Asymptotic behaviour of the stability parameter for a family of singular-limit Hill's equation.

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Abstract

Under some non degeneracy conditions we give asymptotic formulae for the stability parameter of a family of singular–limit Hill's equation which depends on three parameters. We use the blow up techniques introduced in [7]. The main contribution of this paper concerns the study of the non degeneracy conditions. We give a geometrical interpretation of them, in terms of heteroclinic orbits for some related systems. In this way one can determine values of the parameters such that the non degeneracy conditions are satisfied. As a motivation and application we consider the vertical stability of homographic solutions in the three-body problem.

1 Introduction

Given $\alpha \in (0,2)$ we consider the following Hill's equations

$$\ddot{x} - (\lambda_1 + \lambda_2 g^{\alpha-2})x \quad = \quad 0, \tag{1}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$, $g = g(t;\delta)$ is a periodic function that depends on a parameter $\delta \in (0, \delta_0]$ with δ_0 small enough, $g(t;\delta) > 0$ for all t and $g(0;\delta) \to 0$ for $\delta \to 0$. Therefore, the equation (1) has a singularity at $t = 0$ for $\delta = 0$.

The purpose is to study the stability of (1) for small values of $\delta > 0$ under some hypotheses to be specified below.

There is an exhaustive bibliography on the topic of stability of Hill's equation ([5]). The main point of present paper lies in the fact that the family of periodic functions that we consider approaches a singular limit. This is a natural problem which appears in some applications to be described later.

Let $U(z) = z^\alpha V(z)$ be a real function defined on an open interval $(0, z_b)$ where $V(z)$ is an analytic function for $z > 0$ such that

(A1) there exists z_a, $0 < z_a < z_b$, such that $V(z_a) = 0$, $V(z) < 0$ for all $z \in (0, z_a)$ and $V_z(z) > 0$ for all $z \in (0, z_b)$. ($V_z(z)$ stands for the derivative of $V(z)$ with respect to z.)

(A2) $V(z) = \gamma + z^s V_1(z)$, with $\gamma < 0$, $s > \frac{2-\alpha}{2}$, and $V_1(z)$ is an analytic function on an open set J, $J \supset [0, z_a]$.

Let us consider the conservative system

$$\ddot{z} \quad = \quad -U_z(z), \tag{2}$$

with $U(z)$ satisfying (A1) and (A2). We denote the energy of (2) by

$$E \quad = \quad \frac{\dot{z}^2}{2} + U(z). \tag{3}$$

We shall assume the following hypothesis for $g(t;\delta)$

(B) For $\delta > 0$, $g(t;\delta)$ is the periodic solution of (2) on the energy level $E = -\delta$ such that $g(0;\delta) = g_0$, $\dot{g}(0;\delta) = 0$ being g_0 the minimum of $g(t;\delta)$.

Note that for $\delta > 0$, $g(t;\delta)$ satisfying (B) is an even periodic function with period $T = T(\delta)$ that tends to a finite value when δ goes to 0. Moreover, from (3) we have $g_0 = \left(\frac{\delta}{|\gamma|}\right)^{\frac{1}{\alpha}}(1 + O(\delta^{\frac{s}{\alpha}}))$.

The motivation to study the equation (1) comes from a problem on Celestial Mechanics. The planar three body problem with some homogeneous potential has the well known homographic solutions. For these solutions the configuration of the bodies is preserved for all time. In [6] and [7] the linear stability of these homographic solutions was studied after reducing the problem to a four order linear nonautonomous system. However, if we consider the planar homographic solutions in the spatial three body problem, the vertical stability is determined by an equation of the type (1) (see [9]) where the related potential $U(z)$ is the following one

$$U(z) = z^{\alpha}\left(-\frac{1}{\alpha} + \frac{z^{2-\alpha}}{2}\right) \tag{4}$$

with $\alpha \in (0, 2)$. An important particular case is the Newtonian potential which corresponds to (4) with $\alpha = 1$. In this case, $g(t;\delta) = 1 - e\cos t$ where $\delta = (1 - e^2)/2$ and e is the eccentricity associated to the orbit.

We shall write the equation (1) as a first order system

$$\dot{\mathbf{x}} \;=\; A(t)\mathbf{x}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ \lambda_1 + \lambda_2 g^{\alpha-2} & 0 \end{pmatrix} \tag{5}$$

depending on three parameters, λ_1, λ_2 and δ. To simplify the notation the dependence on these parameters will not be explicitly written if there is no confusion. We shall use the same convention for all linear systems which appear in what follows and for their corresponding monodromy matrices.

We note that system (5) is Hamiltonian, with Hamiltonian function

$$H(x_1, x_2, t) \;=\; \frac{1}{2}\left[-(\lambda_1 + \lambda_2 g^{\alpha-2})x_1^2 + x_2^2\right], \;\; \mathbf{x} = (x_1, x_2)^T.$$

Let $\Phi(t)$ be the fundamental matrix of system (5) such that $\Phi(0) = I$, being I the identity matrix. As usual, we define the stability parameter as $\mathrm{tr} = \mathrm{tr}(\Phi(T)) = \mu + 1/\mu$ where μ, $1/\mu$ are the eigenvalues of $\Phi(T)$ (see [8]).

The following theorem gives us the asymptotic behaviour of tr as $\delta \to 0$. In this theorem, we shall assume non degeneracy conditions in the sense that some coefficient is different from zero. The meaning of this coefficient is that the dominant terms on the stability parameter are the expected ones.

Theorem 1. *Let us consider the system (5) where $g(t;\delta)$ satisfies the hypothesis (B) and assume non degeneracy conditions. Let be $\hat{\lambda} = \gamma\frac{(2-\alpha)^2}{8}$ where γ is defined in (A2) and $\beta = \sqrt{1 - \frac{\lambda_2}{\hat{\lambda}}}$. Assume $\lambda_2 \neq 0$ and $\lambda_2 \neq \hat{\lambda}$. Then we have the following asymptotic behaviour for the stability parameter when δ goes to 0*

$$\begin{aligned} \log|tr| \;&=\; k_1 - \frac{2-\alpha}{2\alpha}\beta\log\delta\,(1 + o(1)) + \dots, \quad \textit{if} \;\; \lambda_2 > \hat{\lambda}, \\ tr \;&=\; k_2 + k_3\cos\left(k_4 - \frac{2-\alpha}{2\alpha}\hat{\beta}(1 + o(1))\log\delta\right) + \dots, \quad \textit{if} \;\; \lambda_2 < \hat{\lambda}. \end{aligned}$$

In the last case, $\beta = \hat{\beta} i$. The coefficients k_j, $j = 1, \ldots, 4$ are constants with $k_3 \neq 0$, $k_2 + k_3 < 0$ and $k_2 - k_3 > 0$.

In the case $\lambda_2 > \hat{\lambda}$ (recalling that $\hat{\lambda} < 0$), for δ small enough we have that $|\mathrm{tr}| > 2$, and the system (5) is hyperbolic. If $\lambda_2 < \hat{\lambda}$, tr oscillates between $k_2 + k_3$ and $k_2 - k_3$ as δ tends to zero. Then, depending on the values of $k_2 + k_3$ and $k_2 - k_3$, tr can cross the lines $\mathrm{tr} = -2$ and $\mathrm{tr} = 2$ infinitely many times for δ small enough. This would implies that infinitely many intervals in δ where the system is elliptic alternate with infinitely many hyperbolic intervals.

We remark that in general, $\hat{\lambda}$ depends on two parameters, that is $\hat{\lambda}(\alpha, \gamma) < 0$. However, in the case (4), $\gamma = -1/\alpha$ and $\hat{\lambda} = -\frac{(2-\alpha)^2}{8\alpha}$. According to the theorem, this is the critical value of λ_2 which separates oscillatory and exponential behaviour of tr. In particular, for the Newtonian potential we get $\hat{\lambda} = -1/8$.

Theorem 1 can be applied to a particular family of Ince's equations using the following result.

Lemma 1. *Let us consider the following Ince equation*

$$(1 + a\cos t)\ddot{y} + b\sin t\dot{y} + (c + ad\cos t)y = 0, \tag{6}$$

where a, b, c and d are real parameters, $b = 0$ or $b = -2a$, and $|a| < 1$. Then (6) can be reduced to the equation

$$\ddot{x} - \left(\lambda_1 + \frac{\lambda_2}{1 + a\cos t}\right)x = 0 \tag{7}$$

with $\lambda_1 = -d$, $\lambda_2 = d - c$ if $b = 0$ and, $\lambda_1 = -d - 1$, $\lambda_2 = d - c + 1$ if $b = -2a$. Moreover, (6) is the most general Ince equation that can be written as a Hill equation.

We shall give a proof of this lemma in the appendix 6. In fact, (7) is the equation obtained using the homographic potential (4) with $\alpha = 1$. Now we can apply theorem 1 to the equation (6) with $\delta = (1 - a^2)/2$. We note that (6) depends on three parameters a, c and d and our result applies for $|a| \approx 1$. Moreover, in this case $\hat{\lambda} = -1/8$. In the plane of parameters (c, d), $\lambda_2 = \hat{\lambda}$ defines a line $d - c = -1/8$ if $b = 0$ and $d - c + 1 = -1/8$ if $b = -2a$. This line separates oscillatory from exponential behaviour as $|a|$ goes to 1, under non degeneracy conditions.

In section 2 we prove theorem 1 using the same techniques introduced in [7]. The idea is to perform a blow up of the singularity which allows us to compute the dominant term of $\mathrm{tr}(\Phi(T))$ as δ goes to 0, by using an appropriate approximation of $\Phi(T)$ for δ small enough. Some values of the parameters can cancel that dominant term and then our asymptotic formulae do not hold. The nondegeneracy conditions are introduced in order to prevent this case. The precise definition is postponed to the beginning of section 3 as definition 1, because it requires some linear systems which are introduced in section 2. The rest of section 3 is devoted to the study of the non degeneracy conditions. We shall see that they depend on the existence of heteroclinic orbits in some related systems, in the sense that, if there are not heteroclinic orbits then, the non degeneracy conditions are satisfied and the asymptotic formulae given in theorem 1 hold. In the case of the homographic potential (4), we determine values of the parameters giving rise to these heteroclinic orbits. We shall see that these values define some limit curves in the (λ_1, λ_2)-plane as δ goes to zero. Furthermore, we have computed numerically tr for the homographic potential with different values of α. Note that for general α, the solution g is not available explicitly. Assume that α and δ are fixed. Then the curves $\mathrm{tr} = \pm 2$ define the stability and instability regions in the (λ_1, λ_2)-plane. In section 4 we show how these curves tend to the ones defined by the heteroclinic orbits as δ goes to zero. In section 5 we study the existence of collapsed gaps in the equation (7) for any value of e.

