Computational methods in perturbation theory

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Glossary

Action-angle variables Conjugate set of coordinates for which an integrable Hamiltonian depends only on the actions and, hence, becomes readily integrable.

Canonical transformation A canonical transformation is a change of variables that preserve the Hamiltonian form, that is, it is equivalent to perform the change of variables on the Hamiltonian or on the differential equations.

Conformally symplectic systems Conformally symplectic systems model dissipative mechanical systems with a friction which is proportional to the velocity. In contrast to conservative systems, dissipative systems have attractors.

Delay differential equation A differential equation in which the derivative of the unknown function depends not only of the actual state but also of previous states.

Effective stability A system is considered effectively stable if the time needed to observe significant changes is longer than the expected lifetime of the system itself. The study of the effective stability is part of the Nekhoroshev Theory.

First integral Function that stays constant on each orbit of a differential equation. If there exists a sufficient number of them, the problem becomes integrable.

Hamiltonian system This is a specific class of dynamical systems that are governed by differential equations that can be written as

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

where $q = (q_1, \ldots, q_\ell)$ are the positions, $p = (p_1, \ldots, p_\ell)$ are the momenta, ℓ is the number of degrees of freedom, and $H = H(q, p)$ is a smooth function called the Hamilton function or Hamiltonian.

Invariant manifold Smooth manifold that is invariant by the dynamics. Jointly with fixed points and periodic orbits (and, sometimes, quasi-periodic orbits) they organize many aspects of the dynamics of the system.

KAM Theory This is a collection of results about the preservation of quasi-periodic trajectories under perturbation. The name come from the initials of A.N. Kolmogorov, V.I Arnol'd and J. Moser who where the pioneers in this field.

Nekhoroshev Theory This is a collection of stability results for an open set of initial conditions and for a finite but very long time span, usually exponentially long with respect to the perturbation.

Normal form A simplified version of the equations of motion obtained by means of changes of coordinates.

Quasi-periodic function A function f depending on a variable $t \in \mathbb{R}$ is said to be quasi-periodic with r basic frequencies if there exists a function F defined on $\mathbb{T}^r$ and a vector $\omega \in \mathbb{R}^r$ such that $f(t) = F(\omega t)$. The components of ω are supposed to be linearly independent over the rationals, otherwise the function F can be rewritten with a lower value for r. Note that, if F is smooth and injective, its image is diffeomorphic to a torus of dimension r.

Restricted Three-Body Problem The problem consists in describing the motion of an infinitesimal particle moving under the attraction of two massive bodies. As the particle does not attract the masses, these follow a Keplerian orbit, usually circular (Circular RTBP or CRTBP) or elliptical (Elliptic RTBP or ERTBP).

Definition of the Subject

The equations of motion of many physical systems can be written as the sum of two parts, a first part that is solvable by closed formulas and a second one which is small. Perturbation theory encloses a set of methods to find approximate solutions that are written as a solution of the solvable part plus small corrections coming from the small part. Numerical methods in perturbation theory comprises numerical techniques that compute approximate solutions by taking advantage of the perturbative situation.

Introduction

Nature is full of situations that can be approximated by simple models that can be solved "by hand", using closed expressions. This fact has been extremely useful to understand the basic properties of many physical systems. Better models can be constructed by adding smaller contributions (perturbations) and looking at the effect that these perturbations have on the simple (unperturbed) system. A classical example is given by the motion of the planets in the Solar system. The simple model is obtained by skipping the gravitational attraction between planets which reduces the dynamics to that of several two-body problems (Sun and planet), which can be solved and give rise to the Kepler laws. The perturbation is given by the interactions between planets.

Perturbation theory is a set of methods and techniques to find approximate solutions by starting from solutions of the unperturbed problem and correcting them to obtain better approximations of the full problem. Usually, the approximated solution as a formal power expansion of a small parameter, where the small parameter represents the size of the perturbing term. This chapter surveys numerical techniques that can be applied in combination with perturbation methods to obtain, in an effective way, approximate solutions of the original problem.

