (r_{ij})_{1 \le i,j \le \sigma}$, with non-zero elements given by

$$m_{ij} = \frac{(\sigma - j)!}{(\sigma - i)!} \binom{i-1}{j-1}, \quad i \ge j, \quad \text{and} \quad r_{ij} = (-1)^{i+1} \frac{(i-1)!}{(\sigma - j)!} \binom{j-1}{i-1}, \quad i \le j.$$

To obtain the previous explicit LU-factorization of S_σ we have used the expressions in Lemma 1 of [1] for the determinants that generate the coefficients of L and U.

Hence, the matrix F has block LU-factorization $F = LR$, where

$$\begin{aligned} L &= X_1(L_{d_+} \otimes I_{r_1}) X_1^\top + X_2(L_{d_-} \otimes I_{r_2}) X_2^\top \in \mathbf{BLT}^n_{r_1, r_2, r_1, r_2, \ldots, r_1}, \quad \text{and} \\ R &= X_1(R_{d_+} \otimes I_{r_1}) X_1^\top + X_2(R_{d_-} \otimes I_{r_2}) X_2^\top \in \mathbf{BUT}^n_{r_1, r_2, r_1, r_2, \ldots, r_1}. \end{aligned}$$

The general case follows by induction over $s = (d+1)/2$. If $s = 1$, then $d = d_+ = 1$, $d_- = 0$ and, since F has only one block, trivially admits a block LU-factorization. By induction hypothesis, we assume that if the number of diagonal blocks of F is less or equal than $d = 2s - 3$ then it admits block LU-factorization. Let us prove that if F has $d = 2s - 1$ diagonal blocks it also admits block LU-factorization. We write

$$F = X_1(S_{d_+} \otimes I_{r_1}) X_1^\top + X_2(S_{d_-} \otimes I_{r_2}) X_2^\top + X_3 \tilde{F} X_3^\top,$$

where $X_1 = (X_1^{(1)}, \dots, X_1^{(d_+)}) \in \mathcal{M}_{n,r_1 d_+}$ and $X_2 = (X_2^{(1)}, \dots, X_2^{(d_-)}) \in \mathcal{M}_{n,r_2 d_-}$ are given by

$$X_1^{(k)} = \mathcal{E}\left(\sum_{j=1}^{2(k-1)} r_j, r_1, n - \sum_{j=1}^{2(k-1)} r_j\right), \qquad X_2^{(k)} = \mathcal{E}\left(\sum_{j=1}^{2k-1} r_j, r_2, n - \sum_{j=1}^{2k-1} r_j\right),$$

and $X_3 = (X_3^{(1)}, \dots, X_2^{(d_-)}) \in \mathcal{M}_{n,n-r_1 d_+ - r_2 d_-}$

$$X_3^{(k)} = \begin{cases} \mathcal{E}\left(r_1 + \sum_{j=1}^{k+1} r_j, r_{k+2} - r_1, n - r_{k+2} - \sum_{j=1}^{k+1} r_j\right) & \text{if } k \text{ odd}, \\ \mathcal{E}\left(r_2 + \sum_{j=1}^{k+1} r_j, r_{k+2} - r_2, n - r_{k+2} - \sum_{j=1}^{k+1} r_j\right) & \text{if } k \text{ even}, \end{cases}$$

and $\tilde{F}$ has $\hat{d} = \hat{d}_+ + \hat{d}_- = 2s - 5$ blocks, being $\hat{d}_\pm = d_\pm - 2$.

By induction hypothesis, the matrix $S_{d_+} \otimes I_{r_1}$ and $S_{d_-} \otimes I_{r_2}$ admits LU block decomposition (with squared blocks of size r_1 and r_2, respectively). We denote $S_{d_+} \otimes I_{r_1} = L_{d_+} R_{d_+}$, where $L_{d_+} \in \mathbf{BLT}^{r_1 d_+}_{r_1,\dots,r_1}$, $R_{d_+} \in \mathbf{BUT}^{r_1 d_+}_{r_1,\dots,r_1}$. Similarly, $S_{d_-} \otimes I_{r_2} = L_{d_-} R_{d_-}$, where $L_{d_-} \in \mathbf{BLT}^{r_2 d_-}_{r_2,\dots,r_2}$, $R_{d_+} \in \mathbf{BUT}^{r_2 d_-}_{r_2,\dots,r_2}$.

