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ON THE EXISTENCE OF DOUBLY SYMMETRIC "SCHUBART-LIKE" PERIODIC ORBITS

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Abstract. We give sufficient conditions to ensure the existence of symmetrical periodic orbits for a class of Hamiltonian systems having some singularities. The results are applied to different subproblems of the gravitational n-body problem where singularities appear due to collisions.

1. **Introduction.** The motivation for this paper comes from the gravitational n-body problem. For a given n, besides the collinear and the planar n-body problem, that can be considered as subproblems of the spatial one, there are different invariant problems using suitable symmetries. We are interested in subproblems that can be reduced to two degrees of freedom. Well known examples are the collinear and the isosceles three body problems. In the first one, the three masses move on a line. In the isosceles problem, we consider the three masses m_1, m_2, m_3 with $m_1 = m_2$, at the vertices of an isosceles triangle, such that the distance between m_1 and m_3 is equal to the distance between m_2 and m_3. With suitable initial velocities, m_3 moves along the z-axis and m_1, m_2 move in a symmetric way such that the configuration is always an isosceles triangle. As usual the center of masses is placed at the origin

It is well known that in the collinear three body problem, there exists a symmetrical periodic orbit, called the "Schubart orbit", such that the behaviour of the masses in one period is as follows. Assume the masses are labeled as m_1, m_2, m_3 from left to right, being $m_1 = m_3$, and m_2 located at the origin at $t = 0$. The mass m_2, leaving from the origin, initially moves to the right and collides with m_3, then it goes to the left and it collides with m_1. After that it returns to the origin in a symmetric way. So, in one period, there are two binary collisions. This orbit was computed numerically by Schubart in [12]. Recently, Moeckel ([6]) gave a topological proof of the existence of that orbit. For the isosceles problem, numerical computations (see [13]) give evidence of the existence of a symmetrical "Schubart-like" periodic orbit in the sense that in one period the equal masses in the basis of the triangle have two binary collisions while the third mass goes up and down on the vertical axis and passes through the origin when the other masses are at a maximum distance.

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In this paper we study the existence of doubly symmetric "Schubart-like" periodic orbits in a general setting that includes typical subproblems of the n-body problem. Our goal is to show that many of these subproblems can be studied in a common framework. We shall give sufficient conditions for the existence of "Schubart-like" periodic orbits for a general potential. We remark that these conditions only involve a function of one variable (the potential restricted to the configuration circle). Then, to prove the existence of this kind of orbits in a particular problem reduces to check that the conditions of our theorems hold.

We shall consider a Hamiltonian system having some singularities. When applied to the n-body problem, the singularities correspond to collisions. Assuming that the potential satisfies some hypotheses we shall prove the existence of such a "Schubart-like" periodic orbit. This result will be applied to different subproblems of the n-body problem, in particular, to the isosceles problem.

Let H be a classical Hamiltonian in two degrees of freedom

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T A^{-1}\mathbf{p} - U(\mathbf{q}), \qquad \mathbf{q} = (q_1, q_2) \in D \subset \mathbb{R}^2, \quad \mathbf{p} = (p_1, p_2) \subset \mathbb{R}^2, \quad (1)$$

where A is a constant diagonal matrix, $A = diag(a_1, a_2)$, $a_1, a_2 > 0$. In fact we can think that (1) gives the motion of a particle in the plane (q_1, q_2) under a prescribed potential $U(\mathbf{q})$. We shall consider negative values of the energy h. So, the motion is restricted to the so called Hill's region defined by $h + U(\mathbf{q}) \geq 0$. The boundary of the Hill's region is the so called zero velocity curve. The existence of the zero velocity curve will be guaranteed by the assumptions that we shall make on U.

Moreover we shall require some conditions on U in order that the system has different singularities to be identified with collisions for the subproblems of the n-body problem.

Let us introduce $r^2 = \mathbf{q}^T A\mathbf{q}$ and $\mathbf{s} = \mathbf{q}/r$. Then $\mathbf{s}^T A\mathbf{s} = 1$ and we define θ such that

$$s_1 = \frac{1}{\sqrt{a_1}}\cos\theta, \qquad s_2 = \frac{1}{\sqrt{a_2}}\sin\theta. \quad (2)$$

We shall assume that there exist some constants θ_a, θ_b, with $0 < \theta_b - \theta_a \leq \pi$ such that the domain D is defined as $D = \{(r,\theta) \mid r > 0,\ \theta_a < \theta < \theta_b\}$. Moreover we assume that the potential $U(\mathbf{q})$ satisfies the following assumptions.

Assumptions

A.1. *$U(\mathbf{q})$ is an homogeneous function of degree -1 such that $U(\mathbf{q}) = V(\theta)/r$ where*

$$V(\theta) = \frac{\beta_1}{\sin(\theta_b - \theta)} + \frac{\beta_2}{\sin(\theta - \theta_a)} + \hat{V}(\theta), \quad (3)$$

being $\beta_1 > 0, \beta_2 \geq 0$ constants, where $\beta_2 = 0$ if and only if $\theta_b - \theta_a = \pi$, and $\hat{V}(\theta) > 0$ a smooth (at least $\mathcal{C}^3$) bounded function in $[\theta_a, \theta_b]$. Furthermore, the critical values of $V(\theta)$ are non degenerate, that is, if $V'(\theta_*) = 0$ then $V''(\theta_*) \neq 0$.

A.2. *$V(\theta)$ is symmetrical with respect to $\theta_m := (\theta_a + \theta_b)/2$.*

We note that $A.2$ implies that $\beta_1 = \beta_2$ if $\theta_b - \theta_a < \pi$. Furthermore, if $U(\mathbf{q})$ satisfies the assumptions $A.1$ and $A.2$, then the Hamiltonian system has a singularity at $r = 0$, corresponding to "total collision". However, (3) implies that additional singularities are located at $\theta = \theta_a$ and $\theta = \theta_b$ for any $r > 0$. In our examples coming from the n-body problem, they will correspond to collisions of k bodies where $k < n$. We shall refer to them as a-collisions and b-collisions respectively.

As a first example we consider a generalization of the isosceles three body problem. We put n equal masses $m_1 = m_2 = \ldots = m_n = m$, equally spaced in a circle of radius q_1 centered at the origin in some horizontal plane $z = z_1$. An additional mass $m_{n+1} = \mu$ is placed on the z-axis (see Figure 1). Let us denote by q_2 the signed distance between m_{n+1} and the plane $z = z_1$, such that $q_2 > 0$ ($q_2 < 0$) if m_{n+1} is on the positive (negative) z-axis. As usual the center of masses is placed at the origin. We can chose initial velocities in such a way that m_{n+1} is moving along the z-axis and $m_1, m_2, \ldots, m_n$ move on half lines in such a way that at any time t, they form a regular n-gon.

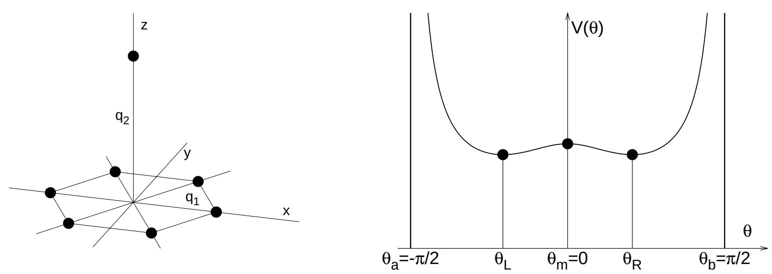


FIGURE 1. Left: A schematic representation of the pyramidal problem for $n = 6$. Right: The potential $V(\theta)$ for the pyramidal problem for $n = 3, \mu = 1.2$.

It is not restrictive to assume that $m = 1$. After some normalizations (preserving the notation q_1, q_2 for the new variables), the motion of the masses is described by a Hamiltonian (1) with $A = I$, $\mathbf{q} = (q_1, q_2) = r(\cos\theta, \sin\theta)$ and, the potential $U(\mathbf{q}) = V(\theta)/r$, being

$$V(\theta) = \frac{S_n}{4\cos\theta} + \frac{\mu}{\sqrt{1 + (n/\mu)\sin^2\theta}}, \qquad -\pi/2 < \theta < \pi/2, \tag{4}$$

where S_n is a constant depending on n, to be defined in (43) (see Figure 1). The values $\theta_b = \pi/2$ and $\theta_a = -\pi/2$, correspond to a collision of masses $m_1, \ldots, m_n$ while m_{n+1} remains in the upper and lower z-axis respectively. We call this problem, the n-pyramidal problem. In particular, if $n = 2$, we get the classical planar isosceles problem.

For a Hamiltonian system defined by (1), with a potential V satisfying $A.1$ and $A.2$, a "Schubart-like" periodic orbit will be a symmetric periodic orbit whose behaviour in one period can be described as follows. At initial time, we can assume $\theta = \theta_m$. In the first quarter of period, θ increases until it reaches $\theta = \theta_b$. In the next quarter of period, the motion is obtained by reversing time, so, θ decreases until it reaches $\theta = \theta_m$. In the second half of the period, the motion is symmetric with respect $\theta = \theta_m$. That is, for the variable θ one has

$$\begin{aligned} \theta(T/4 + t) &= \theta(T/4 - t), \qquad && 0 \le t \le T/4, \\ \theta(T/2 + t) &= 2\theta_m - \theta(t), \qquad && 0 \le t \le T/2. \end{aligned}$$

We note that in fact the orbit is determined by its behaviour in a quarter of period. This is why we name doubly periodic these kind of orbits. A precise definition will be given in section 3.

Our main results are the following

Theorem 1.1. *Let $V(\theta)$ be given by (3), satisfying the assumptions A.1 and A.2. Assume $V(\theta)$ has exactly three non-degenerate critical points at $\theta = \theta_L, \theta_m, \theta_R$ such that $\theta_a < \theta_L < \theta_m < \theta_R < \theta_b$. Moreover we assume that the following conditions are satisfied*

$$3V(\theta_R) - 2V(\theta_m) > 0, \tag{5}$$

$$\cos(\theta_b - \theta)\hat{V}(\theta) - \sin(\theta_b - \theta)\hat{V}'(\theta) > 0, \quad \theta_R \le \theta \le \theta_b, \tag{6}$$

$$G(\theta) := \frac{1}{\theta_R - \theta_m} - \frac{(\theta - \theta_m)}{2}\sqrt{\frac{2(\theta_R - \theta)}{\theta_R - \theta_m}} + 2\frac{V'(\theta)}{V(\theta_m)} > 0, \quad \theta_m \le \theta \le \theta_R. \tag{7}$$

Then, there exists a "Schubart-like" periodic orbit.

Theorem 1.2. *Let $V(\theta)$ satisfying the assumptions A.1 and A.2, being θ_m the unique critical point of $V(\theta)$. Assume that $\theta_b - \theta_a < \pi$, and the following condition is satisfied*

$$\cos(\theta_b - \theta)\hat{V}(\theta) - \sin(\theta_b - \theta)\hat{V}'(\theta) > 0, \qquad \theta_m \le \theta \le \theta_b. \tag{8}$$

Then there exists a "Schubart-like" periodic orbit.

For the n-pyramidal problem we shall prove that if $n < 473$, $V(\theta)$ in (4) has three non-degenerate critical points. Using theorem 1.1 we shall prove the following result

Theorem 1.3. *Let us consider the n-pyramidal problem for $2 \le n < 473$. Then, there exists a "Schubart-like" periodic orbit.*

For the orbit given by Theorem 1.3, we can assume that at the initial time, all the masses lie in the same plane with m_{n+1} at the center of the n-gon determined by $m_1, \ldots m_n$ and $\dot{q}_2(0) > 0$. In the first quarter of period, m_{n+1} goes up along the z axis and the polygon formed by $m_1, \ldots m_n$ shrinks going to collision. At the moment of collision, one has $\theta = \pi/2$, and m_{n+1} is at a maximum height. In the next quarter of period, m_{n+1} goes down and $m_1, \ldots m_n$ move away and they return to the initial configuration with $q_2(T/2) = 0$, but now $\dot{q}_2(T/2) < 0$, where T denotes the period. In the second half of period the motion repeats in a symmetrical way.

Similar results for other problems will be given in section 4.

To prove Theorems 1.1 and 1.2 we mainly use qualitative methods. A similar approach was used in [11] to study the planar isosceles problem. Moreover, in [11] it is proved the existence of a symbolic dynamics, which depends on the mass parameter, by linking the behaviour of orbits passing near triple collision and near infinity. As a consequence of that symbolic dynamics some families of periodic orbits were obtained. In this paper we are only interested in doubly symmetric "Schubart-like" periodic orbits. However it is also expected to get symbolic dynamics in our examples. A key point is the behaviour of the orbits passing near total collision. In section 2 we shall analyze them by using the well know blow up introduced in [3]. In this way the flow can be extended to the so called total collision manifold, to be denoted as $\mathcal{C}$. The behaviour of the invariant manifolds of the equilibrium points on $\mathcal{C}$, determines the dynamics near total collision. In section 3 we prove the main results concerning the existence of periodic orbits. Finally in section 4 we apply the results to several concrete examples.

2. **The system near total collision.** We shall study the Hamiltonian system given by (1) in a neighbourhood of total collision using the blow up coordinates introduced by McGehee [3] (see also [1], [5]). Let $r, \mathbf{s}, v, \mathbf{u}$ be defined by

$$r^2 = \mathbf{q}^T A\mathbf{q}, \quad \mathbf{s} = \mathbf{q}/r, \quad v = r^{1/2}\mathbf{s}^T\mathbf{p}, \quad \mathbf{u} = r^{1/2}A^{-1}\mathbf{p} - v\mathbf{s},$$

where $A = \mathrm{diag}(a_1, a_2)$, $a_1 > 0, a_2 > 0$. Now we introduce θ, u

$$\mathbf{s} = (A^{-1})^{1/2}(\cos\theta, \sin\theta)^T, \qquad \mathbf{u} = u(A^{-1})^{1/2}(-\sin\theta, \cos\theta)^T .$$

After a scaling of time defined by $d\tau = r^{-3/2}dt$, the equations of the system become

$$\begin{aligned} \frac{dr}{d\tau} &= rv, \\ \frac{dv}{d\tau} &= \frac{v^2}{2} + u^2 - V(\theta), \\ \frac{d\theta}{d\tau} &= u, \\ \frac{du}{d\tau} &= -\frac{vu}{2} + V'(\theta), \end{aligned} \tag{9}$$

where $V'(\theta) = \frac{dV(\theta)}{d\theta}$. The energy relation in the new variables is

$$hr = \frac{1}{2}(v^2 + u^2) - V(\theta). \tag{10}$$

The system (9) can be extended analytically to $r = 0$. The total collision manifold defined as

$$\mathcal{C} := \{(v, \theta, u) \,|\, \theta_a < \theta < \theta_b,\ v^2 + u^2 = 2V(\theta)\}$$

is invariant. A remarkable fact is that the flow on $\mathcal{C}$ is gradient-like with respect to v because the second equation in (9) can be written as $\dfrac{dv}{d\tau} = \dfrac{u^2}{2}$. For any θ_* such that $V'(\theta_*) = 0$, there exist two equilibria of (9): $(r, v, \theta, u) = (0, \pm v_*, \theta_*, 0)$ where $v_* = \sqrt{2V(\theta_*)}$. We shall denote these points as $P_\pm(\theta_*)$ depending on the sign of v.