As a motivating example, let us consider the following differential equation,

$$\begin{aligned} \dot{x} &= (A + \varepsilon Q(\theta))x, \\ \dot{\theta} &= \omega, \end{aligned} \tag{1}$$

where A is a constant $d \times d$ matrix, Q is a $d \times d$ matrix taking values on $\mathbb{T}^r$, $\omega \in \mathbb{R}^r$ and ε is a small but fixed value. The goal is to "solve" this equation, that is, to find a change of variables $x = P(\theta)y$ that reduces this linear system to constant coefficients,

$$\begin{aligned} \dot{y} &= B(\varepsilon)y, \\ \dot{\theta} &= \omega, \end{aligned} \tag{2}$$

where B does not depend on θ. This allows to know, for instance, the stability of the origin of (1) or to write explicitly a fundamental matrix of (1) as $P(\omega t)\exp(Bt)$. To start, let us denote by $\bar{Q}$ the average of Q,

$$\bar{Q} = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} Q(\theta)\, d\theta,$$

and let us define $\tilde{Q}$ as $Q - \bar{Q}$, and $A_0(\varepsilon) = A + \varepsilon\bar{Q}$. Then let us write (1) as

$$\begin{aligned} \dot{x} &= (A_0(\varepsilon) + \varepsilon\tilde{Q}(\theta))x, \\ \dot{\theta} &= \omega. \end{aligned} \tag{3}$$

Assume it is possible to compute a $d \times d$ matrix $P_1(\omega t)$ such that

$$\dot{P}_1 = A_0 P_1 - P_1 A_0 + \tilde{Q}(\omega t).$$

A sufficient condition for P_1 to exists is that the eigenvalues of A_0, λ_1, …, λ_d, satisfy a suitable non-resonance condition,

$$|\lambda_i - \lambda_j + \langle k, \omega\rangle\sqrt{-1}| \geq \frac{c}{|k|^\gamma}, \qquad \forall k \in \mathbb{Z}^r \setminus \{0\}, \quad \text{and for all } i \neq j.$$

for some $c > 0$ and $\gamma > r - 1$ (Jorba and Simó, 1992). Then, it is not difficult to check that the change of variables $x = (I + \varepsilon P_1(\omega t))y$ transforms (4) into

$$\begin{aligned} \dot{y} &= (A_0 + \varepsilon^2 (I + \varepsilon P_1(\theta))^{-1}\tilde{Q}(\theta)P_1(\theta))y, \\ \dot{\theta} &= \omega. \end{aligned} \tag{4}$$

Since this equation is similar to (1) but with ε^2 instead of ε, the inductive scheme seems clear: average the quasi-periodic part of (4) and restart this process to obtain a sequence of matrices $\{P_j\}_j$ and $\{A_j\}_j$ such that the products of the matrices $(I + \varepsilon^{2^{j-1}} P_j)$ converge to some (smooth and regular) matrix P and the matrices A_j converge to some matrix B. This convergence is studied in (Bogoljubov et al., 1976; Jorba and Simó, 1992). Here it is important to note that this process can be carried out by a computer program to obtain the matrices P_1, P_2, …, and A_0, A_1, …, so that it is possible to use, for a value of j sufficiently large, P_j and A_j as good approximations to P and B, respectively. A discussion of the effectivity of this procedure is contained in (Jorba et al., 1997) where, as an example, the following equation is considered,

$$\ddot{x} + (1 + \varepsilon q(t))x = 0,$$

where $q(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$, with $(\omega_1, \omega_2) = (\sqrt{2}, \sqrt{3})$. The previous algorithm has been implemented as a C program for a given (and fixed) value of ε. The program computes and performs a finite number of changes of variables. To simplify and make the program more efficient, the functions involved are expanded as (truncated) Fourier series and their coefficients have been stored as double-precision variables. During the operations, coefficients less than 10^{-20} have been dropped in order to control the size of the Fourier series that appears during the process. Of course, this introduces some (small) numerical error in the results. After four changes of variables, the original equation becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[\begin{pmatrix} 0.0 & b_{12} \\ b_{21} & 0.0 \end{pmatrix} + R(t) \right] \begin{pmatrix} x \\ y \end{pmatrix},$$

where $b_{12} = 1.000000366251255$ and $b_{21} = -0.992421151834871$. The remainder R is very small: the largest coefficient it contains is below 10^{-16}. Note that the accuracy (relative error) of this remainder R is very poor due to the use of double precision arithmetic (15-16 digits) for the coefficients. See (Jorba et al., 1997) for more details.