2 Proof of theorem 1

Using the reversibility of the system, $\Phi(T)$ can be written as

$$\Phi(T) = \Phi^{-1}(-T/2)\Phi(T/2) = L\Phi^{-1}(T/2)L\Phi(T/2) \tag{8}$$

where $L = \mathrm{diag}(-1, 1)$.

In order to prove the theorem we shall work, for $\delta > 0$, with a linear system without any singularity. To this end, we introduce new variables defined by $\mathbf{u} = S(t)\mathbf{x}$, where $S(t) = \mathrm{diag}(1, q)$, $q = q(t; \delta)$ defined as $q := g^{\frac{2-\alpha}{2}}$. We note that, for $\delta > 0$, $S(t)$ is non–singular for all t. By introducing the new time τ via $dt = qd\tau$, the new system is

$$\mathbf{u}' = B(\tau)\mathbf{u}, \quad B(\tau) = q(\dot{S} + SA)S^{-1} = \begin{pmatrix} 0 & 1 \\ \lambda_2 + \lambda_1 q^2 & \dot{q} \end{pmatrix}, \tag{9}$$

where $' = \frac{d}{d\tau}$. We shall denote by $\mathcal{T}(\delta)$, or simply by $\mathcal{T}$, the period of $g(t; \delta)$ in the new time τ.

Let $\Psi(\tau)$ be the fundamental matrix of system (9) such that $\Psi(0) = I$. Then, $\Phi(t) = S^{-1}(t)\Psi(\tau(t))S(0)$. As $S(t)$ is T–periodic, we get that $\Phi(T) = S^{-1}(0)\Psi(\mathcal{T})S(0)$ and so, $\Phi(T)$ and $\Psi(\mathcal{T})$ have the same eigenvalues. Using (8) it is easy to check that

$$\Psi(\mathcal{T}) = L\Psi^{-1}(\mathcal{T}/2)L\Psi(\mathcal{T}/2). \tag{10}$$

Our purpose is to obtain an expression of $\Psi(\mathcal{T}/2)$ for $\delta > 0$ small enough in order to compute the dominant terms of the trace of $\Psi(\mathcal{T})$. It turns out that the technique introduced in [7] can be applied to our problem after some obvious modifications. In the rest of this section we summarize the main steps of this technique and refer to [7] for the details.

We define $Q = -(2-\alpha)g^{-\alpha/2}\dot{g} = -(2-\alpha)q^{-\alpha/(2-\alpha)}\dot{g}$ where $q = g^{(2-\alpha)/2}$ as before. Then, using the time τ, $q(\tau), Q(\tau)$ is a solution of the following system

$$\left\{ \begin{array}{lcl} q' & = & -\dfrac{1}{2}qQ, \\ Q' & = & \dfrac{\alpha}{2(2-\alpha)}Q^2 + (2-\alpha)\hat{q}^{1-\alpha}U_z(\hat{q}), \end{array} \right. \tag{11}$$

where $\hat{q} := q^{2/(2-\alpha)}$. From (3) we get that the system above has a first integral

$$E = \hat{q}^{\alpha}\left(\frac{Q^2}{2(2-\alpha)^2} + V(\hat{q})\right). \tag{12}$$

The behaviour of the orbits of (11) is summarized in figure 1. On the level set $E = 0$ we distinguish two saddle points $P_\pm$ with $(q, Q) = (0, \pm Q_p)$, $Q_p = (2-\alpha)\sqrt{-2\gamma}$, and two heteroclinic orbits γ_0, γ_+. The system (11) restricted to $q = 0$ does not depend on $V(z)$ and it is easily integrated. So we get for γ_0, to be denoted as solution L_1

$$q_{L_1}(\tau) \equiv 0, \qquad Q_{L_1}(\tau) = -Q_p \tanh\left(\frac{\alpha}{2(2-\alpha)}Q_p\tau\right).$$

Let us denote by $q_{L_2}(\tau), Q_{L_2}(\tau)$ the solution of the system (11) restricted to γ_+ such that $Q_{L_2}(0) = 0$. We note that this solution can not be obtained explicitly because it depends on the potential $V(z)$. However for the homographic potential (4) we get this solution explicitly as

$$q_{L_2}(\tau) = q_a/\cosh\left(\frac{2-\alpha}{2}q_a\tau\right), \qquad Q_{L_2}(\tau) = Q_p \tanh\left(\frac{2-\alpha}{2}q_a\tau\right).$$

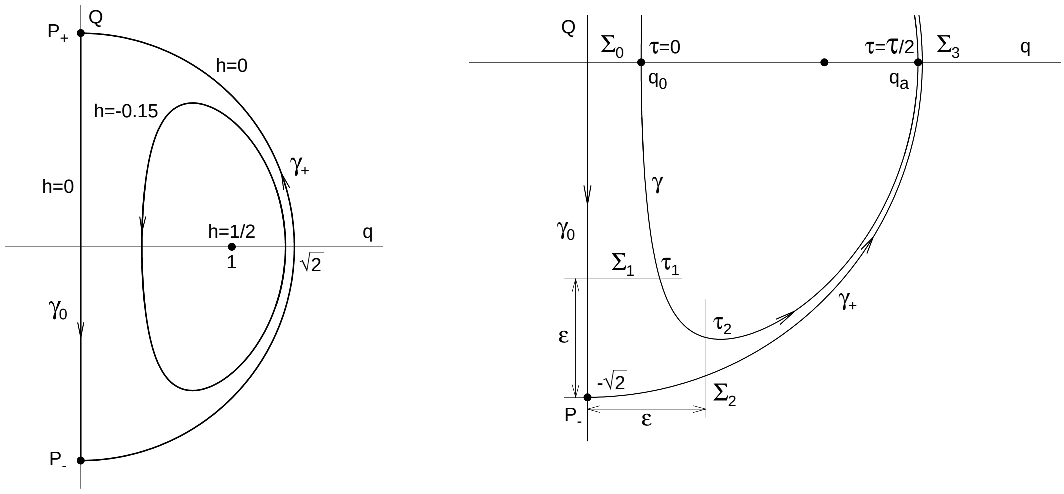


Figure 1: Left: Phase portrait of (11) for $\alpha = 1$ and $U(z) = z(-1 + z/2)$. Right: An illustration of the sections used in the proof.

Notice that $q_{L_2}(0) = z_a^{(2-\alpha)/2} := q_a$.

We note that we are interested in the solutions of (11) near the heteroclinic cycle defined by γ_0, γ_+ and the equilibria $P_\pm$. More concretely, in the level $E = -\delta$.

Given $\epsilon, \epsilon_i, i = 0, \dots, 3$, small enough, we define the following sections (see figure 1 right)

$$\Sigma_0 = \{(q, Q) \,|\, 0 < q < \epsilon_0, Q = 0\}, \quad \Sigma_1 = \{(q, Q) \,|\, 0 < q < \epsilon_1, Q = -Q_p + \epsilon\},$$

$$\Sigma_2 = \{(q, Q) \,|\, q = \epsilon, |Q + Q_p| < \epsilon_2\}, \quad \Sigma_3 = \{(q, Q) \,|\, 0 < q_a - q < \epsilon_3, Q = 0\}.$$

For a fixed value of $\epsilon > 0$, sufficiently small, we can take small enough ϵ_i for $i = 0, \dots, 3$, such that the Poincaré maps $\mathcal{P}_1 : \Sigma_0 \mapsto \Sigma_1$, $\mathcal{P}_2 : \Sigma_1 \mapsto \Sigma_2$, and $\mathcal{P}_3 : \Sigma_2 \mapsto \Sigma_3$ be well defined. Figure 1 right shows an illustration of the sections to be used.

We denote by $\tau_{L_1} > 0$ the time defined by $Q_{L_1}(\tau_{L_1}) = -Q_p + \epsilon$, and $s_{L_2} > 0$ such that $q_{L_2}(-s_{L_2}) = \epsilon$. Using the symmetry of (11) one has $Q_{L_2}(s_{L_2}) > 0$. Note that τ_{L_1} and s_{L_2} are finite and independent of δ once ϵ is fixed.

For a fixed value of $\delta > 0$ small enough, we consider the solution of (11) with $E = -\delta$ such that $(q(0), Q(0)) \in \Sigma_0$, $q(0) = q_0$, the minimum of $q(\tau)$. Using the hypothesis (A2) and (12) we get that $q_0 = q(0) = (\delta/|\gamma|)^{(2-\alpha)/(2\alpha)} \left(1 + O(\delta^{s/\alpha})\right)$. Let τ_1 be the smallest positive time such that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$. In a similar way we define τ_2 such that $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. It is clear that τ_1 and τ_2 depend continuously on δ. Moreover $\tau_1 \to \tau_{L_1}$ and $\mathcal{T}/2 - \tau_2 \to s_{L_2}$ when $\delta \to 0$. We refer to [7] for the proof of the next lemma.