If we restrict the flow to $\mathcal{C}$, the eigenvalues of the linearized system are

$$\lambda_\pm = \frac{1}{4}\left(-v_P \pm \sqrt{v_P^2 + 16V''(\theta_*)}\right), \tag{11}$$

being $v_P = \pm v_*$. The eigenvectors are $(1, \lambda_\pm)$ in coordinates θ, u. Then, if θ_* is a minimum of $V(\theta)$, the corresponding equilibria on $\mathcal{C}$ are saddle points. If θ_* is a maximum of $V(\theta)$, the point is an attractor if $v_P = v^*$, and a repellor if $v_P = -v^*$.

To get the complete picture of the neighbourhood of the equilibria one has to add a double eigenvalue equal to v_P. We shall denote by $W^{u(s)}(P)$ the unstable (stable) invariant manifolds of the equilibrium point P. Furthermore, if P is a saddle point, we shall denote by $W^{u,1}_{\mathcal{C}}(P)$ $(W^{u,2}_{\mathcal{C}}(P))$ the branch of the unstable invariant manifold $W^u(P)$, restricted to $\mathcal{C}$, which leaves a neighbourhood of P with $u > 0$ $(u < 0)$. Similar notation holds for the stable manifolds. Next table summarizes the dimensions of the stable and unstable manifolds of the equilibrium points.

		$W^s_{\mathcal{C}}$	$W^u_{\mathcal{C}}$	W^s	W^u
θ_* minimum of V	$v_P > 0$	1	1	1	3
	$v_P < 0$	1	1	3	1
θ_* maximum of V	$v_P > 0$	2	0	2	2
	$v_P < 0$	0	2	2	2

Moreover, if we fix a negative energy level, for any critical point of V, θ_*, one has an orbit defined by $\theta = \theta_*, u = 0$. They are the so called homothetic orbits. Clearly they belong to $W^s(P_-(\theta_*)) \cap W^u(P_+(\theta_*))$.

The system (9) is not defined for $\theta = \theta_a$ and $\theta = \theta_b$. However it is possible to regularize these singularities by introducing a new variable w and a new time s such that

$$w = uF(\theta), \quad d\tau = F(\theta)ds,$$

with

$$F(\theta) = \frac{f(\theta)}{\sqrt{W(\theta)}}, \qquad W(\theta) = f(\theta)V(\theta), \tag{12}$$

where $f(\theta) = \sin(\theta - \theta_a)\sin(\theta_b - \theta)$ if $0 < \theta_b - \theta_a < \pi$ and $f(\theta) = \sin(\theta_b - \theta)$ if $\theta_b - \theta_a = \pi$. We remark that $W(\theta)$ is positive and bounded for $\theta \in [\theta_a, \theta_b]$. The system (9) becomes

$$\begin{aligned}
\dot{r} &= rvF(\theta),\\
\dot{v} &= F(\theta)\left(2hr - \frac{v^2}{2}\right) + \sqrt{W(\theta)},\\
\dot{\theta} &= w,\\
\dot{w} &= -\frac{vw}{2}F(\theta) + \frac{W'(\theta)}{W(\theta)}\left(f(\theta) - \frac{w^2}{2}\right) + f'(\theta)\left(1 + \frac{f(\theta)}{W(\theta)}(2hr - v^2)\right).
\end{aligned} \tag{13}$$

The energy relation is expressed now as

$$\frac{w^2}{2f(\theta)} - 1 = \frac{f(\theta)}{W(\theta)}\left(rh - \frac{v^2}{2}\right). \tag{14}$$

The flow defined by (13) can be extended to $\theta = \theta_a$ and $\theta = \theta_b$. We note that, if $\theta = \theta_a, \theta_b$, using (14) we obtain $w = 0$. Then for a fixed value of h, the flow is defined in

$$\mathcal{M} := \{(r, v, \theta, w) \,|\, r \geq 0,\ \theta_a \leq \theta \leq \theta_b,\ v \in \mathbb{R},\ w \in \mathbb{R},\ \text{satisfying (14)}\}.$$

We shall keep the same notation for the collision manifold once the regularization of a- and b-collisions was done, that is $\mathcal{C} = \{(r, v, \theta, w) \in \mathcal{M} \,|\, r = 0\}$. This has the effect of gluing lines $\theta = \theta_a$ and $\theta = \theta_b$. In this way we obtain a nice topological representation of $\mathcal{C}$ as a 2-sphere with four holes, that we denote as $B_a^{+,-}$ and $B_b^{+,-}$ for $\theta = \theta_a$, and $\theta = \theta_b$ respectively. The sign refers to the sign of v.

Let us consider an orbit on $\mathcal{C}$. The ω-limit set can be an equilibrium point. Otherwise, as time increases the orbit runs going up to one branch B_a^+ or B_b^+. Figure 2 shows the two branches of the unstable manifold of an equilibrium point with $v < 0$.

On the other hand, we recover the zero velocity curve in the blow up variables as

$$S_0 := \{(r, v, \theta, w) \in \mathcal{M} \,|\, v = 0, w = 0, \theta_a < \theta < \theta_b\}.$$

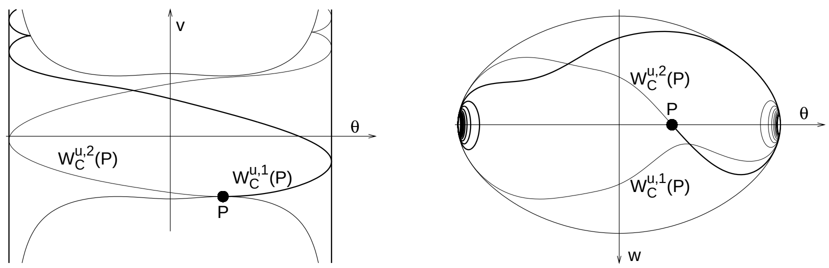


FIGURE 2. The total collision manifold $\mathcal{C}$, $W_{\mathcal{C}}^{u,1}(P)$ (thick lines) and $W_{\mathcal{C}}^{u,2}(P)$ (thin lines) for the pyramidal problem with $n = 4$, $\mu = 1$. Left: The projection on the plane (θ, v). Right: The projection on $(\theta, -w)$.

We introduce the following subsets of $\mathcal{M}$

$$S_b := \{(r, v, \theta, w) \,|\, r \geq 0, v = 0, w = 0, \theta = \theta_b, \},$$

$$S_a := \{(r, v, \theta, w) \,|\, r \geq 0, v = 0, w = 0, \theta = \theta_a, \}.$$

Then S_0, S_a and S_b are fixed by the symmetry

$$L_1 : (r, v, \theta, w) \to (r, -v, \theta, -w),$$

and

$$S_m := \{(r, v, \theta, w) \in \mathcal{M} \,|\, v = 0, \theta = \theta_m\}$$

is fixed by

$$L_2 : (r, v, \theta, w) \to (r, -v, 2\theta_m - \theta, w).$$

Moreover it is easy to check that for the system (13) (and also for (9) by changing w by u), L_1 and L_2 carry orbits to orbits reversing time.

2.1. **Variational equations along an homothetic orbit.** In order to prove the main theorems we shall need, in section 3, some properties of the orbits passing near an homothetic one. To this end we shall study in this section the variational equations along these special orbits.

Let us consider an homothetic orbit with $\theta = \theta_*$, being θ_* a non-degenerate critical point of $V(\theta)$. As we restrict to θ near θ_* we can use variables r, v, θ, u. By taking $u = 0$, the system (9) reduces to

$$\frac{dr}{d\tau} = rv, \qquad \frac{dv}{d\tau} = \frac{v^2}{2} - V(\theta_*)$$

that can be integrated easily. So, we get

$$v(\tau) = -v_* \tanh(v_* \tau/2), \quad r(\tau) = \frac{v_*^2}{2|h| \cosh^2(v_* \tau/2)}. \tag{15}$$

We remark that using this parametrization the homothetic orbit reaches $v = 0$ at $\tau = 0$. The variational equations of (9) along (15) uncouple in two systems

$$\frac{d\xi_1}{d\tau} = v(\tau)\xi_1 + r(\tau)\xi_2, \qquad \frac{d\xi_2}{d\tau} = v(\tau)\xi_2, \tag{16}$$

$$\frac{d\xi_3}{d\tau} = \xi_4, \qquad \frac{d\xi_4}{d\tau} = \beta\xi_3 - \frac{v(\tau)}{2}\xi_4, \tag{17}$$

where $\beta = V''(\theta_*)$. We are interested in the dynamics transversal to the homothetic orbit. Hence, we focus our attention on the solutions of (17).

Assume that θ_* is a maximum of $V(\theta)$ such that the eigenvalues (11) are real. Then, restricted to $\mathcal{C}$, $P_+(\theta_*)$ is an attractor with $\lambda_- < \lambda_+ < 0$, and the weak direction in $\mathcal{C}$, with eigenvalue λ_+, is the one given by the eigenvector $(1, \lambda_+)$. We want to know how this direction evolves from a neighbourhood of $P_+(\theta_*)$ until the plane $v = 0$.

Let us introduce polar coordinates $\xi_3 = R\cos\psi$, $\xi_4 = R\sin\psi$, and $\omega(\tau) = \tanh(v_*\tau/2)$. Then

$$\begin{aligned} \frac{d\psi}{d\tau} &= \beta\cos^2\psi + \frac{v_*\omega}{2}\sin\psi\cos\psi - \sin^2\psi, \\ \frac{d\omega}{d\tau} &= \frac{v_*}{2}(1-\omega^2). \end{aligned} \tag{18}$$

So, we get a planar autonomous system which is $\pi-$periodic in ψ. Hence, to study the solutions of (18) it is sufficient to consider the domain

$$\mathcal{D} := \{(\psi,\omega)\,|\, 0 \le \psi \le \pi,\ -1 \le \omega \le 1\}.$$

The main properties of (18) in $\mathcal{D}$ are the following (see Figure 3)

1. The lines $\omega = \pm 1$ are invariant.
2. There are four equilibrium points in $\mathcal{D}$
$$(\psi_1, 1), \quad (\psi_2, 1), \quad (\pi - \psi_1, -1), \quad (\pi - \psi_2, -1)$$
where $\psi_1 = -\arctan\lambda_+$, $\psi_2 = -\arctan\lambda_-$, $\psi_{1,2} \in (0, \pi/2)$.
3. The points $(\psi_1, 1), (\pi - \psi_1, -1)$ are saddle points, $(\psi_2, 1)$ is an attractor and $(\pi - \psi_2, -1)$ a repellor.
4. If we restrict to $\psi = \pi$, then $d\psi/d\tau = \beta < 0$.

We are interested in the transport, under the variational flow along the homothetic orbit, of the weak attracting direction of $P_+(\theta_*)$, $(1, \lambda_+)$, that is, $\tan\psi(\tau) \to \lambda_+$ as $\tau \to -\infty$. So, we consider the unstable invariant manifold of the point $(\pi - \psi_1, -1)$ and look for the intersection with $\omega = 0$.

Lemma 2.1. *Assume that θ_* is a maximum of V, and let $W^{u,+}$ be the branch of the unstable invariant manifold of the point $(\pi-\psi_1, -1)$, that locally is contained in $\mathcal{D}$. Then $W^{u,+}$ intersects $\omega = 0$ at a point with coordinate ψ such that $\pi/2 < \psi < \pi$.*

Proof. Let us define the curve

$$\gamma := \{(\psi,\omega) \in \mathcal{D}\,|\, \omega = -\lambda_-/\tan\psi,\ \ \pi/2 < \psi < \pi - \psi_2\}$$

and the following region (see Figure 3 (a))

$$\mathcal{R} := \{(\psi,\omega) \in \mathcal{D}\,|\ \max\{-\lambda_-/\tan\psi, -1\} \le \omega \le 0,\ \ \pi/2 \le \psi \le \pi\}.$$

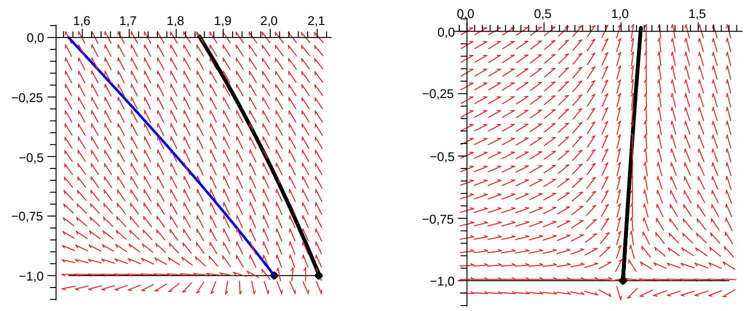
The inner product of the gradient vector of γ and the vector field defined by (18) gives

$$E := \lambda_- + \frac{v_*}{2} - \frac{\beta\lambda_-}{\tan^2\psi}.$$

We recall that $\lambda_- < 0$ and $\beta < 0$. Then, if $\pi/2 < \psi < \pi - \psi_2$, we get $\tan^2\psi > \lambda_-^2$ and

$$E > \lambda_- + \frac{v_*}{2} - \frac{\beta}{\lambda_-} = \frac{1}{\lambda_-}\left(\lambda_-^2 + \frac{v_*}{2}\lambda_- - \beta\right) = 0.$$

Therefore, the orbits through points of γ enter $\mathcal{R}$ for positive time. So, using the property 4. we have that the only way to leave $\mathcal{R}$ is through $\omega = 0$. Then $W^{u,+}$ reaches $\omega = 0$ with $\pi/2 < \psi < \pi$. □



(a) (b)

FIGURE 3. The vector field (18) in the plane (ψ, ω). (a) The case θ_* maximum of V. The equilibrium points $(\pi - \psi_1, -1), (\pi - \psi_2, -1)$, $(\pi - \psi_1 > \pi - \psi_2)$ the curve γ and $W^{u,+}$ are plotted. (b) The case θ_* minimum of V. The equilibrium $(-\psi_1, -1)$ and $W^{u,+}$ are plotted.

The Lemma above says that the direction $(1, \lambda_+)$ of $P_+(\theta_*)$ rotates. This rotation is given by the variation of the angle ψ from the equilibrium $(\pi - \psi_1, -1)$ up to $\omega = 0$ (which corresponds to $v = 0$). The Lemma implies that the angle remains between $\pi/2$ and π.

Remark 1. We can apply the Lemma 2.1 to $\theta_* = \theta_m$ in the case that $V(\theta)$ has three critical points when the eigenvalues (11) at $P_+(\theta_m)$ are real. In this case the orbits of (9) with $r > 0$ which approach $P_+(\theta_m)$ in the direction $(1, \lambda_+)$ will leave a neighbourhood of $P_+(\theta_m)$ by following the homothetic orbit. The Lemma implies that those orbits reach $v = 0$ without crossing $\theta = \theta_m$.

