Superstable periodic orbits of 1d maps under quasi-periodic forcing and reducibility loss*

Àngel Jorba, Pau Rabassa and Joan Carles Tatjer
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mails: angel@maia.ub.es, jcarles@maia.ub.es, prs@maia.ub.es

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Abstract

Let g_α be a one-parameter family of one-dimensional maps with a cascade of period doubling bifurcations. Between each of these bifurcations, a superstable periodic orbit is known to exist. An example of such a family is the well-known logistic map. In this paper we deal with the effect of a quasi-periodic perturbation (with only one frequency) on this cascade. Let us call ε the perturbing parameter. It is known that, if ε is small enough, the superstable periodic orbits of the unperturbed map become attracting invariant curves (depending on α and ε) of the perturbed system. In this article we focus on the reducibility of these invariant curves.

The paper shows that, under generic conditions, there are both reducible and non-reducible invariant curves depending on the values of α and ε. The curves in the space (α, ε) separating the reducible (or the non-reducible) regions are called reducibility loss bifurcation curves. If the map satifies an extra condition (condition satisfied by the quasi-periodically forced logistic map) then we show that, from each superattracting point of the unperturbed map, two reducibility loss bifurcation curves are born. This means that these curves are present for all the cascade.

1 Introduction

The results in this paper are motivated by the study of period doubling bifurcation cascades of invariant curves in a family of autonomous 1D maps under a quasi-periodic perturbation, which is one of the classical routes to chaos. In these maps, the unperturbed (autonomous) part has a period doubling bifurcation cascade and we focus on the effect of the perturbation. Numerical experiments show that, for a fixed value of the perturbing parameter, these maps only have a finite cascade of period doubling bifurcations

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of invariant curves ([Kan83, Kan84]), and this seems related to the possible existence of strange nonchaotic atractors (see [FKP06] and references therein). Our goal is to study the dynamical properties of the invariant curves along the cascade.

1.1 Basic definitions and concepts

Let us start with some basic definitions and concepts. A *quasi-periodically forced one dimensional map* is a map of the form

$$\begin{array}{cccc} F: & \mathbb{T}\times\mathbb{R} & \to & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta+\omega \\ f(\theta,x) \end{pmatrix} \end{array} \tag{1}$$

where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, $f\in C^r(\mathbb{T}\times\mathbb{R},\mathbb{R})$ with $r\geq 1$ and $\omega\in\mathbb{T}\setminus\mathbb{Q}$. A quasi-periodically forced map determines a dynamical system on the cylinder, explicitly defined as

$$\left.\begin{array}{l} \bar{\theta}=\theta+\omega, \\ \bar{x}=f(\theta,x). \end{array}\right\} \tag{2}$$

A continuous function $u:\mathbb{T}\to\mathbb{R}$ is an *invariant curve* of (2) if and only if $u(\theta+\omega)=f(\theta,u(\theta))$, for all $\theta\in\mathbb{T}$. The value ω is known as the *rotation number* of the curve. An equivalent way to define an invariant curve is to require the set $\{(\theta,x)\in\mathbb{T}\times\mathbb{R}\mid x=u(\theta)\}$ to be invariant by F, where F is the function defined by (1). On the other hand, note that F^n is also a quasi-periodically forced map. The rotation number is said to be *Diophantine* if there exist $\gamma>0$ and $\tau\geq 1$ such that

$$|q\omega-p|\geq\frac{\gamma}{|q|^\tau},\qquad\text{for all } (p,q)\in\mathbb{Z}\times\mathbb{Z}\setminus\{0\}.$$

Given a function $u:\mathbb{T}\to\mathbb{R}$, we say that u is a *n-periodic invariant curve* of F if u is invariant by F^n (and there is no smaller n satisfying such condition).

Given $x=u_0(\theta)$ an invariant curve of (2) of class C^r ($r\geq 1$), its linearized normal behaviour is described by the following linear skew product:

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta+\omega, \\ \bar{x} & = & a(\theta)x, \end{array}\right\} \tag{3}$$

where $a(\theta)=D_xf(\theta,u_0(\theta))$ is of class C^{r-1}, $x\in\mathbb{R}$ and $\theta\in\mathbb{T}$.

A linear skew product like (3) is called *reducible* if, and only if, there exists a change of variable $x=c(\theta)y$, continuous with respect to θ, such that (3) becomes

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta+\omega, \\ \bar{y} & = & by, \end{array}\right\}$$

where b does not depend on θ. The constant b is called the *multiplier* of the reduced system. An invariant curve is called *reducible* if its linearized normal behaviour (3) is reducible. An n-periodic invariant curve is reducible if it is reducible for F^n.