Lemma 2. *Let $\varepsilon > 0$ be a fixed small enough value. Then, for any sufficiently small $\delta > 0$ we have*

$$\frac{2}{Q_p + \varepsilon} \log\left(\frac{\varepsilon}{q(\tau_1)}\right) \le \tau_2 - \tau_1 \le \frac{2}{Q_p - \varepsilon} \log\left(\frac{\varepsilon}{q(\tau_1)}\right). \tag{13}$$

Furthermore, $q(\tau_1)$ can be derived from (12) when $Q = -Q_p + \epsilon$, $E = -\delta$. One easily obtains

$$\log\left(\frac{\epsilon}{q(\tau_1)}\right) = -\frac{(2-\alpha)}{2\alpha} \log\delta\,(1 + o(1)) \quad \text{for} \quad \delta \to 0. \tag{14}$$

System (9) can be written as

$$\mathbf{u}' = B(\tau)\mathbf{u}, \qquad B(\tau) \;=\; \begin{pmatrix} 0 & 1 \\ \lambda_1 q^2 + \lambda_2 & -Q/2 \end{pmatrix} := B_a(q(\tau), Q(\tau)). \tag{15}$$

Let $\tilde{\Psi}(\tau_b, \tau_a)$ be the transition matrix from τ_a to τ_b of system (9). Then, we can write

$$\Psi\left(\mathcal{T}/2\right) \;=\; \tilde{\Psi}\left(\mathcal{T}/2, \tau_2\right) \tilde{\Psi}(\tau_2, \tau_1) \tilde{\Psi}(\tau_1, 0), \tag{16}$$

where we recall that τ_1, τ_2 and $\mathcal{T}$ depend on δ. On the interval $[\tau_2, \mathcal{T}/2]$ it will be more convenient to use a new time $s = \tau - \mathcal{T}/2$ similar to the time used along L_2. Let $\Gamma(s)$ be the fundamental matrix of $\mathbf{u}' = B(s + \mathcal{T}/2)\mathbf{u}$ ($'$ denotes here the derivative with respect to s) such that $\Gamma(0) = I$. It is easy to check that

$$\tilde{\Psi}(\mathcal{T}/2, \tau_2) = \Gamma^{-1}(-s_2) = L\Gamma^{-1}(s_2)L, \qquad s_2 = \mathcal{T}/2 - \tau_2.$$

Therefore

$$\Psi(\mathcal{T}/2) = L\Gamma^{-1}(s_2)L\tilde{\Psi}(\tau_2, \tau_1)\tilde{\Psi}(\tau_1, 0).$$

Our purpose is to approximate the transition matrices involved in $\Psi(\mathcal{T})$ by simpler ones. Following [7] we introduce the limit systems

$$\begin{aligned} \mathbf{u}' &= B_{L_1}(\tau)\mathbf{u}, \qquad B_{L_1}(\tau) = B_a(0, Q_{L_1}(\tau)), & (17) \\ \mathbf{u}' &= B_{L_2}(\tau)\mathbf{u}, \qquad B_{L_2}(\tau) = B_a(q_{L_2}(\tau), Q_{L_2}(\tau)) & (18) \end{aligned}$$

and define $Z_1(\tau)$, $Z_2(\tau)$ as the fundamental matrices of (17) and (18) respectively, such that $Z_1(0) = I$ and $Z_2(0) = I$. In the interval $[\tau_1, \tau_2]$ we write

$$\begin{aligned} B(\tau) &= B_a(0, -Q_p) + B_1(\tau), \quad B_a(0, -Q_p) = \begin{pmatrix} 0 & 1 \\ \lambda_2 & \frac{Q_p}{2} \end{pmatrix}, & (19) \\ & B_1(\tau) = \begin{pmatrix} 0 & 0 \\ \lambda_1 q^2 & -(Q + Q_p)/2 \end{pmatrix}. \end{aligned}$$

We note that $B_a(0, -Q_p)$ has eigenvalues $\rho^{\pm} = \frac{Q_p}{4}(1 \pm \beta)$, being $\beta = \sqrt{1 - \frac{\lambda_2}{\hat{\lambda}}}$. The corresponding eigenvectors are $(1, \rho^{\pm})^T$. We assume that $\lambda_2 \neq \hat{\lambda}$, so $\rho^+ \neq \rho^-$ and $B_a(0, -Q_p)$ diagonalizes. Moreover for $\delta > 0$ sufficiently small, $\tilde{\Psi}(\tau_2, \tau_1)$ can be approximated by $\sigma P \mathcal{D} P^{-1}$ (see [7]) where

$$\sigma = \exp\left(\frac{Q_p}{4}(\tau_2 - \tau_1)\right), \quad P = \begin{pmatrix} 1 & 1 \\ \rho^+ & \rho^- \end{pmatrix}, \quad \mathcal{D} = \mathrm{diag}(\sigma^{\beta}, \sigma^{-\beta}).$$

Therefore we can write

$$\Psi(\mathcal{T}/2) = \sigma L Z_2^{-1}(s_{L_2}) L P \mathcal{D} P^{-1} Z_1(\tau_{L_1})(I + \Delta_3) \tag{20}$$

where Δ_3 is a matrix depending on δ, ϵ and on the parameters λ_1, λ_2, α and γ. It has norm $\|\Delta_3\| = o(1)$ for $\delta \to 0$. By substituting (20) in (10) we get

$$\Psi(\mathcal{T}) = \mathcal{M}(I + O), \qquad \mathcal{M} = LE^{-1}\mathcal{D}^{-1}XLX^{-1}\mathcal{D}E,$$

where $E = P^{-1}Z_1(\tau_{L_1})$, $X = P^{-1}LZ_2(s_{L_2})$ and O depends on δ and also on λ_1, λ_2, α and γ. The dependence on ϵ cancels because the matrix $\Psi(\mathcal{T})$ is independent of the arbitrary choice of ϵ. The norm $\|O\|$ is $o(1)$ for $\delta \to 0$. The important fact here is that the matrices E and X do not depend on δ. Now the trace of $\mathcal{M}$ gives us a suitable approximation of the stability parameter tr.

We introduce the following notation

$$E = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Then a simple computation shows that

$$\text{trace}(\mathcal{M}) = \frac{2(\rho^+ - \rho^-)^2}{d_1 d_2}(2e_1e_2x_3x_4\sigma^{2\beta} + 2e_3e_4x_1x_2\sigma^{-2\beta} - (x_1x_4 + x_2x_3)(e_1e_4 + e_2e_3)) \quad (21)$$

where $d_1 = \det Z_1(\tau_{L_1}) \neq 0$, $d_2 = \det Z_2(s_{L_2}) \neq 0$. Using Liouville theorem we can compute

$$d_1 = \left(\cosh\left(\frac{\alpha Q_p}{2(2-\alpha)}\tau_{L_1}\right)\right)^{(2-\alpha)/\alpha}, \qquad d_2 = \frac{q_{L_2}(s_{L_2})}{q_a} = \frac{\epsilon}{q_a}.$$

Remark 1. *If $\lambda_2 > \hat{\lambda}$, then $\rho^\pm \in \mathbb{R}$ and it is clear that all the matrices involved in $\mathcal{M}$ are real. If $\lambda_2 < \hat{\lambda}$, $\rho^\pm$ are conjugate complex numbers and $\beta = \hat{\beta} i$. In this case one has $\bar{E} = \mathcal{F}E$, $\bar{X} = \mathcal{F}X$ and $\bar{\mathcal{D}} = \mathcal{F}\mathcal{D}\mathcal{F}$, where $\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the bar stands for complex conjugate. Then $\mathcal{M}$ is also a real matrix and we get*

$$\textit{trace}(\mathcal{M}) = \frac{8(\rho^+ - \rho^-)^2}{d_1 d_2}(\, Re(e_1e_2x_3x_4\sigma^{2\hat{\beta} i}) - Re(\bar{x}_3x_4)Re(e_1\bar{e}_2)\,) \quad (22)$$

It is clear that in the case $\lambda_2 > \hat{\lambda}$, the dominant term in (21) as δ goes to zero, is $e_1e_2x_3x_4\sigma^{2\beta}$ if and only if $e_1e_2x_3x_4 \neq 0$. We recall from (14) that $\sigma \to \infty$ as $\delta \to 0$. The non degeneracy condition will be defined in order to ensure that this inequality holds.

To finish the proof of theorem 1 we shall assume $e_1e_2x_3x_4 \neq 0$. If $\lambda_2 > \hat{\lambda}$ using (21) we write

$$\text{tr} = c\,\sigma^{2\beta}(1 + \ldots), \qquad c = \frac{4(\rho^+ - \rho^-)^2}{d_1 d_2} e_1e_2x_3x_4 \neq 0.$$

Notice that c does not depend on δ. Taking logarithms and using (14) we obtain

$$\log|\text{tr}| = \log|c| - \beta\frac{(2-\alpha)}{2\alpha}\log\delta\,(1 + o(1)) + \ldots\,.$$

In the case $\lambda_2 < \hat{\lambda}$, using (22)

$$\text{tr} = c_1 \text{Re}(c_0\sigma^{2\hat{\beta}\text{i}}) + c_2 + \ldots \quad (23)$$

for some constants c_i, $i = 0, 1, 2$. We introduce c_3, c_4 as $c_0 = c_3 e^{c_4 \text{i}}$. Then

$$\text{tr} \;=\; c_1c_3\cos\left(c_4 - \frac{(2-\alpha)}{2\alpha}\hat{\beta}\log\delta\,(1 + o(1))\right) + c_2 + \ldots$$

where

$$c_1 = \frac{8(\rho^+ - \rho^-)^2}{d_1 d_2} = -\frac{2Q_p^2\hat{\beta}^2}{d_1 d_2} < 0, \quad c_2 = -c_1|e_1e_2x_3x_4|A, \quad c_3 = |e_1e_2x_3x_4|,$$

for some $A \in \mathbb{R}$, $|A| < 1$. In this case tr oscillates mainly between $c_2 + c_1c_3 < 0$ and $c_2 - c_1c_3 > 0$.

3 The non degeneracy condition

In this section we shall study the non degeneracy condition and we look for the values of the parameters $\alpha, \lambda_1, \lambda_2$ such that the non degeneracy condition is satisfied. To do this we need to know the behaviour of e_1, e_2, x_3 and x_4. These coefficients are related to the elements of matrices $Z_1(\tau_{L_1})$ and $Z_2(s_{L_2})$, which are the fundamental matrices of (17) and (18) respectively.