If θ_* is a minimum of $V(\theta)$, then, restricted to $\mathcal{C}$, $P_\pm(\theta_*)$ are saddle points and there are stable and unstable directions given by the eigenvectors $(1, \lambda_\pm)$ where $\lambda_+ > 0$ and $\lambda_- < 0$. In this case for the system (18) we have $\beta = V''(\theta_*) > 0$ and there are four equilibria in $\mathcal{D}$

$$(\pi + \psi_1, 1), \quad (\psi_2, 1), \quad (-\psi_1, -1), \quad (\pi - \psi_2, -1)$$

where $\psi_1 = -\arctan \lambda_+ \in (-\pi/2, 0)$ and $\psi_2 = -\arctan \lambda_- \in (0, \pi/2)$.

Now, $(\pi + \psi_1, 1), (-\psi_1, -1)$ are saddle points, $(\psi_2, 1)$ is an attractor and $(\pi - \psi_2, -1)$ a repellor.

Lemma 2.2. *Let θ_* be a minimum of V, and let $W^{u,+}$ be the branch of the unstable invariant manifold of the equilibrium point $(-\psi_1, -1)$, that locally is contained in $\mathcal{D}$. Then $W^{u,+}$ intersects $\omega = 0$ at a point with coordinate ψ such that $0 < \psi < \pi/2$.*

Proof. We consider the region (see Figure 3 (b))

$$\mathcal{R} := \{(\psi, \omega) \,|\, 0 < \psi < \pi/2, \ -1 \leq \omega \leq 0\}.$$

The point $(-\psi_1, -1)$ is the unique equilibrium point in $\mathcal{R}$. Moreover, using (18), if we restrict to $\psi = \pi/2$, $\frac{d\psi}{d\tau} = -1$, and $\frac{d\psi}{d\tau} = \beta > 0$ if $\psi = 0$. Then the only way to leave $\mathcal{R}$ is through $\omega = 0$. □

Remark 2. The Lemma 2.2 can be applied to $\theta_* = \theta_L$ or $\theta_* = \theta_m$ when $V(\theta)$ has a minimum at θ_*. Going from a neighbourhood of $P_+(\theta_*)$ until the plane $v = 0$ near the homothetic orbit, the unstable direction given by the eigenvector $(1, \lambda_+)$ rotates but it remains in the first quadrant. This implies that, for points P in the local unstable invariant manifold of $P_+(\theta_*)$ with $u > 0$ ($P \in W^{u,1}_{loc}(P_+(\theta_*))$), near the homothetic orbit, forward in time, the θ, u components remain in the same region until the orbit reaches the plane $v = 0$ and so, $u > 0$ and $\theta > \theta_*$.

We note that $W^s(P_-(\theta_*)) = L_1(W^u(P_+(\theta_*)))$. So, Lemma 2.2 implies that the manifolds $W^s(P_-(\theta_*))$ and $W^u(P_+(\theta_*))$ intersect transversally along the homothetic orbit. In fact this result was proved in [10] in a more general setting.

3. **Main results.**

In this section we shall prove the existence of "Schubart-like" periodic orbits. To this end we shall prove that there exists a point in S_m such that, forward in time, its orbit reaches S_b without crossing the section $w = 0$. Therefore the orbit will be doubly symmetric with respect to the symmetries L_1 and L_2 (see [2], [11]). This is the so called "Schubart-like" periodic orbit. In Figure 4 we plot a pair of "Schubart-like" periodic orbits, using blow up coordinates, for a couple of different potentials with three critical points (Figures 4 (a) and (b)), and, with a unique critical point (Figures 4 (c) and (d)) .

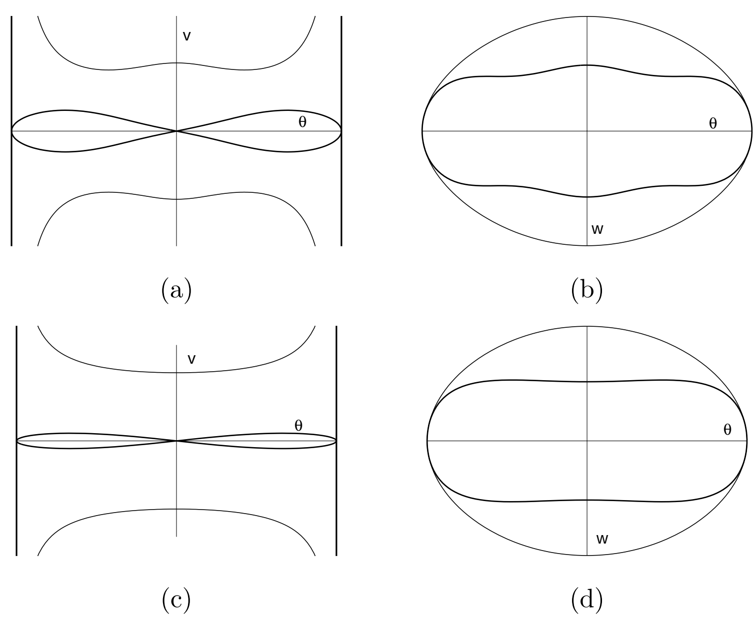


FIGURE 4. Typical "Schubart-like" periodic orbits. We display the projection on the (θ, v)-plane in the plots (a) and (c), and the projection on the (θ, w)-plane in the (b) and (d) ones. The periodic orbit of (a) and (b) corresponds to the n-pyramidal problem for $n = 3$, and $\mu = 1.2$. The one in (c) and (d) corresponds to the $2N$-planar problem for $N = 3$ (to be introduced in section 4).

Theorems 1.1 and 1.2 will follow after next results

Theorem 3.1. *Let $V(\theta)$ be the potential defined in (3) satisfying the assumptions A.1 and A.2 with three critical points at $\theta_L < \theta_m < \theta_R$. Let us assume that $W^{u,1}_{\mathcal{C}}(P_-(\theta_R))$ has a b-collision with $v < 0$ before reaching the plane $v = 0$, and the following conditions are satisfied*

$$3V(\theta_R) - 2V(\theta_m) > 0, \qquad (19)$$

$$G(\theta) := \frac{1}{\theta_R - \theta_m} - \frac{(\theta - \theta_m)}{2}\sqrt{\frac{2(\theta_R - \theta)}{\theta_R - \theta_m}} + 2\frac{V'(\theta)}{V(\theta_m)} > 0, \quad \theta_m \leq \theta \leq \theta_R. \qquad (20)$$

Then there exists a "Schubart-like" periodic orbit.

Theorem 3.2. *Let $V(\theta)$ be a potential satisfying the assumptions A.1 and A.2 with a unique non-degenerate critical point, θ_m. Assume that $W^{u,1}_{\mathcal{C}}(P_-(\theta_m))$ has a b-collision with $v < 0$ before reaching the plane $v = 0$. Then there exists a "Schubart-like" periodic orbit (see Figure 4 (c), (d)).*

The proofs of Theorems 3.1 and 3.2 are postponed to the next subsection. We shall see in this subsection that besides the conditions (19) and (20) (identical to (5) and (7) in Theorem 1.1), the behaviour of $W^{u,1}_{\mathcal{C}}(P_-(\theta_R))$ has a key role in the proof of the existence of the periodic orbits. In the next Proposition one shows that the condition (6) in Theorem 1.1 guaranties that $W^{u,1}_{\mathcal{C}}(P_-(\theta_R))$ has the behaviour required in Theorem 3.1. A similar remark holds for Theorems 3.2 and 1.2.

Proposition 1. *Let $V(\theta)$ be the potential defined in (3) satisfying the assumption A.1. Let θ_* be a non-degenerate critical point of $V(\theta)$ such that $V'(\theta) > 0$ if $\theta_* < \theta < \theta_b$ and, $\theta_* > \theta_b - \pi/2$ if $\pi/2 < \theta_b - \theta_a \leq \pi$. Assume that the following condition is satisfied*

$$\cos(\theta_b - \theta)\hat{V}(\theta) - \sin(\theta_b - \theta)\hat{V}'(\theta) > 0, \qquad \theta \in [\theta_*, \theta_b]. \qquad (21)$$

Then $W^{u,1}_{\mathcal{C}}(P_-(\theta_*))$ has a collision at $\theta = \theta_b$ before reaching the plane $v = 0$ forward in time.

Remark 3. The assumptions on θ_* in Proposition 1 imply that there are not critical points of $V(\theta)$ in the interval (θ_*, θ_b). On the other hand, if $\pi/2 < \theta_b - \theta_a \leq \pi$ we assume $\theta_* > \theta_b - \pi/2$. No restriction on θ_* is needed in the case $0 < \theta_b - \theta_a \leq \pi/2$.

We also remark that these conditions are satisfied under the hypotheses of the Theorems 1.1 and 1.2. Assume that $V(\theta)$ satisfies the hypotheses of Theorem 1.1. Then using that $\theta_b - \theta_a \leq \pi$ we have

$$\theta_b - \pi/2 \leq \theta_m < \theta_R$$

and the Proposition 1 holds for $\theta_* = \theta_R$. In the case of Theorem 1.2 we get $\theta_b - \pi/2 < \theta_m$ and so Proposition 1 holds for $\theta_* = \theta_m$.

Proof. To prove the Proposition we shall use a new variable $g = \frac{v}{\sqrt{W(\theta)}}$ with $W(\theta)$ defined in (12), as it was introduced in [9] for the isosceles problem. Then, the equations (13) restricted to $\mathcal{C}$ are the following

$$\begin{aligned}\dot{g} &= 1-\frac{g^2}{2}f(\theta)-\frac{gw}{2}\frac{W'(\theta)}{W(\theta)},\\ \dot{\theta} &= w,\\ \dot{w} &= -\frac{gw}{2}f(\theta)+f'(\theta)(1-f(\theta)g^2)+\frac{W'(\theta)}{W(\theta)}\left(f(\theta)-\frac{w^2}{2}\right),\end{aligned} \tag{22}$$

and $\mathcal{C}$ becomes

$$\frac{w^2}{2f(\theta)}-1=-\frac{1}{2}f(\theta)g^2. \tag{23}$$

We remark that the flow defined by (22) is not necessarily gradient-like with respect to g. For any critical point of $V(\theta)$, (22) has two equilibrium points. The assumption on θ_* implies that θ_* is a minimum of $V(\theta)$. Let us denote by $P_\pm$ the equilibria of (22) with $\theta=\theta_*$ and $g=\pm\sqrt{2/f(\theta_*)}$ respectively. Then $P_\pm$ are saddle points. We shall prove that $W^{u,1}(P_-)$ has a first b-collision at some point with $g<0$.

An important fact is that, using the variable g, the collision manifold (23) does not depend explicitly on $W(\theta)$ (or, equivalently, on $V(\theta)$) but it depends on the function f. We remark that $f(\theta)=\sin(\theta_b-\theta)$ if $\theta_b-\theta_a=\pi$ and $f(\theta)=\sin(\theta_b-\theta)\sin(\theta-\theta_a)$ otherwise. We recall that these are the suitable functions to regularize the singularities at $\theta_{a,b}$ in both cases.

We introduce $V_s(\theta):=\dfrac{\beta_1}{\sin(\theta_b-\theta)}$ and $W_s(\theta)=f(\theta)V_s(\theta)$, and we write

$$\frac{W'(\theta)}{W(\theta)}=\frac{W_s'(\theta)}{W_s(\theta)}+\Omega(\theta).$$

Therefore, in the case $\theta_b-\theta_a=\pi$ (using $f(\theta)=\sin(\theta_b-\theta)$) we get

$$W_s(\theta)=\beta_1,\qquad \Omega(\theta)=-\frac{(\cos(\theta_b-\theta)\hat{V}-\sin(\theta_b-\theta)\hat{V}')}{f(\theta)V(\theta)}. \tag{24}$$

If $\theta_b-\theta_a<\pi$ (using $f(\theta)=\sin(\theta_b-\theta)\sin(\theta-\theta_a)$) one has

$$\begin{aligned} W_s(\theta)&=\beta_1\sin(\theta-\theta_a),\\ \Omega(\theta)&:=\frac{1}{V(\theta)V_s(\theta)\sin^2(\theta_b-\theta)}\left(-\frac{\beta_1\beta_2\sin(\theta_b-\theta_a)}{\sin^2(\theta-\theta_a)}-\beta_1\mathcal{L}(\theta)\right),\\ \mathcal{L}(\theta)&:=\cos(\theta_b-\theta)\hat{V}(\theta)-\sin(\theta_b-\theta)\hat{V}'(\theta).\end{aligned} \tag{25}$$

If we set $\Omega(\theta)$ equal to zero, (22) reduces to

$$\begin{aligned}\dot{g} &= 1-\frac{g^2}{2}f(\theta)-\frac{gw}{2}\frac{W_s'(\theta)}{W_s(\theta)},\\ \dot{\theta} &= w,\\ \dot{w} &= -\frac{gw}{2}f(\theta)+f'(\theta)(1-f(\theta)g^2)+\frac{W_s'(\theta)}{W_s(\theta)}\left(f(\theta)-\frac{w^2}{2}\right).\end{aligned} \tag{26}$$

If $\theta_b-\theta_a=\pi$, using (24), $W_s'(\theta)=0$ and (26) reduces to

$$\begin{aligned}\dot{g} &= 1-\frac{g^2}{2}f(\theta),\\ \dot{\theta} &= w,\\ \dot{w} &= -\frac{gw}{2}f(\theta)+f'(\theta)(1-f(\theta)g^2),\end{aligned}$$

where $f(\theta) = \sin(\theta_b - \theta)$. It is easy to check that the system above has the following orbits (see Figure 5)

$$g_0(\theta) = j\sqrt{2\sin(\theta_b-\theta)}, \quad w_0(\theta) = -j\cos(\theta_b-\theta)\sqrt{2\sin(\theta_b-\theta)}, \qquad j = \pm 1. \tag{27}$$

In the case $\theta_b - \theta_a < \pi$, from (25) one has $W_s'(\theta) = \beta_1\cos(\theta - \theta_a)$ and (26) has the following orbits

$$g_0(\theta) = j\sqrt{\frac{2\sin(\theta_b-\theta)}{\sin(\theta-\theta_a)}}, \;\; w_0(\theta) = -j\cos(\theta_b-\theta)\sqrt{2\sin(\theta_b-\theta)\sin(\theta-\theta_a)}, \;\; j = \pm 1. \tag{28}$$

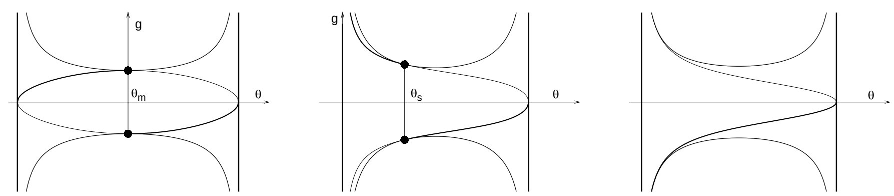


FIGURE 5. Projection on the (θ, g)-plane of orbits of system (26). Thick lines correspond to branches with $w > 0$. Thinner lines represent branches with $w < 0$. Left: Case $\theta_b - \theta_a = \pi$. The orbits (27) and the equilibrium point at $\theta_s = \theta_m$ are plotted. Middle: Case $\pi/2 < \theta_b - \theta_a < \pi$. The projection of orbits (28) and the equilibrium point θ_s are plotted. Right: Case $\theta_b - \theta_a < \pi/2$. The projection of orbits (28).