Moreover, the matrix $\tilde{F}$ admits LU decomposition, say $\tilde{F} = \tilde{L}\tilde{R}$, where

$$\tilde{L} \in \mathbf{BLT}^{n-\hat{d}_+ r_1}_{r_2, r_3 - r_1, r_4 - r_2, \dots, r_{\hat{d}_+} - r_1}, \qquad \tilde{R} \in \mathbf{BUT}^{n-\hat{d}_+ r_1}_{r_2, r_3 - r_1, r_4 - r_2, \dots, r_{\hat{d}_+} - r_1}.$$

Since $X_1^\top X_1 = I_{r_1 d_+}$, $X_2^\top X_2 = I_{r_2 d_-}$, $X_3^\top X_3 = I_{n - r_1 d_+ - r_2 d_-}$, $X_i^\top X_j = 0$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, then F admits a LU block decomposition, $F = \hat{L}\hat{R}$, where

$$\begin{aligned} \hat{L} &= X_1 L_{d_+} X_1^\top + X_2 L_{d_-} X_2^\top + X_3 \tilde{L} X_3^\top \in \mathbf{BLT}^n_{r_1, r_2, r_1, r_3 - r_1, r_2, r_4 - r_2, \dots, r_{2d-4}}, \ \textit{and} \\ \hat{R} &= X_1 R_{d_+} X_1^\top + X_2 R_{d_-} X_2^\top + X_3 \tilde{R} X_3^\top \in \mathbf{BUT}^n_{r_1, r_2, r_1, r_3 - r_1, r_2, r_4 - r_2, \dots, r_{2d-4}}. \end{aligned}$$

Finally, we note that some of the $2d-4$ obtained diagonal blocks in the LU decomposition above are smaller than the original diagonal blocks of F, being the block partition of the decomposition finer than the block partition of F. Therefore, there exists a block LU factorization with the original partition of F, which is the required decomposition $F = L_0 R_0$ of item 2.

Note that, by construction, R_0 has blocks $(R_0)_{i,i+1} = 0$, $1 \le i \le d-1$. Moreover, the diagonal blocks $(R_0)_{i,i}$ are invertible. This follows since $\det(F) \neq 0$ (note that $PF \in \mathbf{BLT}^n_{r_1,\dots,r_d}$ with identity blocks in the diagonal). Denote by $N(t) = \widehat{W}(t) \hat{L} \widehat{W}^{-1}(t) \in \mathbf{BLT}^n_{r_1,\dots,r_d}$, then

$$(N(t))_{i,j} = t^{p(i,j)} \hat{L}_{i,j}, \ 1 \le i \le d, \ 1 \le j \le i, \tag{24}$$

where

$$p(i,j) = j - i + \lfloor \frac{d-1-2j}{4} \rfloor - \lfloor \frac{d-1-2i}{4} \rfloor.$$

If $d = 1 (\bmod 4)$, one has $p(i,i) = 0$ and, for i odd, $p(i,i-1) = 0$. If, on the other hand, $d = 3 (\bmod 4)$, one has $p(i,i) = 0$ and, for i even, $p(i,i-1) = 0$. In all the other cases $p(i,j) \le -1$ in both cases. Hence, when $t \to \infty$, only the diagonal blocks and some blocks of the subdiagonal remain. The properties of the blocks of R_0 stated previously imply the existence of block LU factorization of $R_0 N(t)$ when $t = \infty$ and, hence, when $|t|$ is large enough. We denote it $L_1(t) R_1(t)$. This proves item 3.