To be more precise let $\mathbf{v}_1(\tau)$, $\mathbf{v}_2(\tau)$ be the solutions of (17) such that $\mathbf{v}_1(0) = (1,0)^T$ and $\mathbf{v}_2(0) = (0,1)^T$ respectively. In a similar way, let $\mathbf{w}_1(\tau)$, $\mathbf{w}_2(\tau)$ be the solutions of (18) such that $\mathbf{w}_1(0) = (1,0)^T$ and $\mathbf{w}_2(0) = (0,1)^T$ respectively. Then

$$e_i = \frac{1}{\rho^- - \rho^+} < \mathbf{y}^-, \mathbf{v}_i(\tau_{L_1}) >, \qquad x_{i+2} = \frac{1}{\rho^- - \rho^+} < \mathbf{y}^+, \mathbf{w}_i(s_{L_2}) >, \qquad i = 1,2$$

where $\mathbf{y}^- = (\rho^-, -1)^T$, $\mathbf{y}^+ = (\rho^+, 1)^T$ and $<,>$ stands for the scalar product. In order to simplify the notation from now on we shall use $\mathbf{v}_i, \mathbf{w}_i$, $i = 1,2$ to denote $\mathbf{v}_i(\tau_{L_1})$ and $\mathbf{w}_i(s_{L_2})$, $i = 1,2$ respectively. Therefore, the coefficient of $\sigma^{2\beta}$ in (21) is

$$\begin{aligned} &\frac{4(\rho^- - \rho^+)^2}{d_1 d_2} e_1 e_2 x_3 x_4 = \\ &\frac{4}{d_1 d_2 (\rho^- - \rho^+)^2} \prod_{i=1,2} \|\mathbf{v}_i\| \|\mathbf{w}_i\| < \mathbf{y}^-, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} >< \mathbf{y}^+, \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|} >, \end{aligned} \tag{24}$$

where $\| \; \| = \| \; \|_2$. Notice that $\|\mathbf{v}_i\| \neq 0$, and $\|\mathbf{w}_i\| \neq 0$, for $i = 1,2$.

We recall that the systems (17) and (18) are obtained from (15) by substituting (q, Q) by $(0, Q_{L_1}(\tau))$ and $(q_{L_2}(\tau), Q_{L_2}(\tau))$ respectively. So, we consider (15) and we introduce polar coordinates in (15) as $u_1 = r\cos\varphi$, $u_2 = r\sin\varphi$. Then

$$r' = r\sin\varphi \left((\lambda_1 q^2 + \lambda_2 + 1)\cos\varphi - \frac{Q}{2}\sin\varphi \right), \tag{25}$$

$$\varphi' = (\lambda_1 q^2 + \lambda_2)\cos^2\varphi - \frac{Q}{2}\sin\varphi\cos\varphi - \sin^2\varphi. \tag{26}$$

We write $\mathbf{v}_i(\tau)$, $\mathbf{w}_i(\tau)$, $i = 1,2$ using polar coordinates as $\mathbf{v}_i(\tau) = r_i(\tau)(\cos\varphi_i(\tau), \sin\varphi_i(\tau))$ and $\mathbf{w}_i(\tau) = r_{i+2}(\tau)(\cos\varphi_{i+2}(\tau), \sin\varphi_{i+2}(\tau))$ with $\varphi_{1,3}(0) = 0$ and $\varphi_{2,4}(0) = \pi/2$.

Moreover, if $\mathbf{v}(\tau)$ denotes a solution of (15)

$$< \mathbf{y}^-, \frac{\mathbf{v}(\tau)}{\|\mathbf{v}(\tau)\|} >= \rho^- \cos\varphi(\tau) - \sin\varphi(\tau), \qquad < \mathbf{y}^+, \frac{\mathbf{v}(\tau)}{\|\mathbf{v}(\tau)\|} >= \rho^+ \cos\varphi(\tau) + \sin\varphi(\tau) \tag{27}$$

To cancel (24) we must set to zero some of the factors $<\mathbf{y}^-, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}>$ or $<\mathbf{y}^+, \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}>$. The non degeneracy conditions will be defined in terms of the limit behaviour of that factors.

Remark 2. *If $\lambda_2 < \hat{\lambda}$, $\rho^\pm$ are complex numbers $a \pm bi$ with $a \neq 0, b \neq 0$. Now for any τ, $\rho^- \cos\varphi(\tau) - \sin\varphi(\tau) \neq 0$ and $\rho^+ \cos\varphi(\tau) + \sin\varphi(\tau) \neq 0$. We say that in this case the non degeneracy condition is satisfied and, hence, (23) follows.*

Definition 1. *Assume $\lambda_2 > \hat{\lambda}$. We say that the non degeneracy conditions are satisfied if and only if the following limits exist and*

$$\lim_{\tau \to \infty} < \mathbf{y}^-, \frac{\mathbf{v}_i(\tau)}{\|\mathbf{v}_i(\tau)\|} > \neq 0, \quad \textit{for} \quad i = 1,2, \tag{28}$$

$$\lim_{\tau \to \infty} < \mathbf{y}^+, \frac{\mathbf{w}_i(\tau)}{\|\mathbf{w}_i(\tau)\|} > \neq 0, \quad \textit{for} \quad i = 1,2. \tag{29}$$

Assume that (28) and (29) are satisfied. Then for any finite $\tau > 0$ sufficiently large, (24) is different from zero and the dominant term in (21) is the one corresponding to $\sigma^{2\beta}$.

Lemma 3. *Assume $0 < \alpha < 2$ and $\gamma < 0$.*

(a) If $\lambda_2 > 0$, then

$$\lim_{\tau\to\infty} < \mathbf{y}^-, \frac{\mathbf{v}_i(\tau)}{\|\mathbf{v}_i(\tau)\|} >= (\rho^- - \rho^+)\cos\varphi_+ \neq 0 \tag{30}$$

where $\varphi_+ = \arctan(\rho^+)$ for $i = 1, 2$, and the condition (28) is satisfied.

(b) If $2/3 \leq \alpha < 2$ and $\hat\lambda(\alpha;\gamma) < \lambda_2 < 0$, then (28) is satisfied.

(c) Assume $\alpha \leq 2/3$ and $\gamma = -k\alpha^s(1 + O(\alpha))$ for some $k > 0$ and $-2 < s < 2$. Then there exists a sequence $\{\alpha_k\}_{k\geq 1}$ with $\alpha_1 = 2/3$, $\lim_{k\to\infty}\alpha_k = 0$, and for any α_k there is a finite sequence of λ_2 values

$$\hat\lambda(\alpha;\gamma) = \lambda_{2,k}^{(k)} < \lambda_{2,k-1}^{(k)} < \ldots < \lambda_{2,1}^{(k)} < \lambda_{2,0}^{(k)} = 0$$

such that for $\alpha = \alpha_k$ and $\lambda_2 = \lambda_{2,j}^{(k)}$, $j = 0, 1, \ldots, k-1$ the condition (28) is not satisfied. Moreover, if $\alpha = \alpha_k$ and $\lambda_2 \neq \lambda_{2,j}^{(k)}$, $j = 0, 1, \ldots, k$ then (28) holds.

Notice that the hypothesis on γ in (c) includes in particular, the cases of constant γ independent of α, as well as the case $\gamma = -1/\alpha$ corresponding to the homographic potential.

We have performed some numerical computations in the case $\gamma = -1/\alpha$ which corresponds, in particular, to the homographic case. As we shall see in next section these numerical results support the following conjecture

Conjecture 1. *If $\gamma = -1/\alpha$, the values given in (c) of lemma 3 are*

$$\alpha_k = \frac{2}{2k+1}, \quad \text{and hence} \quad \hat\lambda(\alpha_k; -1/\alpha_k) = -\frac{k^2}{2k+1}, \qquad k \geq 1. \tag{31}$$

Lemma 4. *Let us consider the homographic potential. Assume $\alpha \in (0, 2)$ and $\lambda_2 > \hat\lambda(\alpha)$ are fixed. Let be $\lambda_1^{(0)} = -(\alpha/2)(\rho^+)^2$. Then*

(a) If $\lambda_1 > \lambda_1^{(0)}$, the condition (29) is satisfied.

(b) There exists a decreasing sequence of λ_1 values, $\{\lambda_1^{(k)}\}_{k\geq 0}$, with $\lim_{k\to\infty}\lambda_1^{(k)} = -\infty$ such that the condition (29) is not satisfied. Moreover if $\lambda_1 \neq \lambda_1^{(k)}$ for any $k \geq 0$, then (29) holds.

Remark 3. *We note that for the homographic potential (4), $\gamma = -1/\alpha$ and so, ρ^+ depends on α and λ_2. Therefore, if we fix α, for $\lambda_2 \geq \hat\lambda$, $\lambda_1 = -\frac{\alpha}{2}(\rho^+)^2$ defines a curve in the (λ_1, λ_2)-plane which can be written as*

$$\lambda_2 = -\frac{1}{\alpha}(2\lambda_1 + (2-\alpha)\sqrt{-\lambda_1}), \qquad \text{for} \qquad \lambda_1 \leq -\frac{(2-\alpha)^2}{16}. \tag{32}$$

Figure 2 shows these curves for some values of α. The curves end at the point $(\lambda_1, \lambda_2) = (-(2-\alpha)^2/16, \hat\lambda)$.

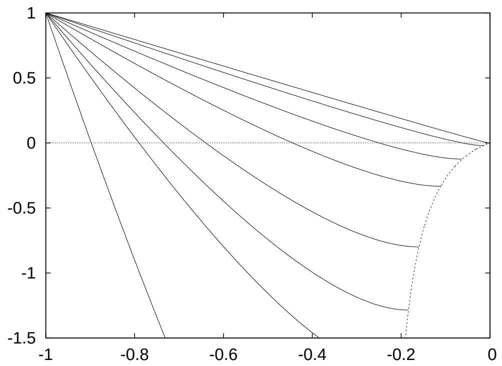


Figure 2: Plot of the curve $\lambda_1 = -\frac{\alpha}{2}(\rho^+)^2$, (32), in the $(\lambda_1, \lambda_2)-$plane for different values of α. From top to bottom the curves correspond to $\alpha = 1.9, 1.5, 1, 2/3, 2/5, 2/7, 2/9, 2/19$. The dashed line gives the end points of these curves.