We note that θ_s is a critical point of $V_s(\theta)$ if and only if $\cos(\theta_b - \theta_s) = 0$, that is, $\theta_s = \theta_b - \pi/2$. Therefore, if $\theta_b - \theta_a \le \pi/2$, the system (26) has no equilibrium points for $\theta_a < \theta < \theta_b$. Otherwise, we get two equilibria such that $(g, \theta, w) = (\pm\sqrt{2/f(\theta_s)}, \theta_s, 0)$ which are saddle points. In particular, if $\theta_b - \theta_a = \pi$, we obtain $\theta_s = \theta_m$ and (27) gives the unstable and stable invariant manifolds of these points (see Figure 5). We remark that $\theta_* > \theta_m$ if $\theta_b - \theta_a = \pi$, and $\theta_* > \theta_s$ if $\pi/2 < \theta_b - \theta_a < \pi$.

Let us consider now the complete system (22) that is we recover the function $\Omega(\theta)$ defined in (24). Now, (27) and (28) do not define orbits of (22) but they are curves lying in the invariant manifold (23). Let us introduce

$$\gamma := \{(g, \theta, w) \,|\, \theta_* \le \theta \le \theta_b,\ g = g_0(\theta),\ w = w_0(\theta), \quad \text{using} \quad j = -1\}$$

and the region

$$\mathcal{R} := \{(g, \theta, w) \,|\, \theta_* \le \theta \le \theta_b, w \ge 0, g \le g_0(\theta),\ j = -1 \text{ and } (23) \text{ holds }\}, \tag{29}$$

where $g_0(\theta), w_0(\theta)$ are defined in (27) or (28), depending on $\theta_b - \theta_a$, with $j = -1$. We note that in a small neighbourhood of P_-, the branch $W^{u,1}(P_-)$ is contained in $\mathcal{R}$. We shall see that the only way to leave the region $\mathcal{R}$ is through $\theta = \theta_b$. From (21) and using (24) and (25) we get $\Omega(\theta) < 0$ if $\theta_* \le \theta \le \theta_b$. Then the vector field defined by (22) on points of γ satisfies

$$\dot{g} = 1 - \frac{g^2 f}{2} - \frac{gw}{2}\frac{W'(\theta)}{W(\theta)} < 1 - \frac{g^2 f}{2} - \frac{gw}{2}\frac{W_s'(\theta)}{W_s(\theta)}. \tag{30}$$

Therefore the orbits of (22) enter $\mathcal{R}$ through γ. However if $\theta_* < \theta < \theta_b$ and $w = 0$, from (23) we obtain $f(\theta)g^2 = 2$. Then the last equation in (22) reduces to $\dot{w} = f(\theta)V'(\theta)/V(\theta) > 0$. Moreover if $\theta = \theta_*$ and $w > 0$ we have $\dot{\theta} > 0$. Then we conclude that the only way to leave the region $\mathcal{R}$ is through $\theta = \theta_b$. Moreover, using the uniqueness of solutions of (22), $W^{u,1}(P_-)$ reaches $\theta = \theta_b$ at some point with $g < 0$, that is with $v < 0$. □

Remark 4. Assume that $V(\theta)$ has three non degenerate critical points at $\theta = \theta_L, \theta_m, \theta_R$, with $\theta_L < \theta_m < \theta_R$. The Proposition 1 holds by taking $\theta_* = \theta_R$. We have seen in the proof of the Proposition that on $\mathcal{C}$, for points in the region $\mathcal{R}$ defined in (29), the orbit exits $\mathcal{R}$ forward in time through $\theta = \theta_b$ with $g < 0$. In particular, this occurs for points on $\mathcal{C}$ with $\theta = \theta_R$, $w > 0$ and

$$g \leq g_0(\theta_R) \tag{31}$$

where $g_0(\theta_R) = -\sqrt{2\sin(\theta_b - \theta_R)}$ in the case $\theta_b - \theta_a = \pi$ and,

$$g_0(\theta_R) = -\sqrt{2\sin(\theta_b - \theta_R)/\sin(\theta_R - \theta_a)}$$

if $\theta_b - \theta_a < \pi$. If we recover the variable v, the condition (31) becomes

$$v \leq \sqrt{W(\theta_R)}g_0(\theta_R) = -\sqrt{2V(\theta_R)}\sin(\theta_b - \theta_R) \tag{32}$$

in both cases $\theta_b - \theta_a = \pi$ and $\theta_b - \theta_a < \pi$.

Remark 5. The Proposition 1 holds also for non symmetrical potentials V.

Proof of Theorem 1.1. It follows from Proposition 1 for $\theta_* = \theta_R$ and Theorem 3.1. □

Proof of Theorem 1.2. It follows from Proposition 1 for $\theta_* = \theta_m$ and Theorem 3.2. □

3.1. **Proof of Theorem 3.1.** From now on if P is a point in $\mathcal{M}$, $\varphi(s; P)$ denotes the solution of (13) such that $\varphi(0; P) = P$. In a similar way $\varphi(\tau; P)$ denotes a solution of (9) when the variables r, v, θ, u are used.

We shall use the following notation for different sections in $\mathcal{M}$. $\mathcal{P}_m$, $\mathcal{P}_R$ and $\mathcal{P}_b$ will denote the sets of points in $\mathcal{M}$ such that $\theta = \theta_m$, $\theta = \theta_R$ and $\theta = \theta_b$ respectively. $\mathcal{V}_0$ will denote the set of points in $\mathcal{M}$ such that $v = 0$. The flow is transversal to $\mathcal{P}_m$ and $\mathcal{P}_R$ except at the equilibria and at the points of the homothetic orbits. Moreover, it is transversal to $\mathcal{V}_0$ except at the points of the curve defined by $u^2 = V(\theta)$ when variables (r, v, θ, u) are used, or equivalently $w^2 = f(\theta)$ after the regularization of a and b-collisions. In fact, on $\mathcal{V}_0$, using (9) we obtain that $\frac{dv}{d\tau} < 0$ if $u^2 < V(\theta)$ and, $\frac{dv}{d\tau} > 0$ otherwise.

Given two sections $\mathcal{P}_1$ and $\mathcal{P}_2$, we denote by $T_{1,2} : \mathcal{P}_1 \mapsto \mathcal{P}_2$ the Poincaré map defined in the following way: if $P \in \mathcal{P}_1$ then $T_{1,2}(P) = \varphi(\hat{\tau}; P)$ where $\hat{\tau} = \min\{\tau > 0 \,|\, \varphi(\tau; P) \in \mathcal{P}_2\}$ if it exists. As usual, in this case we name $\varphi(\hat{\tau}; P)$ the first intersection of the orbit with $\mathcal{P}_2$.

To prove Theorem 3.1 we shall take an arc of points $\Gamma \subset S_m$ with $w > 0$ and we shall prove that forward in time, Γ remains in the region $w > 0$ until it reaches $\mathcal{P}_b$, in such a way that $\hat{\Gamma} := T_{R,b} \circ T_{m,R}(\Gamma)$ is a continuous curve in $\mathcal{P}_b$ which has endpoints with $v > 0$ and $v < 0$ respectively. Therefore, $\hat{\Gamma}$ has a point with $v = 0$ giving rise to the symmetrical periodic orbit. In this way we obtain that there exists a point $P_{op} \in S_m$ and $s_{op} > 0$ such that $\varphi(s_{op}; P_{op}) \in S_b$, and $\varphi(s; P_{op})$ is contained in the region $w > 0$ for any $0 < s < s_{op}$. Then, $\varphi(s; P_{op})$ is periodic with period, in s, equal to $4s_{op}$.

To define the arc Γ we shall study the behaviour of $W^{u,1}_{loc}(P_+(\theta_L))$ for $\theta_L \leq \theta \leq \theta_m$ and $u > 0$. This is done in the Lemma 3.3.

In the Lemma 3.4 we study the Poincaré map $T_{m,R}$. We remark that going from $\mathcal{P}_m$ to $\mathcal{P}_R$, $\theta_m < \theta < \theta_R$, and the orbits in the region $u > 0$ can go to $u < 0$. However we shall see in that Lemma that for points in Γ, the orbit can not cross $u = 0$ before reaching $\mathcal{P}_R$. To prove that we shall construct a two dimensional surface $\hat{\Sigma}$ which prevents that points of Γ cross $u = 0$ with $v \leq 0$. The condition (20) will be used to prove this fact when $v > 0$.

The passage from $\mathcal{P}_R$ to $\mathcal{P}_b$ will be obtained after Lemma 3.5. In fact this Lemma has been stated in a general setting in order to be also used in the proof of Theorem 3.2.

Lemma 3.3. *Let $V(\theta)$ be a potential satisfying the assumptions A.1 and A.2, with three critical points at $\theta_L < \theta_m < \theta_R$ and assume that (19) is satisfied.*

Then, for any point $P \in W^{u,1}_{loc}(P_+(\theta_L))$ with $r > 0$, there exists

$$\tau(P) := \min\{\tau > 0 \,|\, \varphi(\tau; P) \in \mathcal{V}_0 \cup \mathcal{P}_m\}$$

and $u(\tau) > 0$ for any $0 < \tau \leq \tau(P)$.

Moreover, let K be the first intersection of $W^{u,1}_{loc}(P_+(\theta_L))$ with $\mathcal{V}_0 \cup \mathcal{P}_m$. Then K is a continuous curve and for any point $P \in K$ with $\theta \neq \theta_L$, the coordinate u satisfies

$$0 < u < \sqrt{V(\theta_R)} < \sqrt{V(\theta_m)}. \tag{33}$$

Proof. The proof is based in the same idea used in [11] for the isosceles problem. To prove the lemma we only need to consider orbits in the region $\theta_L \leq \theta \leq \theta_m$. So, in order to simplify the computations we shall use the variables (r, v, θ, u). Let be

$$\mathcal{R}_1 := \{(r, v, \theta, u) \in \mathcal{M} \,|\, v \geq 0, \theta_L \leq \theta \leq \theta_m, u \geq 0\}.$$

First we prove that the only way that an orbit with $r > 0$ can exit $\mathcal{R}_1$ is through $\mathcal{V}_0 \cup \mathcal{P}_m$.

It is clear that an orbit in $\mathcal{R}_1$ with $r > 0$ can not exit this region through $r = 0$ because the total collision manifold $\mathcal{C}$ is invariant. Also, it can not leave $\mathcal{R}_1$ through the lines $\theta = \theta_L$ or $\theta = \theta_m$ on $u = 0$, unless the orbit be an homothetic one. Moreover, if we restrict to $u = 0$, from (9) we get $\frac{du}{d\tau} = V'(\theta) > 0$ for $\theta_L < \theta < \theta_m$ and so, the orbits enter to $\mathcal{R}_1$. On the other hand, if $\theta = \theta_L$ and $u > 0$, the third equation in (9) gives $\frac{d\theta}{d\tau} = u > 0$ and the orbits can not exit $\mathcal{R}_1$ through this boundary. Therefore, the only way to exit $\mathcal{R}_1$ is through $\mathcal{V}_0 \cup \mathcal{P}_m$.

Now we shall see that for any point $P \in \mathcal{R}_1$ with $u > 0$, $r > 0$, the orbit $\varphi(\tau; P)$ eventually leaves $\mathcal{R}_1$ for positive time. Indeed, as far as the orbit remains in $\mathcal{R}_1$, $\frac{d\theta}{d\tau} = u > 0$ and so, $\theta(\tau)$ is a bounded increasing function. Assume that $\varphi(\tau; P)$ remains in $\mathcal{R}_1$ for any positive time τ. Then there exists $\lim \theta(\tau)$ as $\tau \to \infty$, and $u(\tau) \to 0$ as $\tau \to \infty$. This implies that $\varphi(\tau; P)$ should approach $u = 0$, $\theta = \theta_m$. However, $\varphi(\tau; P)$ can not tend to the equilibrium $P_+(\theta_m)$ because the stable invariant manifold of that point is contained in $\mathcal{C}$. Therefore, $\varphi(\tau; P)$ should approach the homothetic orbit at $\theta = \theta_m$ which leaves $\mathcal{R}_1$ at some positive time. Then the same is true for $\varphi(\tau; P)$ and we get a contradiction.

The local analysis near $P_+(\theta_L)$ shows that $W^{u,1}_{loc}(P_+(\theta_L)) \subset \mathcal{R}_1$. All the orbits of $W^{u,1}_{loc}(P_+(\theta_L))$ with $r > 0$ eventually leave $\mathcal{R}_1$ through $\mathcal{V}_0 \cup \mathcal{P}_m$ defining some curve in $\mathcal{V}_0 \cup \mathcal{P}_m$. Let us look at the endpoints of that curve.

One of the endpoints is the intersection point, H_L, of the homothetic orbit at $\theta = \theta_L$ with $\mathcal{V}_0$. Using the Remark 2, $W^{u,1}_{loc}(P_+(\theta_L))$ near the homothetic one, intersect transversally $\mathcal{V}_0$ at some point close to H_L with $\theta > \theta_L$ and $u > 0$ (see also Figure 6).

On the other hand, the flow on $\mathcal{C}$ is gradient-like with respect to v so, the $\omega-$limit set of $W^{u,1}_{\mathcal{C}}(P_+(\theta_L))$ is the equilibrium point $P_+(\theta_m)$. We know that, restricted to $\mathcal{C}$, $P_+(\theta_m)$ is an attractor. If the eigenvalues (11) are complex, the orbit tends to $P_+(\theta_m)$ spiraling and then, $W^{u,1}_{\mathcal{C}}(P_+(\theta_L))$ intersect transversally $\mathcal{P}_m$ at some point with $v > 0$ and $u > 0$. The same behaviour occurs for nearby orbits of $W^{u,1}_{loc}(P_+(\theta_L))$ with $r > 0$. If the eigenvalues are real, $W^{u,1}_{\mathcal{C}}(P_+(\theta_L))$ can enter $P_+(\theta_m)$ without crossing the section $\mathcal{P}_m$. In this case, from Remark 1 we have that nearby orbits in $W^{u,1}_{loc}(P_+(\theta_L))$ with $r > 0$ can leave the region $\mathcal{R}_1$ through the section $\mathcal{V}_0$.

We recall that the flow is transversal to $\mathcal{V}_0$ except at the points of the curve $u^2 = V(\theta)$. The inequalities (33) imply that K has no points in that curve and so, in $\mathcal{R}_1$, $W^{u,1}_{loc}(P_+(\theta_L))$ intersect $\mathcal{V}_0$ transversally.