The Solar system

Many physical systems are Hamiltonian, and they can be written in a perturbative way. That is, the Hamiltonian is written in the form $H = H_0 + H_1$, where H_0 is integrable (i.e., solvable by closed formulas) and H_1 is small. A paradigmatic example is the Solar system, where H_0 describes the motion of N particles (the planets), each one being attracted by a fixed central mass (the Sun). Then, the dynamics of the planets is integrable and follows the Kepler laws. In this case, H_1 contains the interactions between planets and that the Sun is not fixed at the centre. Perturbation theory was initially developed to deal with this situation (see the textbooks by Moulton (1914); Brouwer and Clemence (1961); Roy (2004)) and it has been extended to other fields.

The arrival of the electronic computer opened a huge range of new methods to study physical models, and some of them are based on taking advantage of ideas from the classical perturbation theory. For instance, there are several numerical studies of the long term dynamics of the Solar system that exploit this perturbative structure in order to be more efficient both in computer speed and accuracy. There is a limitation of the total integration time that comes from the chaoticity of the system (Laskar, 1989) but the results have a high interest for paleoclimatology: the variations of the Earth orbital elements induce some changes in the Earth climate that have an impact in the sedimentary records. Therefore, it is possible to use the knowledge of the Solar system dynamics for the calibration of geological time scales by correlating the variation of orbital and rotational elements of the Earth with the existing geological records (Westerhold et al., 2008; Hoang et al., 2021).

An option to simplify (and to speed up) the integration of these equations is to use the ideas of perturbation theory to prepare the equations before the numerical integration. One of the main ideas is that the the planetary motion combines quantities (angles) that move fast with quantities with a slow dynamics. A first tool to deal with fast angles is the averaging method. The idea is that these rapidly oscillating terms tend to average out without contributing to the long term evolution, while more slowly varying resonant or secular terms accumulate to give significant contributions. In this direction, a set of very relevant works are (Laskar, 1985, 1986, 1988) (see also references therein) where a sequence of symbolic manipulations on the equations of motion (that have been

expanded in a Taylor-Fourier series in suitable variables) has been done to carry out a normalization procedure (with truncation up to suitable order) to produce a much better form for the differential equations which allows for a fast an efficient numerical integration, see also (Laskar, 1990, 1994). These formal computations are done by an specific computer algebra package, see (Gastineau and Laskar, 2011) for its last version.

A different way of exploiting the perturbative structure is by means of direct numerical integration methods that take advantage of it. A very simple (but very effective) idea is to use the classical representation of the Hamiltonian seen before, $H = H_0 + H_1$, where H_0 contains the uncoupled two body interactions Sun-planet (so it is an integrable Hamiltonian) and H_1 contains the interactions between plants (so it is small). This representation is exploited by Wisdom and Holman (1991) to develop a specific numerical integrator for the planetary problem that preserves the Hamiltonian structure. This idea of writing the Hamiltonian of the Solar system as the sum of these two parts has been used to construct other families of symplectic numerical integrators (McLachlan, 1995; Chambers and Murison, 2000; Laskar and Robutel, 2001; Blanes et al., 2013).