Remark 4. *If $\frac{2}{\alpha}\lambda_1 + \lambda_2 > 0$, it is easy to check that $\lambda_1 > \lambda_1^{(0)}$ and so, (29) is satisfied.*

Next sections are devoted to prove lemmas 3 and 4. The condition (28) involves the solutions of (26) along γ_0. We note that they depend on the parameters α, γ and λ_2. However (29) involves solutions of (26) along γ_+. Therefore they depend on α, γ, λ_2 and λ_1. We shall see that in both cases, the condition fails for the values of the parameters such that the corresponding system has an heteroclinic orbit.

3.1 Heteroclinic orbits for the system along γ_0.

We consider the system (25), (26) with $(q, Q) = (0, Q_{L_1}(\tau))$. We introduce a new variable $u = -Q_{L_1}(\tau)/Q_p$. Then we get the following equations for φ, u

$$\begin{aligned} \varphi' &= \lambda_2 \cos^2\varphi + \frac{Q_p u}{2} \sin\varphi\cos\varphi - \sin^2\varphi, \\ u' &= \frac{\alpha}{2(2-\alpha)} Q_p (1-u^2). \end{aligned} \tag{33}$$

In order to check (28) we must study the solutions of (33) such that $(\varphi(0), u(0)) = (0, 0)$, $(\pi/2, 0)$. Assume that $\tan\varphi(\tau) \to \rho^-$ when $\tau \to \infty$ for one of these solutions. Then (28) does not hold. We shall see that in this case we have an heteroclinic orbit of the system (33). We recall that (28) has been defined for $\lambda_2 > \hat{\lambda}$. However in this section, to look for heteroclinic orbits, it will be convenient to consider also $\lambda_2 = \hat{\lambda}$.

We note that (33) is π periodic in φ. So, we only need to consider (33) in the region

$$\mathcal{R} = \{(\varphi, u) \,|\, \varphi \in [-\pi/2, \pi/2],\ u \in [-1, 1]\}. \tag{34}$$

However to study the existence of heteroclinic orbits it will be more convenient to consider $\varphi \in \mathbb{R}$. We summarize the important marks of the qualitative behaviour of the system (33) in the following properties:

1. The lines $u = \pm 1$ are invariant.

2. If $\lambda_2 > \hat{\lambda}$, (33) has four equilibrium points in $\mathcal{R}$

   $$(\varphi, u) = (\varphi_+, 1), (\varphi_-, 1), (-\varphi_+, -1), (-\varphi_-, -1), \quad \varphi_+ = \arctan(\rho^+), \quad \varphi_- = \arctan(\rho^-)$$

3. If $\lambda_2 > \hat{\lambda}$, $(\varphi_-, 1)$ and $(-\varphi_-, -1)$ are saddle points, $(\varphi_+, 1)$ is an attractor and $(-\varphi_+, -1)$ a repellor. For a saddle point, P, we shall denote by $W^u(P)$ $(W^s(P))$ the branch of the unstable (stable) invariant manifold of P, contained in $-1 \leq u \leq 1$.

4. If $\lambda_2 = \hat{\lambda}$, $\varphi_+ = \varphi_-$ and the system has only two equilibrium points on $\mathcal{R}$ of saddle node type. In this case, $W^u(P)$ $(W^s(P))$ will denote the fast unstable (stable) manifold in $\mathcal{R}$ of the saddle node point.

5. The system has the following symmetry $(\varphi, u, \tau) \mapsto (-\varphi, -u, -\tau)$. Therefore, if $(\varphi(\tau), u(\tau))$ is a solution, then $(-\varphi(-\tau), -u(-\tau))$ is also a solution.

6. For values of $0 < \alpha < 2$, $\gamma < 0$, and $\lambda_2 = 0$, we have $\varphi_+ = \arctan(Q_p/2)$ and $\varphi_- = 0$. In this case $\varphi = 0$ is invariant under the flow defined by (33) and $W^u((-\varphi_-, -1))$ and $W^s((\varphi_-, 1))$ coincide.

Remark 5. *We recall that we are interested in the orbits of the points* $(\varphi, u) = (0, 0)$ *and* $(\pi/2, 0)$*. It is clear that the* $\omega-$*limit set for these orbits is one of the equilibrium points in* $u = 1$*. If these orbits go to an attractor* $(\varphi_+ - k\pi, 1)$ *for some integer* k*, then* $\lim_{\tau\to\infty} \tan\varphi(\tau) = \rho^+$*. In this case,*

$$\lim_{\tau\to\infty} < \mathbf{y}^-, \frac{\mathbf{v}_i(\tau)}{\|\mathbf{v}_i(\tau)\|} >= (-1)^k(\rho^- - \rho^+)\cos\varphi_+ \neq 0 \quad \textit{if} \quad \lambda_2 \neq \hat{\lambda}$$

and (28) is satisfied. However, if one of these orbits goes to a saddle point $(\varphi_- - k\pi, 1)$*, for some integer* k*, then the limit above equals zero and (28) does not hold. In this case using the symmetry and the periodicity of (33), one has an heteroclinic orbit.*

Let us consider $\alpha \in (0, 2)$, $\gamma < 0$ and $\lambda_2 \geq \hat{\lambda}(\alpha, \gamma)$. We shall denote by $\varphi^u(0; \alpha, \gamma, \lambda_2)$, or simply $\varphi^u(0; \lambda_2)$ if α and γ are fixed, the value of φ at the intersection point of $W^u((-\varphi_-, -1))$ with $u = 0$. Notice that using the symmetry and the periodicity of (33), $\varphi^u(0; \alpha, \gamma, \lambda_2) = -k\pi/2$ for some positive integer k if and only if $W^u((-\varphi_-, -1))$ and $W^s((\varphi_- - k\pi, 1))$ coincide.

Lemma 5. *For* $\alpha = 2/3$ *and* $\lambda_2 = \hat{\lambda}(2/3; \gamma)$*,* $W^u((-\varphi_-, -1))$ *reaches* $u = 0$ *with* $\varphi = -\pi/2$*. Then an heteroclinic orbit between the points* $(-\varphi_-, -1)$ *and* $(\varphi_- - \pi, 1)$ *exists. This heteroclinic orbit has an easy analytical expression given by* $u = \rho^-/\tan\varphi$*.*

Proof The lemma follows by checking that $u = \rho^-/\tan\varphi$ is invariant under the flow defined by (33). □

Lemma 6. *Let us assume* $0 < \alpha < 2$*,* $\gamma < 0$*. Then* $\varphi^u(0; \lambda_2)$ *is a continuous increasing function of* λ_2 *for* $\hat{\lambda} \leq \lambda_2 \leq 0$*.*

Proof Let μ_1, μ_2 be such that $\hat{\lambda} \leq \mu_1 < \mu_2 < 0$, we shall prove that $\varphi^u(0, \mu_1) < \varphi^u(0, \mu_2)$.

For fixed α and γ, $\varphi_\pm$ depend on λ_2, so we denote them as $\varphi_\pm(\lambda_2)$. It is simple to check that $-\varphi_-(\mu_1) < -\varphi_-(\mu_2)$. We denote (33) as

$$\varphi' = f(\varphi, u; \lambda_2), \qquad u' = g(u)$$

One has $f(\varphi, u; \mu_1) < f(\varphi, u; \mu_2)$.

Let us consider the unstable invariant manifold of the point $(-\varphi^-, -1)$ for $\lambda_2 = \mu_1$, $W^u_{\mu_1}$, and let Γ be the arc defined by $W^u_{\mu_1}$ for $-1 \leq u \leq 0$. Recall $-\varphi_- < 0$ in the present case. We define $\mathcal{R}_2$ the region in the (φ, u) plane bounded by Γ, $\Gamma_1 = \{(0, u) | -1 \leq u \leq 0\}$, $\Gamma_2 = \{(\varphi, -1) | -\varphi^-(\mu_1) \leq \varphi \leq 0\}$ and $\Gamma_3 = \{(\varphi, 0) | \varphi^u(0, \mu_1) \leq \varphi \leq 0\}$. If we take $\lambda_2 = \mu_2$, the corresponding vector field (f, g) given by (33) on Γ and Γ_1 points inside to the region $\mathcal{R}_2$. Now $W^u_{\mu_2}$ is contained in $\mathcal{R}_2$ for $\tau << 0$ and it leaves $\mathcal{R}_2$ through Γ_3. Therefore $W^u_{\mu_2}$ intersects $u = 0$ at some point $\varphi^u(0, \mu_2) > \varphi^u(0, \mu_1)$. This ends the proof. $\square$

Proof of lemma 3 Let us consider $0 < \alpha < 2$ and $\lambda_2 > 0$. On $\varphi = 0$, we have $\varphi' = \lambda_2 > 0$ and if we restrict to $\varphi = \pi/2$, $\varphi' = -1$. Therefore the ω-limit set of the orbits through $(0, 0)$ and $(\pi/2, 0)$ is the attractor $(\varphi_+, 1)$. Then (30) follows using the remark 5.

To prove (b) we consider first $\alpha = 2/3$. We know after lemma 5 that for $\lambda_2 = \hat{\lambda}(2/3; \gamma)$ there exists an heteroclinic orbit given by $u = \rho^- / \tan\varphi$, where $\rho^- = Q_p/4 = \sqrt{-2\gamma}/3$. In this case we have $\varphi^u(0; \hat{\lambda}(2/3; \gamma)) = -\pi/2$. Using lemma 6, and $\varphi^u(0; 0) = 0$, we conclude that for $\alpha = 2/3$

$$-\pi/2 < \varphi^u(0; \lambda_2) < 0 \qquad \text{for} \qquad \hat{\lambda} < \lambda_2 < 0 \tag{35}$$

and there are not heteroclinic orbits. Moreover this implies that for these values of λ_2 the ω-limit set of $W^u((-\varphi_-, -1))$ is the attractor $(\varphi_+ - \pi, 1)$, and it is easy to check that the ω-limit set of the orbits through $(0, 0)$ and $(\pi/2, 0)$ are the attractors $(\varphi_+ - \pi, 1)$ and $(\varphi_+, 1)$ respectively. Using the remark 5, part (b) for $\alpha = 2/3$ follows.