Now we shall prove (33). From (9) we have that in $\mathcal{R}_1$

$$\frac{du}{d\theta} \leq \frac{V'(\theta)}{u}$$

if $u > 0$. By integrating this inequality on the orbits of $W^{u,1}(P_+(\theta_L))$ until they reach $\mathcal{V}_0 \cup \mathcal{P}_m$, we obtain

$$u^2(\theta) \leq 2\int_{\theta_L}^{\theta} V'(\theta)d\theta \leq 2\int_{\theta_L}^{\theta_m} V'(\theta)d\theta = 2(V(\theta_m) - V(\theta_R))$$

where $V(\theta_R) = V(\theta_L)$ has been used. The condition (19) implies that

$$u^2(\theta) < V(\theta_R) < V(\theta) < V(\theta_m)$$

for any $\theta_L < \theta \leq \theta_m$. □

After Lemma 3.3, if $P \in W^{u,1}_{loc}(P_+(\theta_L))$ with $r > 0$, we have that $\varphi(\tau; P) \subset \mathcal{R}_1$ for any $\tau \leq \tau(P)$. So we can define (see Figure 6)

$$\Sigma := \{\varphi(\tau; P) \,|\, P \in W^{u,1}_{loc}(P_+(\theta_L)),\ \tau \leq \tau(P)\}. \tag{34}$$

Assume that Σ intersects $\mathcal{P}_m$ (in particular this is true if the eigenvalues at $P_+(\theta_m)$ are complex). Then, adding the equilibrium $P_+(\theta_L)$ to Σ we get a 2-dimensional surface, $\hat{\Sigma}$, which separates $\mathcal{R}_1$ in two components (see Figure 6). We shall denote by D the point of $\hat{\Sigma} \cap S_m$ with a larger value of u, that is, D is the nearest point to $P_0 =: \mathcal{C} \cap S_m$ in $\hat{\Sigma} \cap S_m$. We note that the point D belongs to $W^{u,1}(P_+(\theta_L))$.

If the eigenvalues $\lambda_\pm$ at $P_+(\theta_m)$ are real, then, for the flow restricted to $\mathcal{C}$, $P_+(\theta_m)$ is an attracting node. Therefore it can occur that $W^{u,1}_{\mathcal{C}}(P_+(\theta_L))$ tends to $P_+(\theta_m)$ without crossing $\mathcal{P}_m$ (see Remark 1) and the orbits of $W^{u,1}(P_+(\theta_L))$ with $r > 0$ exit the region $\mathcal{R}_1$ through $\mathcal{V}_0$. In this case we define $\hat{\Sigma}$ as the union of Σ, the two equilibria $P_+(\theta_m), P_+(\theta_L)$ and the set of points $\{(r, v, \theta, u) \in \mathcal{M} | v > 0, \theta = \theta_m, u = 0\}$. As before, $\hat{\Sigma}$ separates $\mathcal{R}_1$ in two components but $W^{u,1}(P_+(\theta_L)) \cap \mathcal{V}_0$ is a curve which approaches the point $\theta = \theta_m$, $u = 0$, so, we define D as that endpoint. We remark that in this case, the point D belongs to the homothetic orbit.

Let Γ be the arc of points in S_m between P_0 and P_1, being P_1 a point sufficiently close to D in the segment DP_0 (see Figure 6).

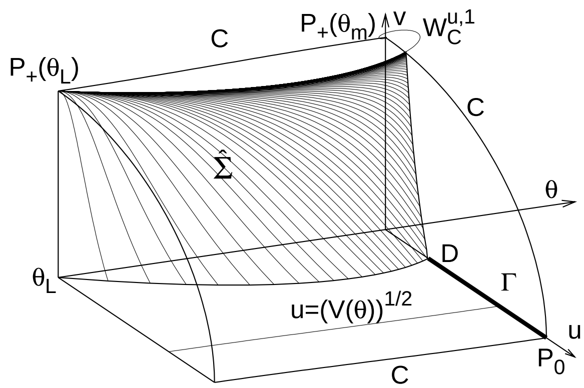


FIGURE 6. The surface $\hat{\Sigma}$ for the n-pyramidal problem with $n = 5$, $\mu = 5$. The branch of the unstable manifold $W^{u,1}_{\mathcal{C}}(P_+(\theta_L))$ has been plotted until it reaches, spiraling, a small neighbourhood of the equilibrium point $P_+(\theta_m)$. Also the curve $u = \sqrt{V(\theta)}$ on $\mathcal{V}_0$ and the arc $\Gamma \subset S_m$ have been plotted.

Lemma 3.4. *Assume the hypotheses of Lemma 3.3 are satisfied and (20) holds. Let us consider the Poincaré map $T_{m,R} : \mathcal{P}_m \mapsto \mathcal{P}_R$. Then, $T_{m,R}(\Gamma)$ is a continuous arc with endpoints in $v < 0$ and $v > 0$ respectively.*

Proof. Let us introduce the following regions

$$\mathcal{R} := \{(r, v, \theta, u) \in \mathcal{M} \,|\, \theta_m \leq \theta \leq \theta_R,\ u \geq 0\},$$
$$\mathcal{R}_2 := \{(r, v, \theta, u) \in \mathcal{R} \,|\, v \leq 0\}.$$

Using the symmetry, $W^{s,1}(P_-(\theta_R)) = L_2(W^{u,1}(P_+(\theta_L)))$. Moreover, $L_2(\hat{\Sigma})$ separates $\mathcal{R}_2$ in two components, one of them containing the arc Γ. We shall prove that for any point P in Γ, the orbit exits $\mathcal{R}_2$ through $\mathcal{P}_R$ with $v < 0$ if P is sufficiently close to D, and $v > 0$ if P is near P_0.

Let P be a point in Γ such that the orbit enters to $\mathcal{R}_2$ for $\tau > 0$ small enough. This holds, for instance, if P is close to D. By construction, the orbit can not tend to the equilibrium $P_-(\theta_R)$ without leaving $\mathcal{R}_2$. Using similar arguments to the ones in the proof of Lemma above, we have that the orbit of the point P must leave the region $\mathcal{R}_2$. However, as far as the orbit of P remains in $\mathcal{R}_2$, the surface $L_2(\hat{\Sigma})$ prevents the orbit to reach the section $u = 0$ with $\theta_m \leq \theta \leq \theta_R$. In fact, for $P \in \Gamma$, the only way to exit $\mathcal{R}_2$ is through $\mathcal{P}_R$ or through $\mathcal{V}_0$. In the second case, this is only possible at points of $\mathcal{V}_0$ with $v(\tau)$ increasing. Using (9) we have that the points of $\mathcal{V}_0$ with positive u, such that $dv/d\tau \geq 0$ satisfy

$$\sqrt{V(\theta)} \leq u \leq \sqrt{2V(\theta)}, \qquad \theta_m \leq \theta < \theta_R. \tag{35}$$

Claim 1: The points on $\mathcal{V}_0$ satisfying (35) leave the region $\mathcal{R}$ through $\mathcal{P}_R$ with $u > 0$.

Assume that the claim is true. Then for any point $P \in \Gamma$, there exists

$$\tau_R(P) = \min\{\tau > 0 \,|\, \varphi(\tau; P) \in \mathcal{P}_R\} > 0$$

such that $\varphi(\tau; P) = (r(\tau), v(\tau), \theta(\tau), u(\tau)) \in \mathcal{R}$ and $u(\tau) > 0$ for all $0 \leq \tau \leq \tau_R(P)$.

The flow is gradient-like with respect to v on $\mathcal{C}$. Then for points $P \in \Gamma$ sufficiently close to $P_0 \in \mathcal{C}$, we obtain $v(\tau_R(P)) > 0$.

Assume that the point D belongs to $W^{u,1}(P_+(\theta_L))$. Using the symmetry, we have that D belongs to $W^{s,1}(P_-(\theta_R))$. Therefore, if $P \in \Gamma$ is sufficiently close to D, $\varphi(\tau; P)$ enters $\mathcal{R}_2$ and follows $\varphi(\tau; D)$ until it reaches a neighbourhood of $P_-(\theta_R)$. Using the local behaviour of the flow near the equilibrium $P_-(\theta_R)$, we conclude that $\varphi(\tau; P)$ reaches $\mathcal{P}_R$ at some point with θ near θ_R and $u > 0$.

In the case that D belongs to the homothetic orbit, for a point $P \in \Gamma$ sufficiently close to D, $\varphi(\tau; P)$ enters $\mathcal{R}_2$, passes near the equilibrium point $P_-(\theta_m)$ and it should leave a neighbourhood of that equilibrium by following closely some orbit on $\mathcal{C}$ with $u > 0$. Now it is sufficient to prove that for orbits on $\mathcal{C}$ going from a small neighbourhood of $P_-(\theta_m)$ until $\mathcal{P}_R$ with $u > 0$, the point on $\mathcal{P}_R$ has negative v. In fact, we shall prove that the value of v at $\mathcal{P}_R$ satisfies the following inequality

$$v \leq -\sqrt{2V(\theta_m)} \cos\left((\theta_R - \theta_m)/2\right) < 0. \tag{36}$$

To prove (36) we restrict (9) to $\mathcal{C}$. Using (10) $\frac{dv}{d\tau} = \frac{u^2}{2}$ and then, as far as $u > 0$

$$\frac{dv}{d\theta} = \frac{u}{2} = \frac{1}{2}\sqrt{2\left(V(\theta) - \frac{v^2}{2}\right)}.$$

If $\theta_m \leq \theta \leq \theta_R$, then $V(\theta) \leq V(\theta_m)$ and

$$\frac{dv}{d\theta} \leq \frac{1}{2}\sqrt{2V(\theta_m)\left(1 - \frac{v^2}{2V(\theta_m)}\right)}.$$

By integration from $\theta = \theta_m$ until $\theta = \theta_R$ on some orbit emanating from the point $P_-(\theta_m)$ where $v = -\sqrt{2V(\theta_m)}$ we get easily (36).

Using the continuity of solutions with respect to initial conditions and the transversality of the flow on $\mathcal{P}_R \cap \{u > 0\}$, we have that $T_{m,R}(\Gamma)$ is a continuous arc in $\mathcal{P}_R$ with endpoints in $v < 0$ and $v > 0$ respectively.

To prove the claim 1 we consider a point $P_i = (r_i, v_i, \theta_i, u_i) \in \mathcal{V}_0$ with $\theta_m \leq \theta_i < \theta_R$, $v_i = 0$ and

$$\sqrt{V(\theta_i)} \leq u_i \leq \sqrt{2V(\theta_i)}. \tag{37}$$

First we obtain a bound of $v(\tau)$ for the orbit of P_i. We recall that $h < 0$. Then, from (9) and (10)

$$\frac{dv}{d\tau} = \frac{u^2}{2} + rh \leq \frac{u^2}{2}.$$

Hence, if $u > 0$ we get

$$\frac{dv}{d\theta} \leq \frac{u}{2} \leq \frac{1}{2}\sqrt{2V(\theta)} \leq \frac{1}{2}\sqrt{2V(\theta_m)}, \qquad \theta_m \leq \theta \leq \theta_R,$$

where (10) has been used to derive $u \leq \sqrt{2V(\theta)}$. By integrating this inequality we get

$$v(\theta) \leq \frac{(\theta - \theta_i)}{2}\sqrt{2V(\theta_m)} \leq \frac{(\theta - \theta_m)}{2}\sqrt{2V(\theta_m)}.$$

Then, as far as $u(\tau) > 0$, the inequality above holds and then the last equation in (9) gives

$$\frac{du}{d\tau} = -\frac{vu}{2} + V'(\theta) \geq -u(\theta - \theta_m)\frac{\sqrt{2V(\theta_m)}}{4} + V'(\theta).$$

Let us consider the autonomous planar vector field

$$\begin{aligned} \frac{d\theta}{d\tau} &= u, \\ \frac{du}{d\tau} &= -u(\theta - \theta_m)\frac{\sqrt{2V(\theta_m)}}{4} + V'(\theta), \end{aligned} \tag{38}$$

in the region

$$\mathcal{D} := \{(\theta, u) \,|\, \theta_m \leq \theta \leq \theta_R,\, u \geq 0\}.$$

In order to prove the claim 1 it is sufficient to prove that the orbit of (38) through a point $(\theta_i, u_i) \in \mathcal{D}$ with u_i satisfying (37), leaves $\mathcal{D}$ through $\theta = \theta_R$ with $u > 0$.

The system (38) has two equilibria in $\mathcal{D}$ at the points $(\theta_m, 0), (\theta_R, 0)$. The linearization of the vector field at $(\theta_R, 0)$ has eigenvalues

$$\mu_\pm = \frac{\sqrt{2V(\theta_m)}}{8}\left(-(\theta_R - \theta_m) \pm \sqrt{(\theta_R - \theta_m)^2 + 32\frac{V''(\theta_R)}{V(\theta_m)}}\right)$$

with eigenvectors $(1, \mu_\pm)$. As $V''(\theta_R) > 0$ we have $\mu_+ > 0, \mu_- < 0$ and so, $(\theta_R, 0)$ is a saddle point. Let us consider in $\mathcal{D}$ the curve, γ, defined by

$$u = \hat{f}(\theta) := \sqrt{\frac{V(\theta_m)}{\theta_R - \theta_m}(\theta_R - \theta)}, \qquad \theta_m \leq \theta \leq \theta_R$$

and the region

$$\mathcal{D}_1 := \{(\theta, u) \in \mathcal{D} \,|\, 0 \leq u \leq \hat{f}(\theta)\}.$$

Claim 2: The curve $u = \sqrt{V(\theta)}$ for $\theta_m \leq \theta \leq \theta_R$ has a unique intersection point with γ at $\theta = \theta_m$ (see Figure 7).

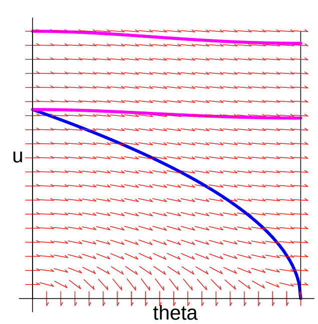


FIGURE 7. The vector field (38) and the curve γ in $\mathcal{D}$ for the pyramidal problem with $n = 3, \mu = 0.5$. We plot θ, $\theta_m \leq \theta \leq \theta_R$, in the horizontal axis and the coordinate u, $0 \leq u \leq \sqrt{2V(\theta_m)}$, in the vertical one. The graphs of $u = \sqrt{V(\theta)}$ and $u = \sqrt{2V(\theta)}$ are plotted using grey (magenta) lines.

To prove claim 2, we only need to check that $\hat{f}(\theta) < \sqrt{V(\theta)}$ for $\theta_m < \theta < \theta_R$, or equivalently,

$$\frac{V(\theta_m)}{\theta_R - \theta_m} < \frac{V(\theta)}{\theta_R - \theta}. \tag{39}$$

We introduce the function $g(\theta) = V(\theta)/(\theta_R - \theta)$. Then

$$\frac{dg}{d\theta} = \frac{h(\theta)}{(\theta_R - \theta)^2}, \quad \text{where} \quad h(\theta) = V'(\theta)(\theta_R - \theta) + V(\theta).$$

We write (20) in the following way

$$G(\theta) = \frac{1}{V(\theta_m)}\left(\frac{V(\theta_m)}{\theta_R - \theta_m} + 2V'(\theta)\right) - \frac{(\theta - \theta_m)}{2}\sqrt{\frac{2(\theta_R - \theta)}{\theta_R - \theta_m}} > 0, \quad \theta_m \le \theta \le \theta_R.$$

Then

$$\frac{V(\theta_m)}{\theta_R - \theta_m} + 2V'(\theta) > 0.$$

Using this inequality and (19) we obtain

$$h(\theta) > -\frac{V(\theta_m)}{2}\frac{(\theta_R - \theta)}{(\theta_R - \theta_m)} + V(\theta) \ge -\frac{V(\theta_m)}{2} + V(\theta) > \frac{V(\theta_R)}{4} > 0\,.$$

Therefore, $g(\theta)$ is an increasing function if $\theta_m < \theta < \theta_R$ and (39) holds and claim 2 is proved.