In the case that $a(\cdot)$ is a C^∞ function and ω is Diophantine, the skew product (3) is reducible if, and only if, $a(\cdot)$ has no zeros [JT08]. Due to this property, the reducibility loss can be characterized as a codimension one bifurcation.

Let us consider a one-parametric family of linear skew-products

$$\left.\begin{array}{rcl}\bar{\theta} & = & \theta+\omega, \\ \bar{x} & = & a(\theta,\mu)x,\end{array}\right\} \tag{4}$$

where ω is Diophantine, μ belongs to an open set of $\mathbb{R}$ and a is a C^∞ function of θ and μ. We say that the system (4) undergoes a *reducibility loss bifurcation* at μ_0 if

1. $a(\cdot,\mu)$ has no zeros for $\mu<\mu_0$,
2. $a(\cdot,\mu)$ has a double zero at θ_0 for $\mu=\mu_0$,
3. $\frac{d}{d\mu}a(\theta_0,\mu_0)\neq 0$.

On the other hand, consider a system like (2) with f a C^∞ function, which depends (smoothly) on a one dimensional parameter μ (we denote this dependence as $f=f_\mu$), having an invariant curve $u=u_\mu$. We will say that *the invariant curve undergoes a reducibility loss bifurcation* if the family of linear skew-products (4), where $a(\theta,\mu)=D_x f_\mu(\theta,u_\mu(\theta))$, undergoes a reducibility loss bifurcation. Note that the reducibility loss takes place when the number of zeros of $\theta\mapsto a(\theta,\mu)$ goes from 0 to 2 when μ crosses μ_0. The number of zeros of a is invariant under linear changes of variables (for more details see Section 3 in [JT08]).

Definition 1.1 *For a given value of μ, if the number of zeros (counting their multiplicity) of $\theta\mapsto a(\theta,\mu)$ is finite, this number is called degree of non-reducibility.*

In what follows, we will refer to degree of non-reducibility simply as degree. Note that, as a is C^∞ and ω is Diophantine, degree zero is equivalent to reducibility.

In this paper we focus on maps of the form

$$\begin{array}{rccc}F_{\alpha,\varepsilon}: & \mathbb{T}\times\mathbb{R} & \to & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix}\theta\\ x\end{pmatrix} & \mapsto & \begin{pmatrix}\theta+\omega \\ f(\theta,x,\alpha,\varepsilon)\end{pmatrix},\end{array} \tag{5}$$

where ω is Diophantine, α and ε are real parameters and f is of the form

$$f(\theta,x,\alpha,\varepsilon)=g(x,\alpha)+\varepsilon h(\theta,x,\alpha,\varepsilon),$$

where g and h are C^∞ functions. The function g is assumed to have a cascade of period doubling bifurcations. This means that we also have a sequence of superstable periodic orbits. We recall that a superstable periodic orbit is a periodic orbit with a critical point, i.e., a point with zero derivative.

If x_0 is an attracting fixed point of $x\mapsto g(x,\alpha_0)$ then for $|\alpha-\alpha_0|$ and $|\varepsilon|$ small enough there exists a unique invariant curve $x=x(\theta,\alpha,\varepsilon)$ of (5) such that it is smooth w.r.t. all its arguments and $x(\theta, \alpha_0, 0) = x_0$ for all $\theta \in \mathbb{T}$ (see Section 2 in [JT08]). Moreover, the function

$$a(\theta, \alpha, \varepsilon) = \frac{\partial f}{\partial x}(\theta, x(\theta, \alpha, \varepsilon), \alpha, \varepsilon),$$

describing the linearized normal behaviour around the invariant curve satisfies, for $\varepsilon = 0$, $a(\theta, \alpha_0, 0) = \frac{\partial g}{\partial x}(x_0, \alpha_0)$. Hence, if $\frac{\partial g}{\partial x}(x_0, \alpha_0) \neq 0$, the map $\theta \mapsto a(\theta, \alpha, \varepsilon)$ has no zeros so the curve is reducible. In general, if $\left|\frac{\partial g}{\partial x}(x_0, \alpha_0)\right| \neq 1$ (i.e., the point is hyperbolic), the implicit function theorem can be applied to prove the existence of an invariant curve close to x_0 for α close to α_0 and ε close to 0.

1.2 Summary of results

One of the goals of this paper is to describe the reducibility of the invariant curves that appear as a perturbation of a superstable fixed point. We have just mentioned that this point can be continued to an invariant curve $x = x(\theta, \alpha, \varepsilon)$ for $|\alpha - \alpha_0|$ and $|\varepsilon|$ small enough. However, the reducibility in this case is not clear: as $a(\theta, \alpha_0, 0) = 0$ (for all θ), it is not immediate to predict in general the number of zeros of $\theta \mapsto a(\theta, \alpha, \varepsilon)$ (note that this can depend on the values of α and ε).