Dynamics near an equilibrium point of a Hamiltonian system

The study of the motion near an equilibrium point of a Hamiltonian system is a very relevant topic of classical mechanics. Without loss of generality it is assumed that the equilibrium point is at the origin, and that the Hamiltonian system H is analytic, with ℓ degrees of freedom, that can be expanded at the origin as $H = H_2 + H_{>2}$ where H_2 is an homogeneous polynomial of degree 2 and $H_{>2}$ contains the higher order terms (the terms of degree 1 vanish since the origin is an equilibrium point). This is a perturbative setting since, near the origin, $H_{>2}$ is much smaller than H_2. It is now very convenient to take advantage of this situation to obtain a quantitative description of the dynamics near the origin. Let us distinguish several cases.

The first case is when the equilibrium point is of elliptic type. This means that, in suitable variables, H_2 can be written as

$$H_2(q,p) = \sum_{j=1}^{\ell} \frac{\omega_j}{2}(q_j^2 + p_j^2),$$

where $\omega = (\omega_1, \ldots, \omega_\ell)$ are the frequencies of the linearized equations at the origin. Let us now assume that this is the non-resonant case, that is, $\langle k, \omega \rangle \neq 0$ for all $k \in \mathbb{Z}^\ell \setminus \{0\}$, and that the Hamiltonian is expanded as $H = H_2(q,p) + H_3(q,p) + \cdots + H_N(q,p) + R_{N+1}(q,p)$, where H_j is an homogeneous polynomial of degree j and R_{N+1} is the remainder (of order $N+1$). An option in this situation is to start a normalizing procedure to simplify, order by order, the power expansion of the Hamiltonian up to degree N, to produce the so-called Birkhoff normal form (Meyer and Offin, 2017). This normal form allows to easily introduce action-angle variables (I, φ) ($I \in \mathbb{R}^\ell$, $\varphi \in \mathbb{T}^\ell$) such that the Hamiltonian takes the form $H = H_0(I) + H_1(I, \varphi)$, where H_0 is a polynomial of degree $N/2$ and H_1 contains the non-normalized part (and it starts at order $N/2+1$). Generically speaking, these normalizing transformations are not convergent and, when N goes to infinity, the normal form is seen as the power series of a C^∞ function. If the normalizing process is stop at a given N then, as the remainder H_1 is of order $N/2+1$, it turns out that in a small enough neighbourhood of the origin this remainder is sufficiently small so that H_0 gives an accurate description of the dynamics. The change of variables that relate the original coordinates with the final action-angle

variables can be obtained in a similar way, so orbits in the normal form can be sent to original coordinates if required. Note that the dynamics given by H_0 is quasi-periodic and this implies that, if the small term H_1 is discarded, the origin is stable. For long term dynamics, this remainder can be relevant and there are procedures to bound its size and, hence, to bound the speed of the possible instability (usually called Arnol'd diffusion (Arnol'd, 1964)) that may take place around the point (Giorgilli et al., 1989; Simó, 1989; Giorgilli and Skokos, 1997; Benettin et al., 1998; Páez and Efthymiopoulos, 2018; Caracciolo and Locatelli, 2020). A similar machinery can be used for conservative maps (Simó and Vieiro, 2009).

An alternative to the computation of the Birkhoff normal form to bound Arnol'd diffusion is to compute formal first integrals (Marchal, 1980), again as a perturbative series that starts with the second degree terms of the Hamiltonian (H_2 above). Bounding the drift of a suitable truncation of these first integrals provides a bound on the diffusion speed Giorgilli (1979, 1988); Celletti and Giorgilli (1991).

It is important to note that, as this procedure is based on the manipulation of homogeneous polynomials, it can be carried out by a computer program. In fact, there is a long tradition in Celestial Mechanics of building algebraic manipulators to deal with these expansions (Broucke and Garthwaite, 1969; Ricklefs et al., 1983; Broucke, 1988; Meyer and Schmidt, 1986; Brumberg et al., 1989; Jorba, 1999). In many cases, floating point coefficients are used for the coefficients of these expansions, but it is possible to use interval arithmetic for them (Jorba, 1999) in order to control the rounding error of the operations, which can allow to produce a computer-assisted proof either to check the non-degeneracy condition of the KAM theorem (Meyer and Offin, 2017) and to prove the existence of invariant tori, or to bound the diffusion speed.