On the points of γ, the scalar product of the vector field (38) and the gradient vector of γ gives

$$-u\hat{f}'(\theta) - u(\theta - \theta_m)\frac{\sqrt{2V(\theta_m)}}{4} + V'(\theta)$$

Now it is easy to check that the expression above is equal to

$$\frac{V(\theta_m)}{2}G(\theta)\,,$$

where $G(\theta)$ is defined in (20). If $G(\theta) > 0$, then the points of γ are exit points of $\mathcal{D}_1$ for the flow defined by (38). Therefore for points (θ_i, u_i) satisfying (37), the orbit leaves $\mathcal{D}$ through $\theta = \theta_R$ with $u \ge 0$. However the local stable invariant manifold of the equilibrium point $(\theta_R, 0)$ is contained in the region $\mathcal{D}_1$. Therefore the orbit of a point (θ_i, u_i) does not tend to the equilibrium and so, at the intersection with $\theta = \theta_R$, one has $u > 0$. This ends the proof of claim 1 and, hence, of Lemma 3.4. □

Remark 6. If point D belongs to $W^{s,1}(P_-(\theta_R))$, by taking $P \in \Gamma$ sufficiently close to D, $T_{m,R}(\Gamma)$ has an endpoint close enough to $P_-(\theta_R)$ with $u > 0$ and $\varphi(\tau; P)$ leaves a neighbourhood of $P_-(\theta_R)$ by following $W_C^{u,1}(P_-(\theta_R))$. Using the Proposition 1 with $\theta_* = \theta_R$, $\varphi(\tau; P)$ intersects $\mathcal{P}_b$ at some point with $v < 0$. Otherwise we have seen that $\varphi(\tau; P)$ reaches $\mathcal{P}_R$ with v satisfying (36). A simple computation shows that in this case, also (32) holds. Using Remark 4, we conclude that $\varphi(\tau; P)$ intersects $\mathcal{P}_b$ at some point with $v < 0$.

Lemma 3.5. *Let $V(\theta)$ be a potential satisfying the assumption A.1, and θ_* a non-degenerate critical point of V such that $V'(\theta) > 0$ if $\theta_* < \theta < \theta_b$. Let $P = (r_i, v_i, \theta_i, w_i) \in \mathcal{M}$ be a point with $\theta_i = \theta_*$, $w_i > 0$. Then, there exists*

$$\hat{s} := \min\{s > 0 \,|\, \varphi(s; P) \in \mathcal{P}_b\}$$

such that $w(s) > 0$ for all $0 \le s < \hat{s}$, where $\varphi(s; P) = (r(s), v(s), \theta(s), w(s))$.

Proof. Let be

$$\mathcal{R} := \{(r, v, \theta, w) \in \mathcal{M} \mid \theta_* \le \theta \le \theta_b,\ w \ge 0\}.$$

Using (13) we have that $\theta(s)$ is an increasing function as far as $w > 0$. So, $\varphi(s; P) \in int(\mathcal{R})$ for $s > 0$ small enough.

It is clear that θ_* is a minimum of $V(\theta)$. Then the equilibria $P_\pm(\theta_*)$ are saddle points. The local study of that equilibria shows that $\varphi(s; P)$ can not reach total collision without going out of $\mathcal{R}$. In fact, the only points in $\mathcal{R}$ such that the orbit goes to total collision without leaving $\mathcal{R}$, are the ones of the homothetic orbit with $\theta = \theta_*$. On the other hand, if $w = 0$ we get from (13) that $\dot{w} = f(\theta)\dfrac{V'(\theta)}{V(\theta)} > 0$ for $\theta_* < \theta < \theta_b$. Then the only way to leave the region $\mathcal{R}$ is through $\mathcal{P}_b$.

We shall use coordinates $r \ge 0$, $v \in \mathbb{R}$ in $\mathcal{P}_b$. Notice that $\mathcal{P}_b$ is not bounded. Using (13) on $\theta = \theta_b$ we have $\dot{\theta} = 0$ and $\ddot{\theta} = \dot{w} = f'(\theta_b) < 0$, that is, if $\theta(\hat{s}) = \theta_b$ for some finite $\hat{s} > 0$, then $\theta(s)$ has a maximum at $s = \hat{s}$. In this case, it is easy to prove that $v(\hat{s})$ and $r(\hat{s})$ are bounded. In fact, using (13) we obtain

$$\dot{v} \le \sqrt{W(\theta)} \le k_0, \qquad \theta_* \le \theta \le \theta_b$$

for some positive constant k_0, and $v(s) \le v(0) + k_0 s$. Then $v(\hat{s}) < \infty$. Moreover

$$\frac{\dot{r}}{r} = vF(\theta) \le k_1(v(0) + k_0 s),$$

where k_1 is a bound of $F(\theta)$. Then

$$r(\hat{s}) \le r(0) \exp\left(k_1 v(0)\hat{s} + \frac{k_1 k_0}{2}\hat{s}^2\right) < \infty.$$

To prove the existence of $\hat{s}$ we shall proceed by contradiction. Assume that $\varphi(s; P) \in int(\mathcal{R})$ for any $s > 0$. In particular, $w(s) > 0$ for all $s > 0$ and $\theta(s)$ is a bounded increasing function. Then

$$\lim_{s \to \infty} \theta(s) = \theta_b \qquad \text{and} \qquad \lim_{s \to \infty} \dot{\theta}(s) = 0.$$

First, we shall prove that $v(s)$ has an upper bound, that is, there exists a constant $v_M > 0$ such that

$$v(s) \le v_M, \qquad \text{for all} \quad s > 0. \tag{40}$$

Using (13) and (14) we write

$$\dot{v} = hr\frac{f(\theta)}{\sqrt{W(\theta)}} + (hr - \frac{v^2}{2})\frac{f(\theta)}{\sqrt{W(\theta)}} + \sqrt{W(\theta)} = hr\frac{f(\theta)}{\sqrt{W(\theta)}} + w^2\frac{\sqrt{W(\theta)}}{2f(\theta)}. \tag{41}$$

We recall that $h < 0$ and hence

$$\dot{v} \le w^2\frac{\sqrt{W(\theta)}}{2f(\theta)}$$

and as far as $w > 0$

$$\frac{dv}{d\theta} \le w\frac{\sqrt{W(\theta)}}{2f(\theta)} \le \sqrt{\frac{V(\theta)}{2}} \tag{42}$$

where the last inequality follows using that $w \le \sqrt{2f(\theta)}$ (see (14)) and the definition of $W(\theta)$. Using the assumption $A.1$ we can write

$$V(\theta) \le \frac{c}{\sin(\theta_b - \theta)}, \qquad \theta_* < \theta < \theta_b,$$

for some positive constant c. Now we integrate (42)

$$v(\theta) \leq v_i + \int_{\theta_i}^{\theta} \sqrt{\frac{V(\theta)}{2}} d\theta.$$

The integral in the inequality above is convergent as θ goes to θ_b. Therefore $v(s)$ has an upper bound.

Let us introduce $\rho := |h|r + v^2/2$. Using (13), (41) and (40) we obtain

$$\dot{\rho} = vw^2 \frac{\sqrt{W(\theta)}}{2f(\theta)} \leq v_M w^2 \frac{\sqrt{W(\theta)}}{2f(\theta)}.$$

Then from $w \leq \sqrt{2f(\theta)}$ we obtain

$$\frac{d\rho}{d\theta} \leq v_M w \frac{\sqrt{W(\theta)}}{2f(\theta)} \leq v_M \sqrt{\frac{V(\theta)}{2}}.$$

As before, this inequality implies that $\rho(s)$ has an upper bound and then, $r(s)$ and $|v(s)|$ are bounded, that is, there exist constants $r_M > 0, v_M > 0$ such that $r(s) \leq r_M$ and $|v(s)| \leq v_M$ for all $s \geq 0$.

To get a contradiction we write

$$\dot{v} = F(\theta)\left(2hr - \frac{v^2}{2}\right) + \sqrt{W(\theta)} \geq F(\theta)\left(2hr_M - \frac{v_M^2}{2}\right) + \sqrt{W(\theta)}.$$

Let c_0 be a constant such that $\sqrt{W(\theta)} \geq c_0 > 0$, for $\theta_* \leq \theta \leq \theta_b$. We recall that F is a continuous function of θ and $F(\theta_b) = 0$. Therefore if $s > s_0$, s_0 sufficiently large we have

$$\dot{v} \geq -\epsilon + \sqrt{W(\theta)} \geq -\epsilon + c_0 > \frac{c_0}{2} > 0$$

for some $\epsilon > 0$ small enough. Using the inequality above we obtain

$$v(s) > v(s_0) + \frac{c_0}{2}(s - s_0)$$

which is a contradiction with the fact that $|v(s)| \leq v_M$ for all $s \geq 0$. □

Remark 7. As additional information we note that if an orbit $\varphi(s; P)$ goes to $\mathcal{P}_b$, the variable ρ introduced in the proof of the Lemma 3.5 is bounded. Using (14) we can write

$$w^2 = 2f(\theta)\left(1 - \rho\frac{f(\theta)}{W(\theta)}\right).$$

Then, if $z := \theta_b - \theta > 0$ is small enough we have that $w = k_0\sqrt{z}(1 + \mathcal{O}(z))$ for some constant $k_0 > 0$. Therefore $\dot{z} = -w = -k_0\sqrt{z}(1 + \mathcal{O}(z))$. The integration of this equation shows easily that the time s to reach $\mathcal{P}_b$ is finite.

To finish the proof of Theorem 3.1 we consider the arc of points Γ defined after Lemma 3.3. Using Lemma 3.4, $T_{m,R}(\Gamma)$ is a continuous arc in $\mathcal{P}_R$ which has endpoints $T_{m,R}(P_0) \in \mathcal{C}$ and $T_{m,R}(P_1)$ with $v > 0$ and $v < 0$ respectively. Using Lemma 3.5 with $\theta_* = \theta_R$ we have that the orbit through $T_{m,R}(P_0)$ intersects $\mathcal{P}_b$ at some point with $v > 0$. Furthermore, the orbit through $T_{m,R}(P_1)$ intersects $\mathcal{P}_b$ at some point with $v < 0$ (see Remark 6). Therefore $T_{R,b} \circ T_{m,R}(\Gamma)$ is a continuous arc in $\mathcal{P}_b$ which has endpoints with $v > 0$ and $v < 0$ respectively, so, it has a point with $v = 0$.

3.2. **Proof of Theorem 3.2.** The proof follows the same steps as the one of Theorem 3.1. However we only need to consider the Poincaré map $T_{m,b}$ from section $\mathcal{P}_m$ to $\mathcal{P}_b$. We introduce the points, $P_0 \in \mathcal{C}$, with coordinates $(r, v, \theta, w) = (0, 0, \theta_m, \sqrt{2f(\theta_m)})$, and $P_1 \in \mathcal{M}$, $(r, v, \theta, w) = (r_1, 0, \theta_m, w_1)$ with $w_1 > 0$ and small enough. We shall denote by Γ the arc of points in S_m between P_0 and P_1. It is clear that the orbit through P_0 is contained in $\mathcal{C}$ and it intersects $\mathcal{P}_b$ at some point with $v > 0$. However, if P_1 is sufficiently close to the homothetic orbit, then $\varphi(s; P_1)$ enters a small neighbourhood of the equilibrium $P_-(\theta_m)$ and it leaves it following $W^{u,1}_{\mathcal{C}}(P_-(\theta_m))$. The Proposition 1 implies that $\varphi(s; P_1)$ intersects $\mathcal{P}_b$ with $v < 0$. Using Lemma 3.5 with $\theta_* = \theta_m$ we have that $T_{m,R}(\Gamma)$ is a continuous arc in $\mathcal{P}_b$ which has a point with $v = 0$. In this way we obtain the existence of a symmetrical"Schubart-like" periodic orbit in this case.

4. **Some examples.** In this section we study some examples of subproblems of the n-body problem which reduce to two degrees of freedom. For all of them the center of masses is assumed to be fixed at the origin. Moreover we only consider negative energy levels. In fact, due to the homogeneity, it is enough to consider $h = -1$.

4.1. **The n-pyramidal problem.** Let us consider the n-pyramidal problem defined in the Introduction. Using coordinates q_1, q_2, the motion of the masses is described by the following Hamiltonian system

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) &= \frac{1}{2}\mathbf{p}^T A^{-1}\mathbf{p} - U(\mathbf{q}), \qquad A = diag(1, \mu/(n+\mu)), \\ & U(q_1, q_2) = \frac{S_n}{4q_1} + \frac{\mu}{\sqrt{q_1^2 + q_2^2}}, \qquad S_n = \sum_{k=2}^{n} \frac{1}{\sin l_k}, \end{aligned} \tag{43}$$

where $l_k = (k-1)\pi/n$. The variables r, θ introduced in (2) are related to q_1, q_2 in the following way

$$q_1 = r\cos\theta, \qquad q_2 = \sqrt{\frac{n+\mu}{\mu}}\, r\sin\theta.$$

Then the potential (43) is equal to $V(\mathbf{q}) = V(\theta)/r$ where

$$V(\theta) = \frac{S_n}{4\cos\theta} + \frac{\mu}{\sqrt{1 + (n/\mu)\sin^2\theta}}, \qquad -\pi/2 < \theta < \pi/2, \tag{44}$$

It is clear that V satisfies the assumptions $A.1$ and $A.2$ with $\theta_a = -\pi/2$ and $\theta_b = \pi/2$.

Lemma 4.1. *1. If $2 \le n < 473$, then $V(\theta)$ has 3 non-degenerate critical points, a maximum at $\theta = 0$ and two minima at $\pm\theta_R$, where*

$$\tan^2\theta_R = \frac{\mu}{n+\mu}\left(\left(\frac{4n}{S_n}\right)^{2/3} - 1\right). \tag{45}$$

2. If $n \ge 473$, then $V(\theta)$ has a unique non-degenerate critical point at $\theta = 0$.

Proof. It is easy to check that the solutions of $V'(\theta) = 0$ in $(-\pi/2, \pi/2)$ are $\theta = 0$ and $\pm\theta_R$, with θ_R satisfying (45). So, we only need to prove that $S_n/(4n) < 1$ if $n < 473$ and, $S_n/(4n) > 1$ if $n \ge 473$. In [7] (Lemma 1) an asymptotic expansion for

n	$S_n/4n$	$F(n)$
2	0.12500000000000000	0.12499630655842160
3	0.19245008972987526	0.19244992012126177
4	0.23927669529663688	0.23927667710807045
5	0.27527638409423470	0.27527638093453236
15	0.45089776896032865	0.45089776895981726
25	0.53226020510510395	0.53226020510509538
35	0.58582851712693898	0.58582851712693856
45	0.62583348544207622	0.62583348544207633

TABLE 1. Some values of $S_n/4n$ and $F(n)$

S_n is given. Moreover the authors give the following formula which has a relative error less than 10^{-16} for $n \geq 47$

$$\frac{S_n}{4} \approx \frac{n}{2\pi}\left(\gamma + \log\frac{2n}{\pi}\right) - \frac{\pi}{144n} + \frac{7\pi^3}{86400n^3} - \frac{31\pi^5}{7620480n^5} := \tilde{A}_n, \tag{46}$$

where $\gamma \approx 0.5772156649...$ is the Euler-Mascheroni constant. In fact, (46) gives a sufficient good approximation even for $n \geq 2$. Table 1 shows $\frac{S_n}{4n}$ numerically computed in front of $F(n) := \frac{\tilde{A}_n}{n}$ for some values of n, $2 \leq n \leq 45$.