In general, one should expect curves in the parameter space (α, ε) where the degree changes (these curves correspond to values of (α, ε) for which $\theta \mapsto a(\theta, \alpha, \varepsilon)$ has a double zero). We will refer to them as *curves of change of degree*. Among them, the most interesting ones are the curves corresponding to a transition between degree 0 (no zeros) and degree two (two zeros): when ω is Diophantine and a is C^∞, this is a transition from reducible to non-reducible. In this case, we will refer to these curves as *reducibility loss bifurcation curves*. In this paper we prove, under very general conditions, the existence of curves of change of degree.

Next, we focus on the application of the previous result to the superstable orbits that appear along a cascade of period doubling bifurcations. The main difficulty is to check the hypohteses for the iterated map F^{2^n}. Here we give suitable conditions on F such that the hypotheses are satisfied by F^{2^n} (for all n) and for a set of full Lebesgue measure of values of ω. These results are illustrated with the quasi-periodically forced logistic map.

In previous works [JRT12, RJT13] we have considered the role of reducibility in the behaviour of the period doubling cascade. The reason is that, while the bifurcations of reducible curves are studied with very standard tools (they are similar to the bifurcations of periodic orbits), the bifurcations of non-reducible curves are completely different. For instance, in some situations a non-reducible invariant curve is destroyed giving rise to a SNA or a chaotic attractor [GOPY84, PNR01, FKP06]. Hence, it is natural to consider the role of non-reducible curves in the finite character of the cascade.

In this paper we prove the existence of reducibility and non-reducibility regions near values of the parameters for which there exist superstable periodic orbits.

2 Curves of change of degree

In the following theorem we deal with the existence of curves of change of degree. As it has been mentioned before, when the maps are C^∞ and ω is Diophantine, the curves corresponding to a transition between degrees 0 and 2 are also reducibility loss bifurcation curves.

Theorem 2.1 *Consider the map*

$$F_{\alpha,\varepsilon}: \begin{array}{ccc} \mathbb{T}\times\mathbb{R} & \to & \mathbb{T}\times\mathbb{R} \\ \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta+\omega \\ f(\theta,x,\alpha,\varepsilon) \end{pmatrix}, \end{array}$$

where ω is irrational, α and ε are real parameters, and f is of the form

$$f(\theta,x,\alpha,\varepsilon) = g(x,\alpha) + \varepsilon h(\theta,x,\alpha,\varepsilon),$$

where g and h are C^r ($r\geq 3$). Moreover, for $\alpha=\alpha_0$, g has a superattracting nondegenerate fixed point at x_0. In other words,

1. $g(x_0,\alpha_0)=x_0$.
2. $\dfrac{\partial g}{\partial x}(x_0,\alpha_0)=0$.
3. $\dfrac{\partial^2 g}{\partial x^2}(x_0,\alpha_0)\dfrac{\partial g}{\partial \alpha}(x_0,\alpha_0)+\dfrac{\partial^2 g}{\partial x\partial\alpha}(x_0,\alpha_0)\neq 0$.

Let H be the function

$$\theta\mapsto H(\theta) = h(\theta-\omega,x_0,\alpha_0,0)\frac{\partial^2 g}{\partial x^2}(x_0,\alpha_0)+\frac{\partial h}{\partial x}(\theta,x_0,\alpha_0,0).$$

Then, for each simple zero θ_0 of $H'(\theta)$ and for $|\varepsilon|$ small enough, there exists a function $\alpha=\alpha(\varepsilon)$ such that $\alpha(0)=\alpha_0$ and the curve $(\alpha(\varepsilon),\varepsilon)$ is a curve of change of degree for the unique invariant curve $x=z(\theta,\alpha,\varepsilon)$ obtained as the continuation of the invariant curve $z(\theta,\alpha_0,0)\equiv x_0$. Moreover,

$$\alpha'(0) = -\frac{H(\theta_0)}{\dfrac{\partial^2 g}{\partial x^2}(x_0,\alpha_0)\dfrac{\partial g}{\partial\alpha}(x_0,\alpha_0)+\dfrac{\partial^2 g}{\partial x\partial\alpha}(x_0,\alpha_0)}. \tag{6}$$

Proof: Without loss of generality, we can assume that $x_0=0$ and $\alpha_0=0$. A straightforward application of the Implicit Function Theorem ensures that $x=g(x,\alpha)$ defines, for $|\alpha|$ small, a function $x=p(\alpha)$ such that $p(\alpha)=g(p(\alpha),\alpha)$ and $p(0)=0$ (i.e., $x=p(\alpha)$ is the curve of –attracting– fixed points of g w.r.t. the parameter α near $\alpha=0$). Moreover, under these conditions it is known that there exists an invariant curve $x=z(\theta,\alpha,\varepsilon)$ such that $z(\theta,\alpha,0)\equiv p(\alpha)$.