One checks the indentity $Pe^{tL^*}\hat{L} = \overline{\overline{L}}(t)\overline{\overline{R}}(t)$ directly. Finally, since $N(t)$ has limit when $|t| \to \infty$ and the limit is invertible, it follows that $L_1(t)$ also has limit when $|t| \to \infty$. Then

$$\overline{\overline{L}}(t)_{i,j} = t^{q(i,j)}(L_0 L_1(t))_{i,j},$$

where

$$q(i,j) = \lfloor \frac{d-1-2i}{4} \rfloor - \lfloor \frac{d-1-2j}{4} \rfloor .$$

One checks that $q(i,j) \leq -1$ if $i > j$, which implies that $\overline{\overline{L}}(t)$ tends to I_n when $|t| \to \infty$. □

Remark C.1. If in the proof of Lemma C.1, we put $A_j = 0$, for all j, that is, $\tilde{D}_{j,j} = \alpha_j I_{n_j} + L_j^*$, the thesis is also true and, moreover, $\bar{R}(t) = \mathrm{Diag}(e^{t\alpha_1}\overline{\overline{R}}_{1,1}, \ldots, e^{t\alpha_m}\overline{\overline{R}}_{m,m})$.

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List of Figures

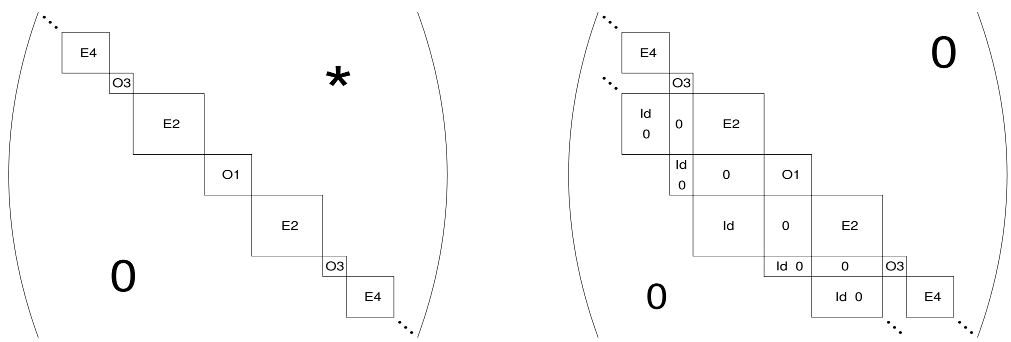

Figure 1: Left: Typical form of a block $Y_{j,j}$ of $\omega(X_0)$. Right: Reordered Jordan normal form used in the proof of Theorem 4.1. In both plots, the blocks labelled by Ok, $k = 1, 3, \dots$ (resp. by Ek, $k = 2, 4, \dots$), have simple eigenvalues $\lambda \in \mathrm{Spec}(X_0)$ with the same real part and odd (resp. even) multiplicity $\mathrm{mult}(\lambda) \geq k$. In the right plot, Id refers to the identity matrix of the corresponding dimension.

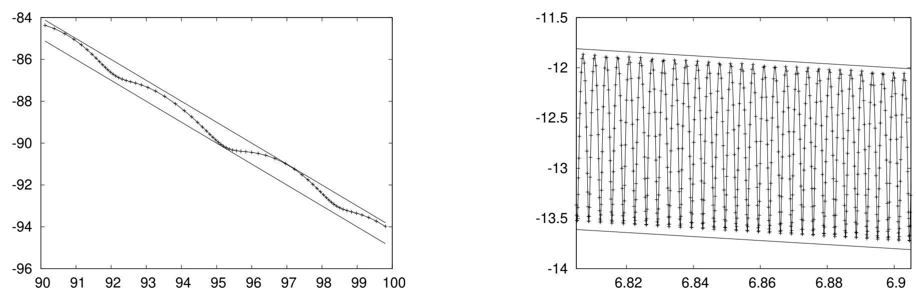

Figure 2: We consider X_0 from Example 4.2. Left: We display $\log((X(t))_{4,3})$ as a function of t. The straight lines correspond to $y = 5 - x$ and $y = 6 - x$. Right: We display $\log((X(t))_{6,5})$ as a function of $\log(t)$. The straight lines correspond to $y = -2x$ and $y = 1.8 - 2x$.