It is easy to check that $F(n)$ is an increasing function of n. Moreover $F(472) = 0.9999086486....$ and $F(473) = 1.000245484....$ In particular $S_n/(4n) \neq 1$ for any integer $n \geq 2$. This proves the existence of the critical points in both cases 1. and 2. Moreover they are non-degenerate

$$V''(0) = n\left(\frac{S_n}{4n} - 1\right) \neq 0, \quad V''(\theta_R) = \frac{3(n+\mu)S_n}{4(\mu + n\sin^2\theta_R)}\frac{\sin^2\theta_R}{\cos^3\theta_R} \neq 0.$$

□

After Lemma 4.1, if $2 \leq n < 473$ there exist three equilibrium on $\mathcal{C}$ with $v < 0$

$$L_-(v,\theta,w) = (-\sqrt{2V(\theta_R)}, -\theta_R, 0), \quad M_-(v,\theta,w) = (-\sqrt{2V(\theta_R)}, \theta_R, 0),$$
$$E_-(v,\theta,w) = (-\sqrt{2V(0)}, 0, 0),$$

and the symmetrical ones L_+, M_+, E_+ with $v > 0$. If $n \geq 473$ only $E_{-,+}$ remains. We note that $E_{-,+}$ corresponds to a planar configuration with n masses in the vertices of a regular polygon and the mass μ in the center. Moreover, at $M_{-,+}$ the masses are in a pyramidal configuration where the basis is a regular n-gon. At $L_{-,+}$ the pyramid is inverted.

Proof of Theorem 1.3. We shall prove that (5), (6) and (7) are satisfied.

A simple computation shows that

$$\cos(\theta_b - \theta)\hat{V}(\theta) - \sin(\theta_b - \theta)\hat{V}'(\theta) = \frac{(n+\mu)\sin\theta}{(1+(n/\mu)\sin^2\theta)^{3/2}} > 0, \quad \theta \in [\theta_R, \pi/2].$$

Then (6) holds.

To prove (5), we write

$$V(\theta) = n\left(\frac{z_n}{\cos\theta} + \frac{\delta}{(1+\delta^{-1}\sin^2\theta)^{1/2}}\right) \tag{47}$$

where $z_n = S_n/(4n)$ and $\delta = \mu/n$. Then

$$3V(\theta_R) - 2V(0) = n\mathcal{F}(z_n;\delta)$$

where

$$\mathcal{F}(z;\delta) := 3\sqrt{h(z;\delta)}(z + \delta z^{1/3}) - 2z - 2\delta, \qquad h(z;\delta) = \frac{1+\delta z^{-2/3}}{1+\delta}.$$

We assume $2 \le n < 473$, then $1/8 \le z_n < 1$ (see the proof of Lemma 4.1). A simple computation shows that $\mathcal{F}(1/8;\delta) > 0$ for any $\delta > 0$. Furthermore,

$$\frac{\partial \mathcal{F}(z;\delta)}{\partial z} = -2 + 3\sqrt{h(z;\delta)} > 0, \qquad \text{if} \quad z < 1, \quad \delta > 0.$$

Then, $\mathcal{F}(z_n;\delta) > 0$ for any $1/8 \le z_n < 1$ and $\delta > 0$, and (5) is satisfied for $2 \le n < 473$.

Now we prove (7). The function G in (7) reduces to

$$G(\theta;\delta) = \frac{1}{\theta_R} - \frac{\theta}{\sqrt{2}}\sqrt{1 - \frac{\theta}{\theta_R}} + 2\frac{V'(\theta)}{V(0)}.$$

Notice that G depends on the parameters δ and n. However, to our purposes, it is sufficient to remark the dependence on δ. For $0 \le \theta \le \theta_R$ we have

$$\frac{\theta}{\sqrt{2}}\sqrt{1 - \frac{\theta}{\theta_R}} \le c_1\theta_R$$

where $c_1 = \sqrt{2/3}/3$.

Moreover, if $0 \le \theta \le \theta_R$,

$$V'(\theta) = n\sin\theta\left(\frac{z_n}{\cos^2\theta} - \frac{\cos\theta}{(1+\delta^{-1}\sin^2\theta)^{3/2}}\right) \ge -\frac{n\sin\theta\cos\theta}{(1+\delta^{-1}\sin^2\theta)^{3/2}}.$$

From (45) and using $z_n \ge 1/8$,

$$\tan^2\theta_R = \frac{\delta}{1+\delta}(z_n^{-2/3} - 1) \le \frac{3\delta}{1+\delta}.$$

Then

$$\theta_R \le g(\delta), \qquad g(\delta) = \arctan\sqrt{\frac{3\delta}{1+\delta}}$$

and we get

$$G(\theta;\delta) \ge \frac{1}{\theta_R} - c_1\theta_R - \frac{2}{(z_n+\delta)}\frac{\sin\theta\cos\theta}{(1+\delta^{-1}\sin^2\theta)^{3/2}} \ge \frac{1}{\theta_R} - c_1\theta_R - \frac{16}{(1+8\delta)}\frac{\sin\theta\cos\theta}{(1+\delta^{-1}\sin^2\theta)^{3/2}}.$$

Let be

$$h_1(\theta;\delta) := \frac{16\sin\theta\cos\theta}{(1+\delta^{-1}\sin^2\theta)^{3/2}}.$$

We introduce $y = \sin^2\theta$ and then we can write

$$h_1(\theta;\delta) = 16\sqrt{h_2(y;\delta)}, \qquad h_2(y;\delta) = \frac{y(1-y)}{(1+\delta^{-1}y)^3}.$$

Let us fix $\delta > 0$. For $0 \le y \le 1$, $h_2(y;\delta)$ has an absolute maximum at $y = y_1(\delta) = \delta + 1 - \sqrt{(\delta+1)^2 - \delta}$. Then for any $0 \le \theta \le \theta_R$

$$\frac{h_1(\theta;\delta)}{1+8\delta} \le \frac{16}{1+8\delta}\sqrt{h_2(y_1(\delta);\delta)} := H(\delta).$$

It is not difficult to check that $H(0) = 0$, $\lim_{\delta\to\infty} H(\delta) = 0$ and $H(\delta)$ has a maximum at $\delta = (-2+\sqrt{22})/24$.

Now we can reduce to study a function of one variable

$$G(\theta;\delta) \geq R(\delta) - H(\delta),$$

where $R(\delta) = 1/g(\delta) - c_1 g(\delta)$ is a positive decreasing function of δ for $\delta > 0$. The function $R(\delta) - H(\delta)$ has an absolute minimum at $\delta_{min} = 0.4992352186751065...$ (computed using an ad hoc program in PARI) and

$$G(\theta;\delta) \geq R(\delta_{min}) - H(\delta_{min}) > 0.$$

This ends the proof. □

4.2. **The $2N$-Planar problem.** Let us consider $n = 2N$, $N \geq 2$, equal masses $m_1 = m_2 = \ldots = m_n = m$, in the plane. Using polar coordinates (r, θ), let $\mathcal{L}_j$ $j = 1, \ldots, n$ be the half line which start at the origin defined by $\theta = \theta_j$ where $\theta_1 = \pi/n$ and $\theta_j = \theta_{j-1} + \pi/N$, $j = 2, \ldots, n$. We put a first mass m_1 in the infinite sector bounded by $\mathcal{L}_n$ and $\mathcal{L}_1$. A second mass m_2 is placed in the second sector, bounded by $\mathcal{L}_1$ and $\mathcal{L}_2$, and symmetric to m_1 with respect to $\mathcal{L}_1$. In a similar way we put m_3 in the third sector symmetrical to m_2 with respect to $\mathcal{L}_2$. We proceed in the same way for the rest of the masses (see Figure 8). By taking suitable velocities the masses preserve these symmetries for all t and so it is sufficient to know the motion of the first mass to describe the motion of the system. We note that in this particular configuration, besides the total collision at the origin, m_1 can only collide with m_2 and m_n on the half lines $\mathcal{L}_1$ and $\mathcal{L}_n$ respectively. In fact, taking into account all the masses, they correspond to N simultaneous binary collisions.

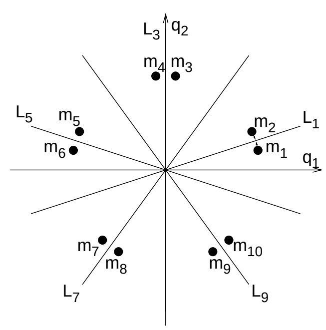


FIGURE 8. A schematic representation of the case $n = 2N$ for $N = 5$.

It is not restrictive to assume $m = 1$. Then the motion of m_1 is described by a Hamiltonian system defined by

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T\mathbf{p} - U(\mathbf{q}), \qquad U(q_1, q_2) = 2\sum_{k=2}^{n} \frac{1}{r_{1k}} \tag{48}$$

where $\mathbf{q} = (q_1, q_2)$ is the position of m_1, r_{1k} is the distance from m_1 to m_k. We note that the singularities due to binary collisions correspond to $r_{12} = 0$ and $r_{1n} = 0$ respectively.

Using polar coordinates $q_1 = r\cos\theta$, $q_2 = r\sin\theta$, for $-\pi/n \leq \theta \leq \pi/n$ we get

$$\begin{aligned} r_{1k} &= 2r\sin l_k, \qquad \text{if} \quad k \quad \text{is odd}, \\ r_{1k} &= 2r\sin(l_k - \theta), \qquad \text{if} \quad k \quad \text{is even}. \end{aligned}$$

where $l_k = \pi(k-1)/n$, and

$$\begin{aligned} V(\theta) &= S_N + \sum_{j=1}^{N} \frac{1}{\sin(l_{2j}-\theta)} \\ &= \frac{1}{\sin(\pi/n-\theta)} + \frac{1}{\sin(\pi/n+\theta)} + S_N + \sum_{j=2}^{N-1} \frac{1}{\sin(l_{2j}-\theta)}, \end{aligned} \tag{49}$$

where $S_N = \sum_{j=2}^{N} \frac{1}{\sin(\pi(j-1)/N)}$. In the case $N=2$, $S_N = 1$ and we take the sum in (49) equal to zero. We note that $\sin(l_{2j}-\theta) > 0$ for $-\pi/n \le \theta \le \pi/n$, if $2 \le j \le N-1$.

Lemma 4.2. *For any $n \ge 4$, $V(\theta)$ defined in (49) satisfies the assumptions A.1 and A.2 and it has a unique non-degenerate critical point in the interval $(-\pi/n, \pi/n)$, at $\theta = 0$.*

Proof. Using the following identity we obtain easily that $V(\theta)$ is an even function

$$\sum_{j=2}^{N-1} \frac{1}{\sin(l_{2j}-\theta)} = \sum_{j=2}^{[N/2]} F_j(\theta) + \frac{i_s}{\cos\theta}$$

where $i_s = 0$ if N is even, $i_s = 1$ if N is odd and,

$$F_j(\theta) = \frac{1}{\sin(l_{2j}-\theta)} + \frac{1}{\sin(l_{2j}+\theta)}.$$

Then $V'(0) = 0$. Furthermore, we can write

$$V'(\theta) = \sum_{j=1}^{N} \frac{\cos(l_{2j}-\theta)}{\sin^2(l_{2j}-\theta)}, \quad V''(\theta) = \sum_{j=1}^{N} \frac{1+\cos^2(l_{2j}-\theta)}{\sin^3(l_{2j}-\theta)}.$$

Then $V''(\theta) > 0$ for any $\theta \in (-\pi/n, \pi/n)$ and so, $V(\theta)$ has a unique non-degenerate critical point at $\theta = 0$. □

We note that if $\theta = 0$, the masses are at the vertices of a regular polygon.

Theorem 4.3. *For any $n \ge 4$, $n = 2N$, the $2N$-Planar problem has a "Schubart-like" periodic orbit.*

Proof. We only need to check the condition (8) and apply Theorem 1.2.

If $n \ge 6$, we have

$$\hat{V}(\theta) = S_N + \sum_{j=2}^{N-1} \frac{1}{\sin(l_{2j}-\theta)}.$$

Then we get

$$\cos(\theta_b-\theta)\hat{V}(\theta) - \sin(\theta_b-\theta)\hat{V}'(\theta) = S_N\cos(\pi/n-\theta) + \sum_{j=2}^{N-1} \frac{\sin(l_{2j}-\pi/n)}{\sin^2(l_{2j}-\theta)} > 0,$$

for $\theta \in [0, \pi/n]$, and (8) is satisfied. If $n = 4$, $\hat{V}(\theta) = S_N$, and (8) reduces to $S_N \cos(\pi/n-\theta) > 0$ for any $\theta \in [0, \pi/n]$. □

We remark that the existence of the "Schubart-like" periodic orbit for $n = 4$ was proved in [8] using a different method. In the same paper, the authors also compute numerically that orbit for other values of n.

4.3. **The double polygonal problem.** We consider n-equal masses $m_1 = m_2 = \ldots = m_n = m$ equally spaced in a circle of radius q_1 centered at the origin, and n additional masses $\mu_1 = \mu_2 = \ldots = \mu_n = m$ equally spaced in a second circle of radius q_2 but rotated an angle π/n with respect to the masses $m_1, \ldots, m_n$. We shall assume that each mass is moving on a straight line in such a way that the configuration of the masses is always equal to two regular polygons (see Figure 9). So, the $2n$ masses can collapse at the origin giving rise to total collision. However, the n masses on a polygon can collide while the others remain at a positive distance of the origin. So, two additional singularities are found depending on the polygon which collapses to the origin.

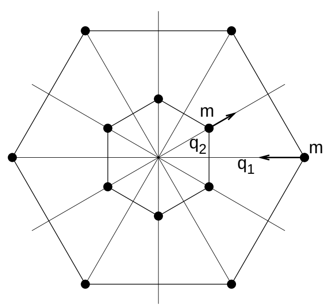


FIGURE 9. The masses for the double polygonal problem with $n = 6$.