Let us rescale the parameter α as $\alpha = \varepsilon\beta$. It is not difficult to see that z can be written as

$$z(\theta, \alpha, \varepsilon) = \left[\frac{\partial g}{\partial \alpha}(0,0)\beta + h(\theta - \omega, 0, 0, 0)\right]\varepsilon + O(\varepsilon^2).$$

The linearized normal behaviour along this curve is described by a linear skew product (4), where

$$a(\theta, \beta, \varepsilon) = \frac{\partial f}{\partial x}(\theta, z(\theta, \varepsilon\beta, \varepsilon), \varepsilon\beta, \varepsilon).$$

It is not difficult to see that

$$\begin{aligned} a(\theta, \beta, \varepsilon) \;&=\; \Bigg[h(\theta - \omega, 0, 0, 0)\frac{\partial^2 g}{\partial x^2}(0,0) + \frac{\partial h}{\partial x}(\theta, 0, 0, 0) \\ &\quad + \left(\frac{\partial^2 g}{\partial x^2}(0,0)\frac{\partial g}{\partial \alpha}(0,0) + \frac{\partial^2 g}{\partial x \partial \alpha}(0,0)\right)\beta\Bigg]\varepsilon + O(\varepsilon^2). \end{aligned}$$

The condition for a to gain (or lose) a zero in θ is to have a double zero. Hence, we ask for the conditions

$$h(\theta-\omega, 0, 0, 0)\frac{\partial^2 g}{\partial x^2}(0,0) + \frac{\partial h}{\partial x}(\theta, 0, 0, 0) + \left(\frac{\partial^2 g}{\partial x^2}(0,0)\frac{\partial g}{\partial \alpha}(0,0) + \frac{\partial^2 g}{\partial x \partial \alpha}(0,0)\right)\beta + O(\varepsilon) = 0,$$

$$\frac{\partial h}{\partial \theta}(\theta - \omega, 0, 0, 0)\frac{\partial^2 g}{\partial x^2}(0,0) + \frac{\partial^2 h}{\partial x \partial \theta}(\theta, 0, 0, 0) + O(\varepsilon) = 0.$$

If θ_0 is a simple zero of the second equation and β_0 is given by

$$\beta_0 = -\frac{h(\theta_0 - \omega, 0, 0, 0)\dfrac{\partial^2 g}{\partial x^2}(0,0) + \dfrac{\partial h}{\partial x}(\theta_0, 0, 0, 0)}{\dfrac{\partial^2 g}{\partial x^2}(0,0)\dfrac{\partial g}{\partial \alpha}(0,0) + \dfrac{\partial^2 g}{\partial x \partial \alpha}(0,0)},$$

applying the Implicit Function Theorem we have that there exist functions $\theta = \theta(\varepsilon)$ and $\beta = \beta(\varepsilon)$ such that $\theta(0) = \theta_0$, $\beta(0) = \beta_0$ and

$$a(\theta(\varepsilon), \beta(\varepsilon), \varepsilon) = 0, \qquad \frac{\partial a}{\partial \theta}(\theta(\varepsilon), \beta(\varepsilon), \varepsilon) = 0,$$

for $|\varepsilon|$ small enough. ■

2.1 A particular class of maps

We are interested in the application of these results to two paradigmatic examples: the quasiperiodically forced logistic map,

$$\left.\begin{array}{rcl} \bar{\theta} &=& \theta + \omega \\ \bar{x} &=& \alpha x(1-x)(1 + \varepsilon\cos(2\pi\theta)) \end{array}\right\}, \tag{7}$$

and the quasiperiodically driven logistic map,

$$\left.\begin{array}{rcl}\bar{\theta} & = & \theta+\omega \\ \bar{x} & = & \alpha x(1-x)+\varepsilon\cos(2\pi\theta)\end{array}\right\}. \tag{8}$$

In both cases, $\omega \in \mathbb{T}\setminus\mathbb{Q}$. Therefore, we rewrite the previous theorem for a more concrete class of maps that includes these examples.