Now we assume $2/3 < \alpha < 2$. We shall prove that

$$-\pi/2 < \varphi^u(0; \hat{\lambda}(\alpha; \gamma)) < 0. \tag{36}$$

Then, using the same argument as before, (35) holds and (b) will be proved. To prove (36) we take $\lambda_2 = \hat{\lambda}(\alpha; \gamma)$. We recall that in this case, the equilibrium point $(-\varphi_-, -1)$ is of saddle node type. It is easy to check that the fast unstable direction is given by the vector $\mathbf{V}_1 = (-2(2 - \alpha)Q_p, \alpha(16 + Q_p^2))^T$. Let us introduce

$$\chi = \{(\varphi, u) \,|\, u = f(\varphi),\ -\pi/2 < \varphi \leq -\varphi_-\}, \quad f(\varphi) = \frac{\rho^-}{\tan\varphi}, \quad \rho^- = \frac{Q_p}{4} = \frac{(2-\alpha)\sqrt{-2\gamma}}{4} \tag{37}$$

We remark that if $2/3 < \alpha < 2$, χ is not an orbit for the flow defined by (33) (see figure 3). However, it is clear that $f(-\varphi_-) = -1$ and $\lim_{\varphi \to -\pi/2} f(\varphi) = 0$.

Let $\mathcal{R}$ be the region in the (φ, u) plane bounded by χ, $\chi_1 = \{(\varphi, u) \,|\, -\pi/2 \leq \varphi \leq 0, u = 0\}$, $\{(\varphi, u) \,|\, -\varphi_- \leq \varphi \leq 0, u = -1\}$ and $\chi_2 = \{(\varphi, u) \,|\, \varphi = 0, -1 \leq u \leq 0\}$. We shall prove that in a neighbourhood of the equilibrium $(-\varphi_-, -1)$, $W^u((-\varphi_-, -1))$ is contained in $\mathcal{R}$, and the only way to leave $\mathcal{R}$ is through the $u = 0$ axis. This means that (36) holds.

Let be $(\varphi, u) \in \chi$. The tangent vector to χ is $\mathbf{V}_2 = (1, -Q_p/(4\sin^2\varphi))^T$. Moreover, on χ the vector field defined by (33) can be written as

$$\mathbf{V}_3 = F\left(-\sin^2\varphi, \frac{\alpha\sqrt{-2\gamma}}{2}\right)^T \qquad \text{where} \qquad F = 1 - \frac{Q_p^2}{16\tan^2\varphi}.$$

Notice that for $-\pi/2 < \varphi < -\varphi_-$, one has $F > 0$. A simple computation shows that

$$\mathbf{V}_2 \wedge \mathbf{V}_3 = \frac{\sqrt{-2\gamma}}{4} F(3\alpha - 2) > 0 \qquad \text{and} \qquad \lim_{\varphi \to -\pi/2} \frac{\sqrt{-2\gamma}}{4} F(3\alpha - 2) > 0 \qquad \text{if} \qquad \alpha > 2/3.$$

This implies that if $(\varphi, u) \in \chi$, the orbit goes inside $\mathcal{R}$ for positive time. Moreover, $\varphi' < 0$, along χ_2. Therefore if (φ, u) is a point in the interior of $\mathcal{R}$, the only way to leave the region $\mathcal{R}$ for positive time is through $u = 0$.

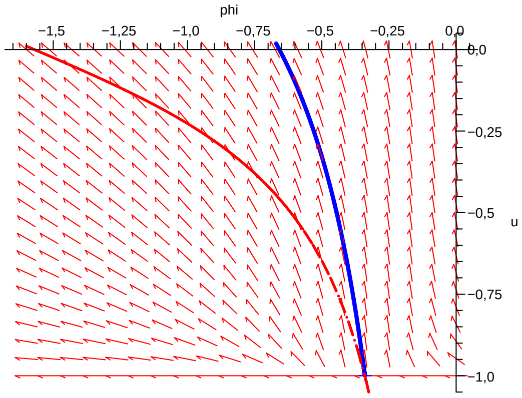


Figure 3: Plot of vector field (33) for $\alpha = 1$, $\gamma = -1$ and $\lambda_2 = \hat{\lambda}(\alpha;\gamma)) = -1/8$. The dashed line is the graph of the function $u = f(\varphi)$ and the continuous line shows $W^u((-\varphi_-, -1))$.

Furthermore, $\mathbf{V}_{2,-} = \big(1, -(16+Q_p^2)/(4Q_p)\big)^T$ is the tangent vector to χ at the equilibrium $(-\varphi_-, -1)$. Then $\mathbf{V}_{2,-} \wedge \mathbf{V}_1 = (16+Q_p^2)(3\alpha-2)/2 > 0$, if $\alpha > 2/3$. Therefore, in a neighbourhood of $(-\varphi_-, -1)$, $W^u((-\varphi_-, -1))$ is contained in $\mathcal{R}$. This ends the proof of (b).

From now on we shall consider $\gamma = \gamma(\alpha) = -k\alpha^s(1+O(\alpha))$ for some $k > 0$ and $-2 < s < 2$. First we take $\lambda_2 = \hat{\lambda}(\alpha;\gamma)$. We recall that $\hat{\lambda}(\alpha;\gamma) = -Q_p^2/16$. From (33) for $u^2 \le 1$ we get

$$\begin{aligned}\varphi' &= -\frac{1}{2}(1+c^2) - \frac{1}{2}(c^2-1)\cos 2\varphi + cu\sin 2\varphi \le -\frac{1}{2}(1+c^2) + \sqrt{\frac{1}{4}(c^2-1)^2 + c^2u^2}\,, \qquad (38)\\ u' &\le \frac{2\alpha c}{2-\alpha},\end{aligned}$$

where $c = Q_p/4$. We note that if $u^2 < 1$, then $\varphi' < 0$. Moreover on the region $\mathcal{B} = \{(\varphi, u) \mid -1/\sqrt{2} \le u \le 0\}$ we get $\varphi' \le -\frac{1}{2}(c^2+1-\sqrt{c^4+1})$. So, in the region $\mathcal{B}$, the vector field defined by (33) can be bounded by the following one

$$\begin{aligned}\hat{\varphi}' &= -\frac{1}{2}(c^2+1-\sqrt{c^4+1}), \qquad (39)\\ \hat{u}' &= \frac{2\alpha c}{2-\alpha}.\end{aligned}$$

Then

$$\frac{\hat{u}'}{\hat{\varphi}'} = -\frac{2\alpha}{(2-\alpha)}(c + c^{-1} + \sqrt{c^2+c^{-2}})\,. \qquad (40)$$

If $\gamma = -k\alpha^s(1+O(\alpha))$ we get $c = k_1\alpha^{s/2}(1+O(\alpha))$ for some constant $k_1 > 0$ and then the right hand side of (40) goes to 0 as α goes to 0. Once W^u enters $\mathcal{B}$ the vector field (33) is bounded by the constant one defined by (39). Then for any $M > 0$ large enough there exists α_M small enough such that for any $0 < \alpha \le \alpha_M$

$$\varphi^u(0;\alpha,\gamma,\hat{\lambda}) < -M, \quad \gamma = \gamma(\alpha), \quad \hat{\lambda} = \hat{\lambda}(\alpha,\gamma).$$

However, from (b) we have for $\alpha = 2/3$, $\varphi^u(0; 2/3, \gamma, \hat{\lambda}) = -\pi/2$. Using the continuity of $\varphi^u(0;\alpha,\gamma,\hat{\lambda})$ with respect to α, for $\alpha > 0$, we get the existence of a sequence $\{\alpha_k\}_{k\ge 0}$, with

$\alpha_1 = 2/3$ and $\lim_{k\to\infty} \alpha_k = 0$, such that

$$\varphi^u(0; \alpha_k, \gamma, \hat{\lambda}) = -k\pi/2, \quad k \geq 1,$$

where $\gamma = \gamma(\alpha_k)$, $\hat{\lambda} = \hat{\lambda}(\alpha_k, \gamma)$.

Let us fix now $\alpha = \alpha_k$. From (a) we know that for $\lambda_2 > 0$ there are not heteroclinic orbits. So, we only need to consider $\hat{\lambda}(\alpha_k, \gamma) \leq \lambda_2 \leq 0$. Let us consider the function $\varphi^u(0; \lambda_2)$. Then $\varphi^u(0; \hat{\lambda}) = -k\pi/2$ and $\varphi^u(0; 0) = 0$. After lemma 6 we know that $\varphi^u(0; \lambda_2)$ is an increasing continuous function of λ_2 for $\lambda_2 \in [\hat{\lambda}, 0]$. Therefore, there exists $\{\lambda_{2,j}^{(k)}\}_{j=0,1\ldots k}$ such that $\varphi^u(0; \lambda_{2,j}^{(k)}) = -j\pi/2$, that is, if $\lambda_2 = \lambda_{2,j}^{(k)}$, $j = 0, 1 \ldots k-1$, there exists an heteroclinic orbit and (28) is not satisfied. Moreover, if $\lambda_2 \in [\hat{\lambda}, 0]$, $\lambda_2 \neq \lambda_{2,j}^{(k)}$, $\varphi^u(0; \lambda_2) \neq -m\pi/2$ for any positive integer m. Following the remark 5, we have now that (28) holds. $\square$

We have performed some numerical computations using the homographic potential (4) for $\alpha \in (0, 2)$. For the system (33) we have computed the unstable invariant manifold $W^u((-\varphi_-, -1))$ up to its intersection with $u = 0$. We know by lemma 5 that for $\alpha = 2/3$ and $\lambda_2 = \hat{\lambda}(2/3) = -1/3$, there is an heteroclinic orbit between the equilibrium points $(-\varphi_-, -1)$ and $(\varphi_- - \pi, 1)$. Figure 4 shows $W^u((-\varphi_-, -1))$ in $-1 \leq u \leq 0$, for $(\alpha, \lambda_2) = (2/3, \hat{\lambda}(2/3)), (2/5, \hat{\lambda}(2/5))$. In both cases we have heteroclinic orbits. We note that in the case $\alpha = 2/5$ after lemma 3 there exists $\lambda_{2,1}$, $\hat{\lambda}(2/5) < \lambda_{2,1} < 0$ such that $\varphi^u(0; 2/5, \lambda_{2,1}) = -\pi/2$.