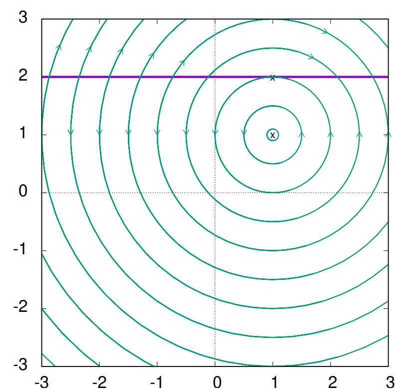

Figure 3: Phase portrait of the system (21) for $c = d = 2$. The triangular equilibrium matrices A_x correspond to the line of fixed points $y = 2$. The matrix C corresponds to the fixed point at $(1, 1)$.

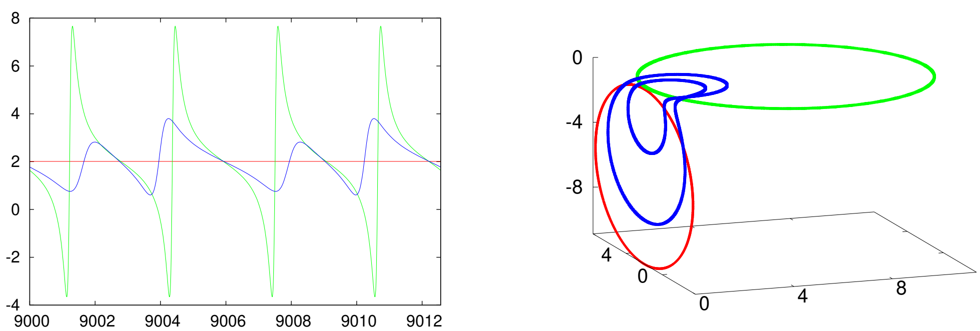

Figure 4: Left: Evolution of the coefficient x_{11} of the trajectory of $X_{0,\epsilon}$ as a function of the integration time t for $\epsilon = 10^{-2}$ (green), $\epsilon = 0$ (blue) and $\epsilon = 10^{-2}$ (red). For $\epsilon = 10^{-2}$, x_{11} tends to be π-periodic, reaching a sequence of maxima close to 8. For $\epsilon = 0$, x_{11} tends to be 2π-periodic. For $\epsilon > 0$, x_{11} tends to 2. Right: Projection onto the coordinates $(x_{2,1}, x_{2,2}, x_{3,2})$ of the ω-limit for $X_{0,10^{-2}}$ (red), $X_{0,-10^{-2}}$ (green), and the periodic orbit of $X_{0,0}$ (blue). The π-periodic orbit obtained as ω-limit for $\epsilon > 0$ (resp. for $\epsilon < 0$) are observed in the coordinates plane $x_{2,1} = 0$ (resp. $x_{3,2} = 0$). For $\epsilon = 0$ the ω-limit is the 2π-periodic orbit of $X_{0,0}$.

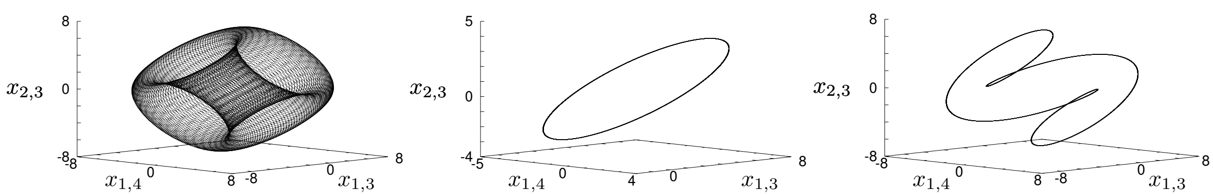

Figure 5: From left to right, we show the orbit of X_0 in cases (i) $\alpha = \pi/2,\ \beta = \pi/4$, (ii) $\alpha = \pi/2,\ \beta = \pi/2$ and (iii) $\alpha = \pi/4$, $\beta \approx 0.361367$. In the first case the orbit is dens in a 2D torus. In the cases (ii) and (iii) the orbit is a periodic orbit.