Corollary 2.1 *Consider the map*

$$\begin{array}{rccc} F_{\alpha,\varepsilon}: & \mathbb{T}\times\mathbb{R} & \rightarrow & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix}\theta \\ x\end{pmatrix} & \mapsto & \begin{pmatrix}\theta+\omega \\ f(\theta,x,\alpha,\varepsilon)\end{pmatrix}, \end{array}$$

where ω is irrational, α and ε are real parameters, and f is of the form

$$f(\theta,x,\alpha,\varepsilon)=g(x,\alpha)+\varepsilon h(\theta,x,\alpha,\varepsilon),$$

where g and h are C^r ($r\geq 3$), g has a critical point x_0 that does not depend on α. Moreover, x_0 is a fixed point for $\alpha=\alpha_0$ and h is of the form

$$h(\theta,x,\alpha,\varepsilon)=h_0(x,\alpha)+h_c(x,\alpha)\cos 2\pi\theta+h_s(x,\alpha)\sin 2\pi\theta+O(\varepsilon).$$

More concretely, we ask for the conditions

1. $\dfrac{\partial g}{\partial x}(x_0,\alpha)=0$, $\dfrac{\partial^2 g}{\partial x^2}(x_0,\alpha)\neq 0$, *for all* α.
2. $g(x_0,\alpha_0)=x_0$, $\dfrac{\partial g}{\partial \alpha}(x_0,\alpha_0)\neq 0$.
3. $|h_c(x_0,\alpha_0)|+|h_s(x_0,\alpha_0)|+\left|\dfrac{\partial h_c}{\partial x}(x_0,\alpha_0)\right|+\left|\dfrac{\partial h_s}{\partial x}(x_0,\alpha_0)\right|\neq 0.$

Then, except may be for two values of ω, there exist exactly two functions $\alpha=\alpha^+(\varepsilon)$ and $\alpha=\alpha^-(\varepsilon)$, for $|\varepsilon|$ small enough, such that $\alpha^\pm(0)=\alpha_0$ and $(\alpha^\pm(\varepsilon),\varepsilon)$ are curves of change of degree between degrees 0 and 2.

Proof: As hypotheses 1, 2 and 3 of Theorem 2.1 can be easily checked, we only have to look for the zeros of H'. So, as

$$\begin{aligned} H(\theta) = {} & [h_0(x_0,\alpha_0)+h_c(x_0,\alpha_0)\cos 2\pi(\theta-\omega)+h_s(x_0,\alpha_0)\sin 2\pi(\theta-\omega)]\frac{\partial^2 g}{\partial x^2}(x_0,\alpha_0) \\ & +\frac{\partial h_0}{\partial x}(x_0,\alpha_0)+\frac{\partial h_c}{\partial x}(x_0,\alpha_0)\cos 2\pi\theta+\frac{\partial h_s}{\partial x}(x_0,\alpha_0)\sin 2\pi\theta, \end{aligned}$$

we have that $H'(\theta)=2\pi a_c(\omega)\cos 2\pi\theta+2\pi a_s(\omega)\sin 2\pi\theta$, where

$$\begin{aligned} a_c(\omega) &= (h_c(x_0,\alpha_0)\sin 2\pi\omega+h_s(x_0,\alpha_0)\cos 2\pi\omega)\frac{\partial^2 g}{\partial x^2}(x_0,\alpha_0)+\frac{\partial h_s}{\partial x}(x_0,\alpha_0), \\ a_s(\omega) &= (-h_c(x_0,\alpha_0)\cos 2\pi\omega+h_s(x_0,\alpha_0)\sin 2\pi\omega)\frac{\partial^2 g}{\partial x^2}(x_0,\alpha_0)-\frac{\partial h_c}{\partial x}(x_0,\alpha_0). \end{aligned}$$

The hypotheses of the theorem ensure that $a_c(\omega)$ and $a_s(\omega)$ cannot be both zero except, may be, for two values of ω. Therefore, there exist two simple zeros of $H'(\theta)$ and then Theorem 2.1 implies the existence of two curves of change of degree. ■

The two values of ω excluded in the statement of Corollary 2.1 correspond to common zeros of $a_c(\omega)$ and $a_s(\omega)$. In a general example it is very unlikely to have values of ω that are, at the same time, zeros of $a_c(\omega)$ and $a_s(\omega)$. For instance, the next result is valid for all $\omega \in \mathbb{T} \setminus \mathbb{Q}$.