Another interesting situation is when the equilibrium point has some unstable directions. This is what happens near the collinear points of the restricted three-body problem (RTBP for short, see Meyer and Offin (2017)) or in some problems of reaction dynamics in chemical physics (Uzer et al., 2002). The neighbourhood of the collinear points of the RTBP are interesting for space missions, since they are a suitable place for some space missions. In these cases, it is necessary to find trajectories that stay near these points and this can be done by the Lindstedt-Poincaré method which is a recursive procedure to find perturbative expansions for periodic and quasi-periodic orbits near a known solution, that can be implemented in a computer program. Using floating points numbers as coefficients of the expansion simplifies the coding and also produces a very efficient algorithm (Richardson, 1980; Gómez et al., 1985, 1997, 1998; Jorba and Masdemont, 1999). Another option to study the dynamics is the so-called reduction to the centre manifold (Gómez et al., 1991), which is a partial normal form process that uncouples the hyperbolic directions from the elliptic ones, and that it allows to restrict the Hamiltonian to these elliptic directions to visualise the dynamics (Jorba and Masdemont, 1999). In some situations, a complete normal form is preferred, for instance to analyse passages near the hyperbolic point (Duarte and Jorba, 2021). When studying the reaction rate of chemical reactions, similar calculations appear (Jaffé et al., 2005; Haller et al., 2011). As it has been mentioned before, these numerical computations can be carried out using interval arithmetic so that the result can be used to produce a computer-assisted proof (Capiński and Roldán, 2012).

A difficulty of the previous approaches is when the interesting region is beyond the radius of convergence of the previous expressions. An option is to use different sets of coordinates (Skokos and Dokoumetzidis, 2000; Páez and Locatelli, 2015).

Time-dependent perturbations

There are several physical models that are written as an autonomous (usually dominant) part plus a time dependent (usually small) part. The goal is to study the effect of this time-dependent part on the dynamics. A typical question is how this perturbation affects the stability. Well-known problems of this king are the effective stability of the triangular points in the Elliptic RTBP, which can be seen as an autonomous part plus a time dependent part coming from the eccentricity of the primaries. A standard tool is the computation of suitable normal forms, which is quite similar to the autonomous case (Jorba and Simó, 1994; Lhotka et al., 2008). Another important modification of the RTBP are the Bicircular Problem (BCP, Huang (1960); Cronin et al. (1964)) and the Quasi-Bicircular Problem (QBCP, Andreu (2002)). These are periodically time-dependent perturbations of the Earth-Moon Circular RTBP that take into account the direct effect of the Sun on the infinitesimal particle. In these models, the equilibrium points are replaced by periodic orbits, and the dynamics around these periodic orbits can also be studied by means of suitable normalization procedures around periodic orbits (Simó et al., 1995; Jorba et al., 2020). As it happens in the autonomous case, the stability can also be studied by computing approximate first integrals, that now depend on time in a periodic way, see (Gabern and Jorba, 2001) for an example in the Sun-Jupiter-Saturn case. It is interesting to note that, under generic conditions of non-degeneracy and non-resonance Jorba and Villanueva (1997), a periodic orbit of the autonomous system becomes quasi-periodic when perturbed periodically, with two frequencies: the one of the periodic orbit plus the one of the perturbation. These quasi-periodic orbits can also be computed taking advantage of the perturbative situation (Castellà and Jorba, 2000).

There are some physical models that are written as time-dependent quasi-periodic perturbations of autonomous models (Gómez et al., 2002). In these models, the simplest solution is a quasi-periodic orbit, that can be computed by means of suitable numerical methods (Castellà and Jorba, 2000; Jorba and Olmedo, 2009). Its linear stability can be analysed by the method discussed in the Introduction, or by means of direct numerical methods (Jorba, 2001). Then, it is possible to start a normalising procedure to have a nonlinear description of the surrounding trajectories (Gabern and Jorba, 2005), or to compute approximate first integrals. There are some models for chemical reactions that also include time-dependent perturbations (Zhang and de la Llave, 2018).