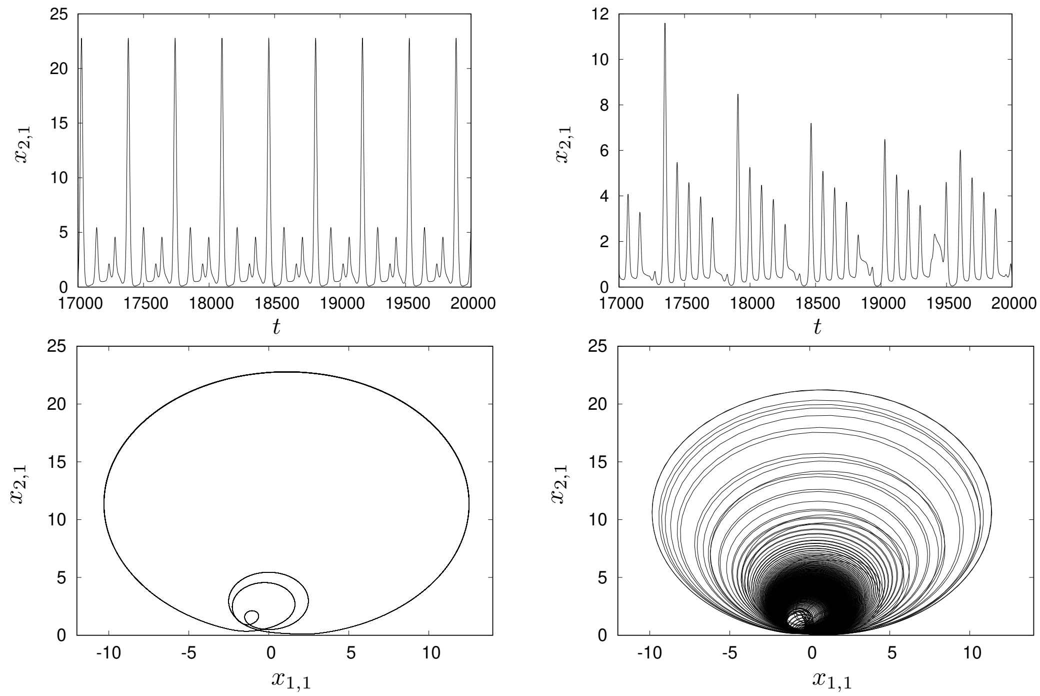

Figure 6: We display the evolution of $x_{2,1}$ with respect the time integration (top) and the projection onto the $(x_{1,1}, x_{2,1})$-plane. Left: Orbit of $X_0^{\mathbb{Q}}$. Right: Orbit of $X_0^{\mathbb{R}\setminus\mathbb{Q}}$.