Corollary 2.2 *For $\alpha = \alpha_0 = 2$ and $x = x_0 = \frac{1}{2}$, both the quasi-periodically forced logistic map (7) and the quasi-periodically driven logistic map (8) have two curves $\alpha^{\pm}(\varepsilon)$, defined in the parameter space (ε, α) for $|\varepsilon|$ small enough, such that*

1. $\alpha^{\pm}(0) = \alpha_0 = 2$.
2. *These curves are curves of change of degree, between degree 0 and 2.*
3. *For the quasi-periodically forced logistic map (7), $\frac{d}{d\varepsilon}\alpha^{\pm}(0) = \pm 2$.*
4. *For the quasi-periodically driven logistic map (8), $\frac{d}{d\varepsilon}\alpha^{\pm}(0) = \pm 4$.*

Proof: To apply Corollary 2.1 we check that $\frac{\partial g}{\partial x}(x_0, \alpha_0) = 0$, $\frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) = -4$ and $\frac{\partial g}{\partial \alpha}(x_0, \alpha_0) = \frac{1}{4}$. Moreover, $\frac{\partial^2 g}{\partial x \partial \alpha}(x_0, \alpha_0) = 0$. For the quasi-periodically forced logistic map (7), we have that $h_0(x_0, \alpha_0) = 0$, $h_c(x_0, \alpha_0) = \frac{1}{2}$ and $h_s(x_0, \alpha_0) = 0$. Then, $H(\theta) = -2\cos(2\pi(\theta - \omega))$ and H' has the two zeros $\theta = \omega$ and $\theta = \omega + \frac{1}{2}$, and using (6) we obtain $\frac{d}{d\varepsilon}\alpha^{\pm}(0) = \pm 2$. For the quasi-periodically driven logistic map (8), we have that $h_0(x_0, \alpha_0) = 0$, $h_c(x_0, \alpha_0) = 1$ and $h_s(x_0, \alpha_0) = 0$. Then, $H(\theta) = -4\cos(2\pi(\theta - \omega))$ and H' has the two zeros $\theta = \omega$ and $\theta = \omega + \frac{1}{2}$, and using (6) we obtain $\frac{d}{d\varepsilon}\alpha^{\pm}(0) = \pm 4$. ■

We note that the values $\frac{d}{d\varepsilon}\alpha^{\pm}(0)$ have been computed in [RJT13] by means of purely numerical methods.

3 Cascades of reducibility loss bifurcation curves

In this section we focus on the effect of a quasi-periodic perturbation on the superstable points that appear along a bifurcation cascade and, more concretely, on the reducibility of the invariant curves that are born at these points as a response to the quasi-periodic excitation. The main difficulty to apply the previous results to a superstable periodic point is that the hypotheses have to be verified on $F_{\alpha,\varepsilon}^{2^n}$ (for all n).

Theorem 3.1 *Consider the map*

$$F_{\alpha,\varepsilon} : \begin{array}{ccc} \mathbb{T} \times \mathbb{R} & \to & \mathbb{T} \times \mathbb{R} \\ \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta + \omega \\ f(\theta, x, \alpha, \varepsilon) \end{pmatrix} \end{array},$$

where ω is irrational, α and ε are real parameters, and f is of the form

$$f(\theta, x, \alpha, \varepsilon) = g(x, \alpha) + \varepsilon h(\theta, x, \alpha, \varepsilon),$$

where g and h are C^r ($r \geq 3$), g is a unimodal map with a (unique) critical point x_0 that does not depend on α, and h is of the form

$$h(\theta, x, \alpha, \varepsilon) = h_0(x, \alpha) + h_c(x, \alpha)\cos 2\pi\theta + h_s(x, \alpha)\sin 2\pi\theta + O(\varepsilon).$$

We assume that there is a sequence of values of the parameter $\{\alpha_n\}_{n\geq 0}$ such that, for each n, $g(\cdot, \alpha_n)$ has a superattracting periodic orbit of period 2^n (and not 2^{n-1}). Let us write g_n to denote g^{2^n}. More concretely, we ask for the conditions

1. $\dfrac{\partial g}{\partial x}(x_0, \alpha) = 0$, $\dfrac{\partial^2 g}{\partial x^2}(x_0, \alpha) \neq 0$, *for all α.*
2. $g_n(x_0, \alpha_n) = x_0$, $\dfrac{\partial g_n}{\partial \alpha}(x_0, \alpha_n) \neq 0$, *for all $n \geq 0$.*
3. $|h_c(x_0, \alpha_n)| + |h_s(x_0, \alpha_n)| \neq 0$, *for all $n \geq 0$.*

Then, there exists a set $\Omega \subset \mathbb{T} \setminus \mathbb{Q}$ such that, if $\omega \in \Omega$, for each n there exist exactly two functions $\alpha = \alpha_n^+(\varepsilon)$ and $\alpha = \alpha_n^-(\varepsilon)$, for $|\varepsilon|$ small enough (depending on n), such that $\alpha_n^\pm(0) = \alpha_n$ and $(\alpha_n^\pm(\varepsilon), \varepsilon)$ are curves of change of degree between degrees 0 and 2. The set Ω is of full Lebesgue measure, more concretely the set $(\mathbb{T} \setminus \mathbb{Q}) \setminus \Omega$ is countable.