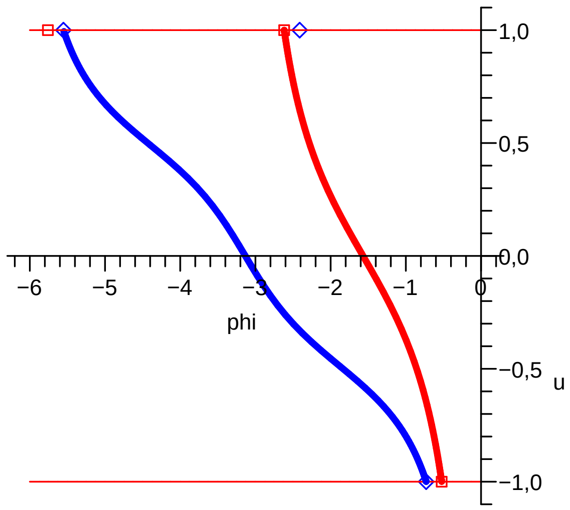


Figure 4: Plot of $W^u((-\varphi_-, -1))$ for the system (33) in the $(\varphi, u)-$plane, with parameters $(\alpha, \lambda_2) = (2/3, -1/3), (2/5, -4/5)$. The equilibrium points are denoted with a box for $\alpha = 2/3$ and a diamond for $\alpha = 2/5$.

The values α_k such that $\varphi^u(0; \alpha_k, \hat{\lambda}(\alpha_k)) = -k\pi/2$ for some positive integers k have been computed numerically. It turns out that all the computed values satisfy (31). So, these results support the conjecture 1. We note that after lemma 3, for $\alpha = \alpha_k$ there are exactly k values of the parameter λ_2 greater than $\hat{\lambda}(\alpha; \gamma)$ such that there exist an heteroclinic orbit of (33). However the numerical computations also show that for $\alpha_{k+1} < \alpha < \alpha_k$ there are exactly $k+1$ values of λ_2, $\hat{\lambda}(\alpha) < \lambda_2 \leq 0$ giving rise to an heteroclinic orbit.

3.2 Heteroclinic connections for the system on γ_+. The homographic potential

In this section we shall consider the homographic potential (4). In order to study the non degeneracy condition (29) one has to consider the system (25), (26), with $(q, Q) = (q_{L_2}(\tau), Q_{L_2}(\tau))$. As before we introduce $u = Q_{L_2}(\tau)/Q_p$. Then we get the following system for φ, u

$$
\begin{aligned}
\varphi' &= \left(\frac{2}{\alpha}\lambda_1(1-u^2)+\lambda_2\right)\cos^2\varphi - \frac{Q_p u}{2}\sin\varphi\cos\varphi - \sin^2\varphi, \qquad (41)\\
u' &= \frac{2-\alpha}{\sqrt{2\alpha}}(1-u^2).
\end{aligned}
$$

We remark that γ_+ depends strongly on the function $V(z)$ which defines the potential. To get the system (41) we have used the particular form of $V(z)$ in the homographic case.

The condition (29) involves the solutions of (41) such that $(\varphi(0), u(0)) = (0, 0)$, $(\pi/2, 0)$. Now, if $\tan\varphi(\tau) \to -\rho^+$ as $\tau \to \infty$ for one of these solutions, (29) is not satisfied. As before, we will see that in this case there exists an heteroclinic orbit of (41).

The system (41) can be analyzed in the same way as (33). However, we recall that (41) depends on the parameters α, λ_2 and λ_1. Moreover, if $\lambda_2 > \hat{\lambda}$ (41) has four equilibrium points in the region $\mathcal{R}$ (defined in (34))

$$
(\varphi, u) = (\varphi_+, -1), (\varphi_-, -1), (-\varphi_+, 1), (-\varphi_-, 1) \qquad (42)
$$

and $(\varphi_+, -1), (-\varphi_+, 1)$ are saddle points, $(-\varphi_-, 1)$ is an attractor and $(\varphi_-, -1)$ a repellor. For a saddle point, P, $W^u(P)$ stands for the branch of the unstable invariant manifold of P, contained in $-1 \leq u \leq 1$. We also denote by $\varphi^u(0; \alpha, \lambda_2, \lambda_1)$, or simply $\varphi^u(0; \lambda_1)$ if α and λ_2 are fixed, the value of φ at the intersection point of $W^u((\varphi_+, -1))$ with $u = 0$. The symmetry given in the property 5. also holds for (41). Therefore, $\varphi^u(0; \alpha, \lambda_2, \lambda_1) = -k\pi/2$ for some positive integer k, if and only if an heteroclinic orbit exists between $(\varphi_+, -1)$ and $(-\varphi_+ - k\pi, 1)$. We remark that the existence of such an heteroclinic orbit implies that (29) is not satisfied.

Remark 6. *To check (29) we are interested in the orbits of the points* $(0, 0)$ *and* $(\pi/2, 0)$. *If the* ω*-limit set for that orbits is an attractor* $(-\varphi_- - k\pi, 1)$ *for some integer* k, *then*

$$
\lim_{\tau\to\infty} < \mathbf{y}^-, \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|} >= (-1)^k(\rho^+ - \rho^-)\cos\varphi_- \neq 0, \quad \text{if} \quad \lambda_2 \neq \hat{\lambda}
$$

and (29) is satisfied. However, if the ω*-limit set of one of these orbits is a saddle point* $(-\varphi_+ - k\pi, 1)$, *then the limit above is zero.*

Proof of lemma 4 Let us denote the equations (41) as

$$
\varphi' = f(\varphi, u; \lambda_1), \qquad u' = g(u).
$$

We assume that α and λ_2 are fixed. So, the equilibrium points are fixed. We claim

1. $f(\pi/2, u; \lambda_1) = -1 < 0$ for any u, λ_1
2. $\varphi^u(0; \lambda_1)$ is a continuous increasing function of λ_1
3. Let $M > 0$ be large enough. Then there exists $\lambda_1 = \lambda_1(M) < 0$ such that $\varphi^u(0; \lambda_1) < -M$ and $\lambda_1(M)$ tends to $-\infty$ as M goes to ∞.

If $\lambda_1 = \lambda_1^{(0)}$, it is easy to check that $W^u((\varphi_+,-1))$ has the simple analytical expression $u = -(1/\rho^+)\tan\varphi$. Then $\varphi^u(0;\lambda_1^{(0)}) = 0$. Using the symmetry, there is an heteroclinic orbit between the points $(\varphi_+,-1)$ and $(-\varphi_+,1)$. Therefore (29) is not satisfied.

Let us consider now $\lambda_1 > \lambda_1^{(0)}$. Using claims 1. and 2. we have

$$0 = \varphi^u(0;\lambda_1^{(0)}) < \varphi^u(0;\lambda_1) < \pi/2$$

Therefore $W^u((\varphi_+,-1))$ goes to the attractor $(-\varphi_-,1)$ as $\tau\to\infty$ and (a) is proved.

From claims 2. and 3. we get a decreasing sequence of λ_1 values $\{\lambda_1^{(k)}\}_{k\geq 0}$, such that $\lim_{k\to\infty}\lambda_1^{(k)} = -\infty$ and $\varphi^u(0;\lambda_1^{(k)}) = -k\pi/2$ and then there exists an heteroclinic connection between points $(\varphi_+,-1)$ and $(-\varphi_+-k\pi,1)$. This implies that (29) does not hold and part (b) is proved.

Now we prove claim 2. We note that $\varphi_\pm$ depend on α and λ_2. So, if we fix these parameters the equilibrium points are fixed. We shall prove that if $\mu_1 < \mu_2$ then $\varphi^u(0;\mu_1) < \varphi^u(0;\mu_2)$. Let $W^u_{\mu_1}$ be the unstable invariant manifold of $(\varphi_+,-1)$ for $\lambda_1 = \mu_1$ and Γ the arc defined by $W^u_{\mu_1}$ for $-1\leq u\leq 0$. Let $\mathcal{R}_3$ be the region bounded by Γ, $\varphi = \pi/2$ and the lines $u=-1$ and $u=0$. It is clear that $f(\varphi,u;\mu_1) < f(\varphi,u;\mu_2)$. Therefore the vector field (f,g) for $\lambda_1=\mu_2$ on Γ, points inside $\mathcal{R}_3$. The same is true on $\varphi=\pi/2$. Moreover using the linear approximation of $W^u_{\mu_2}$ we can see easily that in a small neighbourhood of $(\varphi_+,-1)$, $W^u_{\mu_2}$ is contained in $\mathcal{R}_3$. Therefore $W^u_{\mu_2}$ leaves $\mathcal{R}_3$ through a point on $u=0$ with φ greater than $\varphi^u(0;\mu_1)$.

Now we prove claim 3. Let $M>0$ be large enough and $K>0$ such that $-(2K+1)\pi/2 < -M < -(2K-1)\pi/2$. We shall assume $\lambda_1<0$. Let us introduce $L = -\lambda_1/\alpha - \lambda_2$ and the region $\mathcal{B} = \{(\varphi,u)\,|\, -1/\sqrt{2}\leq u\leq 0\}$. If $(\varphi,u)\in\mathcal{B}$ from (41) we get

$$\varphi' \leq -L\cos^2\varphi - \frac{Q_p u}{2}\sin\varphi\cos\varphi - \sin^2\varphi \leq -\frac{L+1}{2} + \sqrt{\frac{(L-1)^2}{4} + \frac{Q_p^2u^2}{16}}.$$

Then, if we take $L\geq \max\{3/4, Q_p^2/16\}$ we get

$$\varphi' \leq -\frac{1}{2}(L+1-\sqrt{L^2+1}) \leq -\frac{1}{4}.$$

However, if $(\varphi,u)\in\mathcal{B}$, $\varphi\in[-\pi/2,\pi/2]$, using that

$$\varphi' \leq -\frac{L+1}{2} - \frac{L-1}{2}\cos 2\varphi + \frac{Q_p}{4}$$

it is easy to get $\varphi'\leq -K^2$ by removing adequate small neighbourhoods of $\varphi=\pm\pi/2$. Moreover, from (41), $u'\leq (2-\alpha)/\sqrt{2\alpha}$.