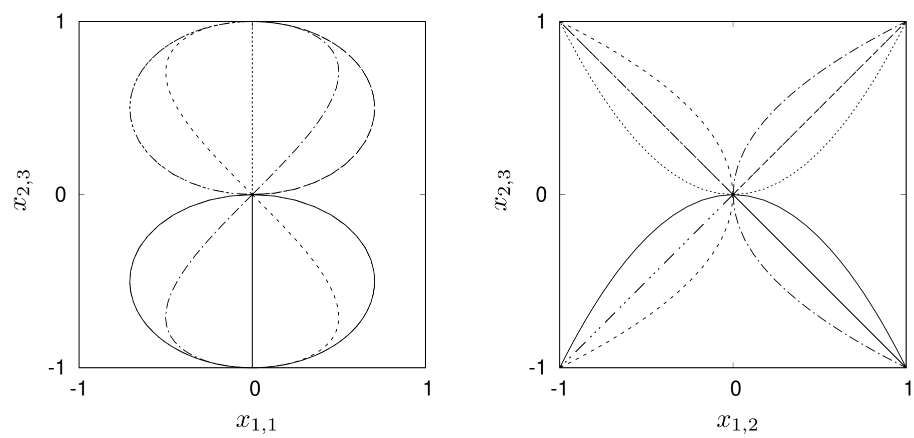

Figure 7: Homoclinic and heteroclinic orbits to $X_{\pm,\pm}$. We represent some components of the solution $X(t)$ of the QR-flow starting at X_i, $1 \leq i \leq 8$. In both plots we display the coefficient $x_{2,3}$ in the y-axis. Left: In the x-axis we display the coefficient $x_{1,1}$. The four equilibrium matrices are projected onto the points $(0,1)$ and $(0,-1)$. The homoclinic orbits appear as ellipses tangent to the origin. Each curve corresponds to two homoclinic or heteroclinic orbits (they are superimposed in this projection). Right: In the x-axis we display the coefficient $x_{1,2}$. The four equilibrium matrices are at the vertices of the square. The four homoclinic orbits start in one of the vertices, go to the center of the square and return. The four heteroclinic orbits are clearly observed.

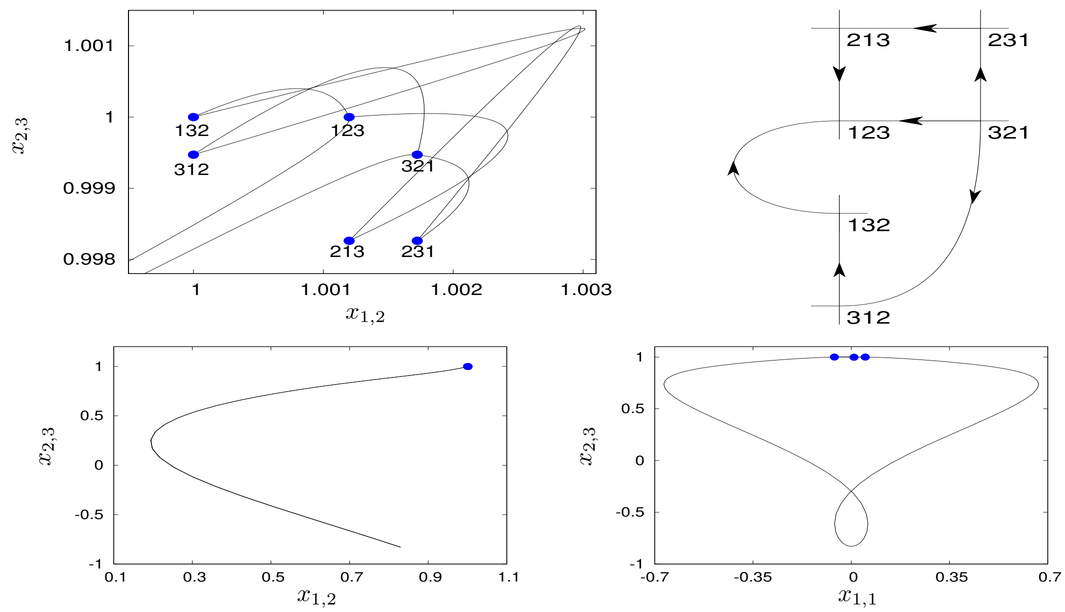

Figure 8: Homoclinic and heteroclinic orbits to $X^{+,+}_{i,j,k}$ (labelled by ijk in the top plots). In the bottom plots the points represent projections of more than one equilibrium matrix. Top left: Heteroclinic orbits to these equilibria. The two pieces of orbits which go to the bottom left vertex of the window correspond to the same heteroclinic orbit from $X^{+,+}_{321}$ to $X^{+,+}_{123}$. Top right: a scheme of the local homo/heteroclinic structure. Bottom left: The same of top plot, but in a larger window, to see the full heteroclinic orbit from $X^{+,+}_{321}$ to $X^{+,+}_{123}$. Bottom right: In the displayed projection each pair of points on the same horizontal coordinate in the top left plot project onto the same point.