The dynamics of these quasi-periodic models is more complex due to the high number of resonances. Therefore, their are very difficult to understand: as the model usually includes different effects, it can be difficult to find the relevance of each of them on the final properties of the system. The perturbative approach is to use a sequence of increasingly accurate (and increasingly complicated) models. First, the simplest model is studied in detail, and then this information is used to study the next model. This involves continuation methods, bifurcation analysis, normal forms, etc. Again, the result of this study is taken as starting point for the next model. At each stage new phenomena appear. At the end, therefore, it is possible to find (and understand) some phenomena present in the most realistic model but not present (even in a qualitative form) in the most simplified one. Examples of works using this approach are Gómez et al. (1985, 1987, 1991, 1993); Jorba (2000).

Finally, there are also some academical examples of quasi-periodic time dependent perturbations of autonomous systems. For a very accurate numerical perturbative treatment of a quasi-periodically forced pendulum, see Simó (1994).

Quasi-periodic motions and KAM Theory

In many situations, KAM methods are constructive and, in some situations, they can be used as a computational tool (Guzzo et al., 2020). For instance, it is possible to use the original Kolmogorov scheme (Kolmogorov, 1954) to construct invariant tori near some Trojan asteroid in the Circular RTBP and to use this construction to show the effective stability of the asteroid (Gabern et al., 2005). The explicit construction of a normal form around a periodic orbit of an autonomous system is described in Jorba and Villanueva (1998).

On the other hand, rigorous numerical methods can be used to validate the hypotheses required for a suitable version of a theorem and then to show the existence of plenty of quasi-periodic motions near a known periodic orbit (Kapela and Simó, 2017). These ideas are quite general and are being applied to many situations. For instance, to study the existence of quasi-periodic motions in realistic celestial mechanics problems (Robutel, 1995) like the Sun-Jupiter-Saturn system (Locatelli and Giorgilli, 2000, 2005, 2007) or the Sun-Jupiter-Saturn-Uranus system (Sansottera et al., 2011). These normalizing techniques can also be used to prove the stability for very long times (Giorgilli and Locatelli, 2009; Martínez and Simó, 2013; Giorgilli et al., 2017).

A problem of great practical interest is the motion of particles around the Earth. This, of course, includes artificial satellites but also particles of space debris whose dynamics is becoming very important due to the danger they pose for operating satellites and spacecrafts. Their motion is close to Keplerian, and the main perturbations come from the non-symmetric mass distribution of the Earth, the gravitational effects of Sun and Moon and the Solar radiation pressure. These orbits can be studied numerically but, since they are close to Keplerian motion, they can also be studied using a combination of numerical and perturbation methods (Lara et al., 2012, 2014, 2018, 2020), (Celletti and Galeş, 2014, 2018; Celletti et al., 2020), (Gachet et al., 2017).

More recently, a new point of view has appeared. Assume that, by means of numerical methods, we have been able to obtain an accurate approximation to a periodic or quasi-periodic orbit of a Hamiltonian system H, and the goal is to show that there exists a true periodic or quasi-periodic orbit nearby. The idea is to think that the result of the computation is a true orbit of a nearby Hamiltonian system, $\hat{H}$, and then write H as $\hat{H} + (H - \hat{H})$, where $H - \hat{H}$ should be small. This is again a perturbative situation, of a very special kind. Under generic assumptions it is possible to prove that, if the computation of the initial periodic or quasi-periodic orbit is sufficiently accurate (this is like saying that $H - \hat{H}$ is small enough), then there is a true periodic or quasi-periodic orbit nearby (de la Llave et al., 2005). An advantage of this approach is that the hypotheses can be checked on the initial Hamiltonian (i.e., no need to use action-angle variables). This kind of results are key ingredients to produce computer-assisted proofs and to validate numerical computations (that is, to have a true bound on the error of the numerical computation). This allows to be confident of the results in some difficult situations when it is nearly impossible to have a reliable control of the errors by other means. For examples, see (Figueras and Haro, 2012; Figueras et al., 2017).