To prove the theorem we will show first (in Lemma 3.1) that if $F_{\alpha,\varepsilon}$ has a special form the result holds. The proof will be completed showing that the 2^n-th iterate of the map in the Theorem has this special form.

Lemma 3.1 *Consider the map*

$$\begin{array}{rccc} F_{\alpha,\varepsilon}: & \mathbb{T}\times\mathbb{R} & \to & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta + 2^n\omega \\ f(\theta, x, \alpha, \varepsilon, \omega) \end{pmatrix}, \end{array}$$

where ω is irrational, α and ε are real parameters, and f is of the form

$$f(\theta, x, \alpha, \varepsilon, \omega) = g(x, \alpha) + \varepsilon h(\theta, x, \alpha, \varepsilon, \omega),$$

where g and h are C^r ($r \geq 3$), g has a critical point x_0 that does not depend on α, and h is of the form

$$h(\theta, x, \alpha, \varepsilon, \omega) = h_0(x, \alpha, \omega) + h_{n,c}(x, \alpha, \omega)\cos 2\pi\theta + h_{n,s}(x, \alpha, \omega)\sin 2\pi\theta + O(\varepsilon),$$

where h_0, $h_{n,c}$ and $h_{n,s}$ are trigonometric polynomials in ω, satisfying

$$\begin{pmatrix} h_{n,c}(x, \alpha, \omega) \\ h_{n,s}(x, \alpha, \omega) \end{pmatrix} = \left(\bar{b}(x, \alpha) I + \sum_{j=1}^{2^n - 1} b_j(x, \alpha) R_j(-\omega) \right) \begin{pmatrix} h_c(x, \alpha) \\ h_s(x, \alpha) \end{pmatrix},$$

where I is the identity matrix and $R_j(\varphi)$ is a rotation of angle $2\pi j\varphi$. Moreover, x_0 is a nondegenerate critical point and fixed point for $\alpha = \alpha_0$ of the map g,

1. $\frac{\partial g}{\partial x}(x_0, \alpha) = 0$, $\frac{\partial^2 g}{\partial x^2}(x_0, \alpha) \neq 0$, *for all* α,

2. $g(x_0, \alpha_0) = x_0$, $\frac{\partial g}{\partial \alpha}(x_0, \alpha_0) \neq 0$,

and

3. $\bar{b}(x_0, \alpha_0) \neq 0$,

4. $|h_c(x_0, \alpha_0)| + |h_s(x_0, \alpha_0)| \neq 0$.

Then, except may be for a finite number of values of ω*, there exist exactly two functions* $\alpha = \alpha^+(\varepsilon)$ *and* $\alpha = \alpha^-(\varepsilon)$*, for* $|\varepsilon|$ *small enough, such that* $\alpha^{\pm}(0) = \alpha_0$ *and* $(\alpha^{\pm}(\varepsilon), \varepsilon)$ *are curves of change of degree between degrees 0 and 2.*

Proof: To apply Theorem 2.1 we have to construct the function H of its statement: in this case, H reads as

$$\begin{aligned} H(\theta) = {} & [h_0(x_0, \alpha_0, \omega) + h_{n,c}(x_0, \alpha_0, \omega)\cos 2\pi(\theta - 2^n\omega) + h_{n,s}(x_0, \alpha_0, \omega)\sin 2\pi(\theta - 2^n\omega)] \\ & \times \frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) + \frac{\partial h_0}{\partial x}(x_0, \alpha_0, \omega) + \frac{\partial h_{n,c}}{\partial x}(x_0, \alpha_0, \omega)\cos 2\pi\theta + \frac{\partial h_{n,s}}{\partial x}(x_0, \alpha_0, \omega)\sin 2\pi\theta. \end{aligned}$$

We note that H' is of the form $H'(\theta) = 2\pi a_c(\omega)\cos 2\pi\theta + 2\pi a_s(\omega)\sin 2\pi\theta$, where

$$\begin{aligned} \begin{pmatrix} a_c(\omega) \\ a_s(\omega) \end{pmatrix} &= \frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) \begin{pmatrix} \sin 2\pi 2^n\omega & \cos 2\pi 2^n\omega \\ -\cos 2\pi 2^n\omega & \sin 2\pi 2^n\omega \end{pmatrix} \begin{pmatrix} h_{n,c}(x_0, \alpha_0, \omega) \\ h_{n,s}(x_0, \alpha_0, \omega) \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{\partial h_{n,s}}{\partial x}(x_0, \alpha_0, \omega) \\ -\frac{\partial h_{n,c}}{\partial x}(x_0, \alpha_0, \omega) \end{pmatrix}. \end{aligned}$$