It is not restrictive to take $m = 1$. Moreover with a suitable change of time, the motion is described by the Hamiltonian

$$
\begin{aligned}
H(\mathbf{q}, \mathbf{p}) &= \frac{1}{2}\mathbf{p}^T\mathbf{p} - U(\mathbf{q}), \\
U(\mathbf{q}) &= \frac{S_n}{4}\left(\frac{1}{q_1} + \frac{1}{q_2}\right) + \sum_{k=1}^{n} \frac{1}{r_k},
\end{aligned}
\tag{50}
$$

where $r_k = (q_1^2 + q_2^2 - 2q_1q_2c_k)^{1/2}$, $c_k := \cos l_{2k}$, $l_k = (k-1)\pi/n$ and S_n defined in (43). Let be $q_1 = r\cos\theta$, $q_2 = r\sin\theta$. Then $U(\mathbf{q}) = V(\theta)/r$ where

$$
V(\theta) = \frac{S_n}{4}\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right) + \sum_{k=1}^{n}\frac{1}{\sigma_k}, \quad \sigma_k = (1 - c_k\sin(2\theta))^{1/2}, \quad 0 < \theta < \pi/2. \tag{51}
$$

We note that $\theta = 0$ corresponds to a n-collision of the masses μ_i and $\theta = \pi/2$ to a n-collision of the masses m_i. Trivially $V(\theta)$ satisfies the assumption $A.2$.

Lemma 4.4. *If $n = 2$, the potential (51) has a unique non-degenerate critical point at $\theta_m = \pi/4$. Otherwise, $V(\theta)$ has three non-degenerate critical points $\theta_L < \theta_m < \theta_R$, where $\theta_R \in (\pi/4, \arctan(2))$.*

Proof. The case $n = 2$ is trivial because $V(\theta)$ reduces to

$$
V(\theta) = \frac{1}{4\cos\theta} + \frac{1}{4\sin\theta} + 2. \tag{52}
$$

Let us consider $n \geq 3$. Then

$$V'(\theta) = \frac{S_n}{4\cos\theta}\left(\tan\theta - \frac{1}{\tan^2\theta}\right) + \cos(2\theta)\sum_{k=1}^{n}\frac{c_k}{\sigma_k^3} = S_n\cos(2\theta)(-h(\theta) + g_n(\theta)), \quad (53)$$

where

$$h(\theta) := \frac{1 + \sin\theta\cos\theta}{\sin^2(2\theta)(\sin\theta + \cos\theta)}, \qquad g_n(\theta) := \frac{1}{S_n}\sum_{k=1}^{n}\frac{c_k}{\sigma_k^3}.$$

Using the symmetry, it is sufficient to prove that $V(\theta)$ has a unique critical point in the interval $(\pi/4, \pi/2)$. Clearly $V'(\pi/4) = 0$. Other critical points must satisfy the following equation

$$g_n(\theta) = h(\theta). \quad (54)$$

For $\pi/4 < \theta < \pi/2$, $h(\theta)$ is a positive increasing function. Furthermore we can write

$$g_n(\theta) = \frac{1}{S_n}\left(2\mathcal{B}_n(\theta) - \frac{i_s}{(1 + \sin(2\theta))^{3/2}}\right), \qquad \mathcal{B}_n(\theta) := \sum_{k=1}^{[n/2]}\frac{c_k}{\sigma_k^3}, \quad (55)$$

where $i_s = 0$ is n is even and, $i_s = 1$ if n is odd. By derivation with respect to θ

$$\mathcal{B}_n'(\theta) = 3\cos(2\theta)\sum_{k=1}^{[n/2]}\frac{c_k^2}{\sigma_k^5} < 0, \qquad g_n'(\theta) < 0, \quad \pi/4 < \theta < \pi/2$$

and $g_n(\pi/2) = \frac{1}{S_n}\sum_{k=1}^{n} c_k = 0$. Notice that $\mathcal{B}_n(\pi/2) = 0$ if n is even and $\mathcal{B}_n(\pi/2) = 1/2$ if n is odd. Then, g_n and $\mathcal{B}_n$ are positive decreasing functions of θ. We conclude that (54) has at most one solution in the interval $(\pi/4, \pi/2)$.

If $n = 3$, we compute directly from (55)

$$g_3(\pi/4) = \frac{21}{8\sqrt{6}} > \frac{3}{2\sqrt{2}} = h(\pi/4).$$

For $n \geq 4$, it is not difficult to prove that

$$\begin{aligned} \mathcal{B}_n(\theta) &= \sum_{k=1}^{[n/4]} c_k F(c_k, \theta), && \text{if } n \text{ is even}, \\ \mathcal{B}_n(\theta) &> \sum_{k=1}^{[n/4]} c_k F(c_k, \theta), && \text{if } n \text{ is odd}, \end{aligned}$$

where

$$F(x, \theta) := \frac{1}{(1 - x\sin(2\theta))^{3/2}} - \frac{1}{(1 + x\sin(2\theta))^{3/2}}.$$

Another useful expression for F is the following

$$F(x, \theta) = \frac{2u(3 + u^2)}{(1 - u^2)^{3/2}[(1 + u))^{3/2} + (1 - u))^{3/2}]}, \qquad u := x\sin(2\theta). \quad (56)$$

We note that $F(x, \theta) > 0$ if $\pi/4 < \theta < \pi/2$, and $0 < x \leq 1$. Let us fix $\pi/4 < \theta < \pi/2$. It is easy to check that $\frac{d}{dx}F(x, \theta) > 0$ if $0 \leq x \leq 1$. Moreover $0 < c_k \leq c_1 = \cos(\pi/n) < 1$, if $1 \leq k \leq [n/4]$. Then

$$\mathcal{B}_n(\theta) > c_1 F(c_1, \theta)$$

and

$$g_n(\theta) > \frac{1}{S_n}\left(2c_1 F(c_1, \theta) - \frac{1}{(1 + \sin(2\theta))^{3/2}}\right)$$

Then using (56) and taking into account that $c_1 \geq 1/\sqrt(2)$ for $n \geq 4$ we get

$$g_n(\pi/4) > \frac{1}{S_n 2\sqrt{2}} \left(\frac{n^3 4 c_1^2 (3 + c_1^2)}{\pi^3} - 1 \right) \geq \frac{1}{S_n 2\sqrt{2}} \left(\frac{7n^3}{\pi^3} - 1 \right) > h(\pi/4).$$

Therefore, for any $n \geq 3$, (54) has a unique solution $\theta_R \in (\pi/4, \pi/2)$.

Using (53)

$$V''(\theta) = -2S_n \sin(2\theta)[-h(\theta) + g_n(\theta)] + S_n \cos(2\theta)[-h'(\theta) + g_n'(\theta)].$$

Then

$$V''(\pi/4) = -2S_n[-h(\pi/4) + g_n(\pi/4)] < 0, \quad V''(\theta_R) = S_n \cos(2\theta_R)[-h'(\theta_R) + g_n'(\theta_R)] > 0$$

and the critical points are non-degenerate.

Let us fix $\theta \in (\pi/4, \pi/2)$. For $1 \leq k \leq [n/2]$, we have $|c_k| \leq c_1 \cos(\pi/n)$. Then

$$g_n(\theta) \leq \frac{2}{S_n} \mathcal{B}_n(\theta) \leq \frac{n}{S_n} \frac{c_1}{\sigma_1^3}.$$

Using (46), we obtain $g_n(\theta) \to 0$ as $n \to \infty$. Therefore, $\theta_R \to \pi/4$ as $n \to \infty$.

In order to prove that $\theta_R < \arctan(2) =: \hat{\theta}$, it is not difficult, using rough estimates, to obtain that $h(\hat{\theta}) > g_n(\hat{\theta})$ for n large enough. For small values of n we check numerically the condition. In the Figure 10 we plot θ_R for small values of n. The maximum corresponds to $n = 7$. □

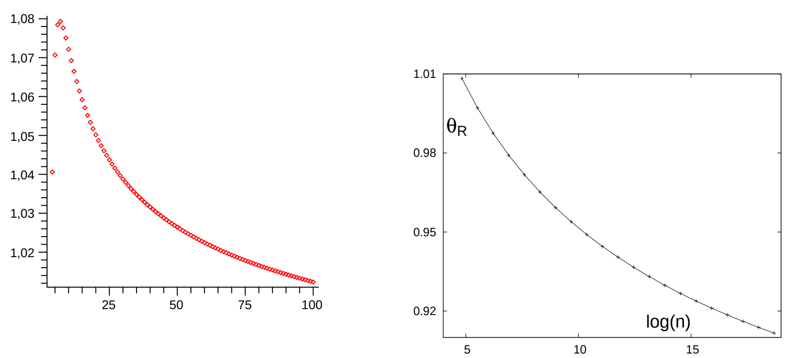


FIGURE 10. Left: The critical point θ_R for $4 \leq n \leq 100$. Right: Values of θ_R as a function of $\log(n)$ for the two n-gons problem. Numerical exploration for large n give evidence that $\theta_R - \pi/4$ behaves as $\mathcal{O}\left(1/\sqrt{\log(n)}\right)$.

Theorem 4.5. *For $n = 2$, the double polygonal problem has a "Schubart-like" periodic orbit.*

Proof. We apply Theorem 1.2. Using (52) and taking $\hat{V}(\theta) = 2$ the left hand part of (8) reduces to $2 \sin \theta$ and then (8) is trivially satisfied. □

For $n \geq 3$ one has to check the conditions (5), (6) and (7) in order to apply the Theorem 1.1.

The condition (6) is trivially satisfied for any $n \geq 3$. In fact, using that $\hat{V}(\theta) = \sum_{k=1}^{n} 1/\sigma_k$, we obtain

$$\sin\theta \hat{V}(\theta) - \cos\theta \hat{V}'(\theta) = \sum_{k=1}^{n} \frac{1}{\sigma_k^3}(\sin\theta - c_k\cos\theta).$$

If $\theta \in [\pi/4, \pi/2]$ then

$$\sin\theta - c_k\cos\theta \geq \cos\theta(1 - c_k) > 0$$

and the condition (6) holds for any $n \geq 3$.

The inequalities (5) and (7) are more cumbersome. Using rough estimates one can prove that they hold for small values of n. However, we shall illustrate numerically the behaviour of the quantities involved in (5) and (7).

First, (5) becomes

$$\begin{aligned} & 3V(\theta_R) - 2V(\theta_m) \\ = \;& S_n\left(\frac{3}{4\cos\theta_R} + \frac{3}{4\sin\theta_R} - \sqrt{2}\right) + \sum_{k=1}^{n}\left(\frac{3}{(1-c_k\sin(2\theta_R))^{1/2}} - \frac{2}{(1-c_k)^{1/2}}\right) \\ = \;& S_n\left(\frac{3}{4\cos\theta_R} + \frac{3}{4\sin\theta_R} - \sqrt{2}\right) + 2\sum_{k=1}^{[n/2]}\left(\frac{3}{(1-c_k\sin(2\theta_R))^{1/2}} - \frac{2}{(1-c_k)^{1/2}}\right) \\ & + i_s\left(\frac{3}{(1+\sin(2\theta_R))^{1/2}} - \sqrt{2}\right), \end{aligned}$$

where $i_s = 0$ if n is even and $i_s = 1$ if n is odd. After the proof of Lemma 4.4 we know that θ_R goes to $\pi/4$ as n goes to infinity. Then, if n is large enough, (5) is satisfied. In Figure 11 we plot the left hand side of (5) for $3 \leq n \leq 50$. The numerics suggests that it is a positive increasing function of n.

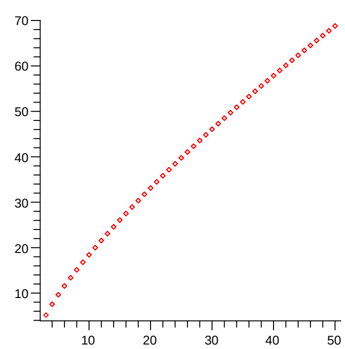


FIGURE 11. The left hand side of (5) for $3 \leq n \leq 50$.

The condition (7) becomes

$$G(\theta) = \frac{1}{\theta_R - \pi/4} - \frac{(\theta - \pi/4)}{2}\sqrt{\frac{2(\theta_R - \theta)}{\theta_R - \pi/4}} + 2\frac{V'(\theta)}{V(\pi/4)} > 0.$$

Plots of the function $G(\theta)$ for different values of n are given in the Figure 12. They show that G has a minimum value in the interval $(\pi/4, \theta_R)$ which decreases as n increases. For $n \geq 34$ the minimum of G becomes negative and then the condition (7) is not satisfied.

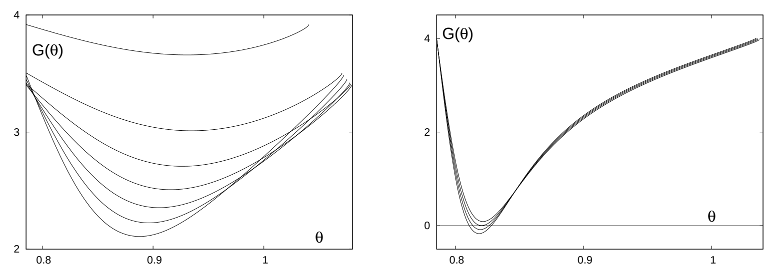


FIGURE 12. Left: The function G for $n = 4, \dots, 10$. The minima decrease when n increases. The values of the function are large (> 26.91821) for $n = 3$ and, hence, they are not shown. Right: The function G for $n = 32, 33, 34$ and 35. For $n \leq 33$ the minimum is positive.

However we recall that Theorem 1.1 gives *sufficient* conditions for the existence of a "Schubart-like" periodic orbit. It can happen that the positivity of $G(\theta)$ in (7) is not a *necessary* condition. As an illustration of this fact we have computed without any problem periodic orbits of the double polygonal problem for values of n up to 1000. Figure 13 shows the (θ, w) and (θ, v) projections of the these periodic orbits for $n = 10, 100$ and 1000.

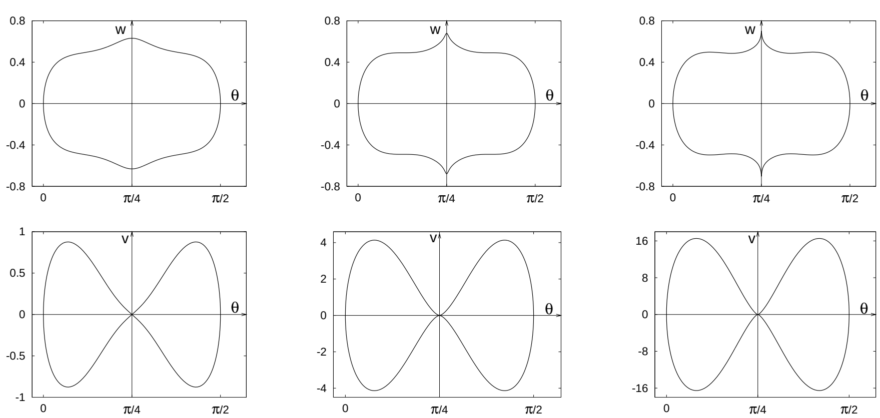


FIGURE 13. "Schubart-like" periodic orbits in the two n-gons case. From left to right the values of n are 10, 100 and 1000. Note that the plots are quite similar in the (θ, w) variables, both in shape and size, while in (θ, v) the shape is similar but the vertical scale changes when increasing n.

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