The parametrization method

This method was used in the 80's to compute invariant manifolds by C. Simó (see also Franceschini and Russo (1981)), but it is also a very good tool to prove their existence as shown by Cabré et al. (2003a,b); Cabré et al. (2005). A very good exposition of the method can be found in the book by Haro et al. (2016).

The idea of the method is to write the manifold of an equilibrium point (or fixed point for a discrete system) in parametric form, to expand it in power series, and to find recursively the coefficients of this expansion by imposing that the manifold is invariant under the dynamics (Simó, 1990). Under suitable hypotheses, this power series is convergent in a neighbourhood of the point giving, at the same time, a computational algorithm and the scheme of a proof. These ideas can be extended for manifolds around periodic orbits or invariant tori (Haro and de la Llave, 2006a,b; Castelli et al., 2015).

A technique that can be combined with the parametrization method is the so-called jet transport. Jet transport is based on using automatic differentiation (Griewank, 2000) on the code of a numerical method for initial value problems of ODEs. The main idea is to replace the basic arithmetic by an arithmetic of (truncated) formal power series in several variables. Note that a formal power series contains the value of a function in a point (the constant term) and their derivatives at this point (the coefficients of each monomial), and the propagation of these power series through the operations of the numerical integration gives the derivative of the final point with respect to the initial data and/or parameters (Berz and Makino, 1998; Alessi et al., 2008). This implies that it is possible to compute derivatives of any order of the flow and, therefore, derivatives of the flow. In particular, this allows to use the parametrization method on a Poincaré section. (Jorba and Nicolás, 2021).

Other situations

This section is devoted to survey some results for general dynamical systems. A first important context is given by delay differential equations. Since the phase space is of infinite dimension, it is very helpful to take advantage of perturbative schemes to avoid brute-force computations. Examples of finding perturbative expansions in some concrete situations are Groothedde and Mireles-James (2017); Yang et al. (2021); Gimeno et al. (2021). Moreover, some of these techniques can also be used for some partial differential equations (Zgliczynski and Mischaikow, 2001; Arioli and Koch, 2010; Breden et al., 2013; Reinhardt and Mireles-James, 2019).

Perturbative techniques can also be adapted to do computations on dissipative systems (Calleja and Celletti, 2010). Among them, there are conformally symplectic systems which are similar to conservative systems, in the sense that the symplectic structure is not preserved but multiplied by a constant. This makes that they can be studied by similar methods. (Calleja et al., 2013; Bustamante and Calleja, 2019, 2021; Calleja et al., 2021).

There are works using perturbative techniques on more general dynamical systems, like computing expansions of invariant manifolds or studying chaos and diffusion (Lessard et al., 2014; Castelli et al., 2015; van den Berg et al., 2016; Gelfreich and Vieiro, 2018; Meiss et al., 2018).

Future directions

Computational methods in perturbation theory are experiencing a big development in the recent years, combined with new theoretical results. In particular, very relevant advances are taking place in some computational methods like the parametrization method and the jet transport technique. They are allowing to develop high order perturbative expansions in new situations. These new tehcniques are expected to have an impact in the design of space missions during the next years (Chen et al., 2020; Kumar et al., 2021). Another problem of high interest in astrodynamics is

the dynamics of space debris (Celletti et al., 2017), which has become very important for space navigation. At present, there is a lot of research going on in this direction so it is expected to have important advances in the forthcoming years.

A field in which there is a lot of activity is the production of computer-assisted proofs (CAP). They allow to produce formal mathematical proofs in situations that are far beyond the reach of the classical mathematical machinery (van den Berg and Jaquette, 2018; González et al., 2021; Valvo and Locatelli, 2021). It is expected that this research line will continue growing. In this direction, the parametrization method is being extended to more situations, giving rise to new methods and applications (Haro and Mondelo, 2021).

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