To simplify the notation, we call $S_m(\varphi)$ to the matrix

$$S_m(\varphi) = \begin{pmatrix} \sin 2\pi m\varphi & \cos 2\pi m\varphi \\ -\cos 2\pi m\varphi & \sin 2\pi m\varphi \end{pmatrix},$$

and note that $S_{m_1}(\varphi)R_{m_2}(-\varphi) = S_{m_1 - m_2}(\varphi)$. Hence,

$$\begin{aligned} \begin{pmatrix} a_c(\omega) \\ a_s(\omega) \end{pmatrix} &= \frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) \left[\bar{b}(x_0, \alpha_0) S_{2^n}(\omega) + \sum_{j=1}^{2^n - 1} b_j(x_0, \alpha_0) S_{2^n - j}(\omega) \right] \begin{pmatrix} h_c(x_0, \alpha_0) \\ h_s(x_0, \alpha_0) \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{\partial h_{n,s}}{\partial x}(x_0, \alpha_0, \omega) \\ -\frac{\partial h_{n,c}}{\partial x}(x_0, \alpha_0, \omega) \end{pmatrix}. \end{aligned}$$

As $\frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) \neq 0$, $\bar{b}(x_0, \alpha_0) \neq 0$ and, at least, one of the values $h_s(x_0, \alpha_0)$, $h_c(x_0, \alpha_0)$ is not identically zero, then $a_c(\omega)$ and $a_s(\omega)$ cannot be both identically zero. As they are trigonometric polynomials of degree 2^n in ω, they can only have a finite number of simultaneous zeros. To end the proof, note that this implies that $H'(\theta)$ has two simple zeros except, may be, for a finite number of values of ω. ∎

Proof of Theorem 3.1: It is enough to prove that, for each n, the number of values of ω for which the functions $\alpha_n^{\pm}$ do not exists is, at most, finite. For the case $n = 0$, this follows from Corollary 2.1.

For $n > 0$, we use the following notation,

$$\begin{array}{cccc} F_{\alpha,\varepsilon}^{2^n}: & \mathbb{T}\times\mathbb{R} & \to & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta + 2^n\omega \\ f_n(\theta, x, \alpha, \varepsilon) \end{pmatrix}. \end{array}$$

It is not difficult to see that

$$\begin{aligned} f_n(\theta, x, \alpha, \varepsilon, \omega) &= g_n(x, \alpha) + \varepsilon h_n(\theta, x, \alpha, \varepsilon, \omega), \\ h_n(\theta, x, \alpha, \varepsilon, \omega) &= h_{n,0}(x, \alpha, \omega) + h_{n,c}(x, \alpha, \omega)\cos 2\pi\theta + h_{n,s}(x, \alpha, \omega)\sin 2\pi\theta + O(\varepsilon), \end{aligned}$$

where

$$\begin{pmatrix} h_{n,c}(x, \alpha, \omega) \\ h_{n,s}(x, \alpha, \omega) \end{pmatrix} = \left(\bar{b}_n(x, \alpha)I + \sum_{j=1}^{2^n-1} b_{n,j}(x, \alpha)R_j(-\omega)\right)\begin{pmatrix} h_c(x, \alpha) \\ h_s(x, \alpha) \end{pmatrix},$$

I is the identity matrix, $R_j(\varphi)$ is a rotation of angle $2\pi j\varphi$ and

$$\bar{b}_n(x, \alpha) = \prod_{j=0}^{n-1} \frac{\partial g_j}{\partial x}(g_j(x, \alpha), \alpha).$$

It is clear that $\bar{b}_n(x_0, \alpha_n) \neq 0$. Then, by Lemma 3.1 we obtain the result. ∎

Corollary 3.1 *In the situation of the previous Theorem, if $F_{\alpha,\varepsilon}$ is C^∞, there exists a set $\Theta \subset \mathbb{T}\setminus\mathbb{Q}$ of full Lebesgue measure such that, if $\omega \in \Theta$, for each n there exist exactly two functions $\alpha = \alpha_n^+(\varepsilon)$ and $\alpha = \alpha_n^-(\varepsilon)$, for $|\varepsilon|$ small enough (depending on n), such that $\alpha_n^{\pm}(0) = \alpha_n$ and $(\alpha_n^{\pm}(\varepsilon), \varepsilon)$ are reducibility loss bifurcation curves. The set Θ is a subset of the Diophantine numbers.*

The set Θ in the corollary appears due to two facts. The first is the equivalence bewteen reducibility and degree zero when the map is C^∞ and ω is Diophantine. The second fact is that this set of Diophantine frequencies has to be intersected with the set Ω in Theorem 3.1 to obtain Θ. Note that Θ also has full Lebesgue measure.

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