Summarizing, if $(\varphi,u)\in\mathcal{B}$ and $\varphi\in[-\pi/2,\pi/2]$, the vector field defined by (41) is bounded by

$$\hat\varphi' = -\frac{1}{4},\qquad \hat u' = \frac{2-\alpha}{\sqrt{2\alpha}},\qquad \text{if}\quad \varphi\in[-\pi/2,-\pi/2+\delta_-]\cup[\pi/2-\delta_-,\pi/2] \tag{43}$$

and by

$$\hat\varphi' = -K^2,\qquad \hat u' = \frac{2-\alpha}{\sqrt{2\alpha}},\qquad \text{if}\quad \varphi\in[-\pi/2+\delta_-,\pi/2-\delta_-]\,, \tag{44}$$

where $\delta_- = \frac{1}{2}\arccos(1-\delta_1)$, $\delta_1 = \frac{2}{L-1}\left(K^2-1+\frac{Q_p}{4}\right)$. We remark that if we take $L>0$ large enough (that is, $\lambda_1<0$ and $|\lambda_1|$ large enough) we can take $\delta_-\leq 1/K^2$, once K is fixed.

Now, if we take initial conditions $\hat{\varphi}(\tau_0) = \pi/2$, $\hat{u}(\tau_0) = -1/\sqrt{2}$, the variation of $\hat{u}$ as $\hat{\varphi}$ goes from $\pi/2$ to $-\pi/2$ under the flow defined by (43) and (44), is

$$\Delta_\pi \hat{u} = \frac{(2-\alpha)}{\sqrt{2\alpha}} \left(8\delta_- + \frac{\pi - 2\delta_-}{K^2} \right) \leq \frac{(2-\alpha)}{\sqrt{2\alpha}} \left(8\delta_- + \frac{\pi}{K^2} \right) = \frac{(2-\alpha)}{\sqrt{2\alpha}} \frac{(8+\pi)}{K^2}.$$

When this orbit intersects the line $\varphi = -(2K+1)\pi/2$ the variation of $\hat{u}$ is bounded by

$$\Delta \hat{u} \leq (K+1)\, \Delta_\pi \hat{u} \leq \frac{2(2-\alpha)}{\sqrt{2\alpha}} \frac{(8+\pi)}{K}.$$

By taking K large enough we get $\Delta \hat{u} \leq 1/\sqrt{2}$. Let us consider $W^u((\varphi_+, -1))$. Once it enters the region $\mathcal{B}$, it is bounded by the orbits of (43), (44). Then it crosses the line $\varphi = -M$ at some point with $u < 0$. This proves claim 3. □

Using the homographic potential (4), we have computed numerically some of the values of the parameters λ_1, λ_2 giving rise to heteroclinic orbits of (41). In the Table 1 we give the first values of λ_1 for $\alpha = 1$ and three different values of λ_2. Figure 5 displays the curves $\lambda_2 = \lambda_1^{(j)}(\lambda_1)$, $0 \leq j \leq 5$. We recall that for $j = 0$ the curve is given analytically in lemma 4 (see also remark 3).

	$\lambda_2 = -0.1$	$\lambda_2 = -0.01$	$\lambda_2 = 1$
$\lambda_1^{(0)}$	$-0.130901699\ldots$	$-0.239895788\ldots$	$-1.000000000\ldots$
$\lambda_1^{(1)}$	$-0.742705098\ldots$	$-0.979687364\ldots$	$-2.250000000\ldots$
$\lambda_1^{(2)}$	$-1.854508497\ldots$	$-2.219478940\ldots$	$-4.000000000\ldots$
$\lambda_1^{(3)}$	$-3.466311896\ldots$	$-3.959270517\ldots$	$-6.250000000\ldots$
$\lambda_1^{(4)}$	$-5.578115294\ldots$	$-6.199062098\ldots$	$-9.000000000\ldots$
$\lambda_1^{(5)}$	$-8.189918693\ldots$	$-8.938853668\ldots$	$-12.25000000\ldots$
$\lambda_1^{(6)}$	$-11.30172209\ldots$	$-12.17864524\ldots$	$-16.00000000\ldots$
$\lambda_1^{(7)}$	$-14.91352549\ldots$	$-15.91843682\ldots$	$-20.25000000\ldots$
$\lambda_1^{(8)}$	$-19.02532889\ldots$	$-20.15822839\ldots$	$-25.00000000\ldots$
$\lambda_1^{(9)}$	$-23.63713228\ldots$	$-24.89801997\ldots$	$-30.25000000\ldots$
$\lambda_1^{(10)}$	$-28.74893568\ldots$	$-30.13781155\ldots$	$-36.00000000\ldots$

Table 1: First critical values of λ_1 giving rise to heteroclinic connections for $\alpha = 1$ and three different values of λ_2.

4 Numerical stability/instability regions

We have computed the stability parameter tr, by integrating numerically the differential equation (1). A systematic use has been made of higher order Taylor methods, see [4] and references therein for description and a public available package and [10] for a didactic presentation and examples. The homographic potential (4) has been used in the computations. We recall that for $\alpha = 1$ (Newtonian potential), the solution of (2) on an energy level $E = -\delta$ is obtained explicitly in terms of the eccentricity e, where $\delta = (1-e^2)/2$. For $\alpha \in (0, 2)$, $\alpha \neq 1$, the equation (2) can not be explicitly integrated. However one can define a generalization of the eccentricity (see [7]) through $\delta = (2-\alpha)(1-e^2)/(2\alpha)$. To show the numerical results in this section we shall use indistinctly parameters δ and e.

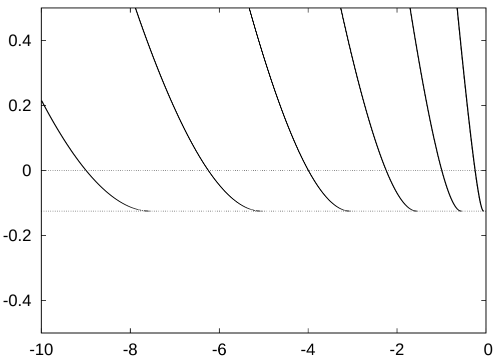


Figure 5: Plot of the curves $\lambda_2 = \lambda_1^{(j)}(\lambda_1)$ for $\alpha = 1$, in the (λ_1, λ_2)-plane. From right to left $j = 0, 1, 2, 3, 4, 5$. The dashed horizontal line is located on $\lambda_2 = -1/8$, the level at which all the $\lambda_1^{(j)}$ end.

First, we show the results for $\alpha = 1$. In figure 6 we show for $\lambda_2 = -0.1$, the stability parameter tr as a function of $-\log_{10}(1-e)$ for $\lambda_1 = \lambda_1^{(0)}$, $\lambda_1^{(1)}$ and nearby values. We recall that for $\lambda_1^{(0)}$, $\lambda_1^{(1)}$, (29) is not satisfied. In figure 6 we observe a constant behaviour of tr as e goes to 1. Moreover, the limit behaviour of tr changes from $-\infty$ to ∞ as λ_1 goes through $\lambda_1^{(0)}$ in a decreasing way. A similar change in the limit behaviour is observed as λ_1 goes through $\lambda_1^{(1)}$.

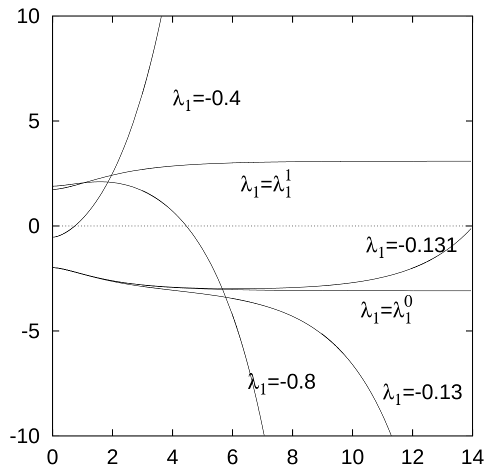


Figure 6: Plot of tr as a function of e (in the horizontal axes we plot $-\log_{10}(1-e)$) for $\lambda_2 = -0.1$ and different values of λ_1

For a fixed value of e, the curves in the $(\lambda_1, \lambda_2)-$plane such that tr$= \pm 2$ define the stability and instability regions in the plane of parameters. We have computed numerically that curves. We recall that for $\alpha = 1$, $\hat{\lambda} = -1/8$. We know from lemma 3 that the condition (28) is satisfied for any λ_1 and $\lambda_2 > \hat{\lambda}$. However, after lemma 4 there are some curves in the (λ_1, λ_2)-plane defined by $\lambda_2 = \lambda_1^{(j)}$, $j = 0, 1, \ldots$ where (29) fails (see figure 5). In figure 7 (a), we plot the curves with tr$= \pm 2$ for $e = 1 - 2^{-9}$ in a neighbourhood of the origin. If $\lambda_1 > 0$, $\lambda_2 > 0$, all points give rise to instability. For $\lambda_1 < 0$ and $\lambda_2 < 0$, there is mainly stability but some instability pockets appear. Instability pockets in Hill's equations are very common, both in the periodic Hill's case (see [1] and references therein) and in the quasi-periodic one (see [3] and references therein). We remark that the stability channels become here narrower as the parameters λ_1, λ_2 move to the second and third quadrant. So, if $\lambda_1\lambda_2 < 0$, there is instability except inside extremely thin tongues where the system is stable. Figure 7 (b) shows a larger neighbourhood of the origin. We observe that the stability channels become very thin close to two "critical lines" (see also the figures 7 (c) and (d)). This fact appears in all families of Hill's equations of the form $\ddot{x} + (a + bp(t))x = 0$ in the (a, b)-plane, see [2]. Figures 8 (a) to (f) show the evolution of the stability and instability regions as e goes to 1. We see that the stability tongues for $\lambda_2 > -1/8$ go quickly to the limit curves $\lambda_2 = \lambda_1^{(j)}$, $j = 0, 1, ...$ For $\lambda_1 < 0$, we observe that the instability pockets in $\lambda_2 < \hat{\lambda}$ change and accumulate to the line $\lambda_2 = -1/8$ as e goes to 1. However, if $\hat{\lambda} < \lambda_2 < 0$ the instability pockets tend to some limit regions bounded by $\lambda_2 = 0$, $\lambda_2 = -1/8$ and $\lambda_2 = \lambda_1^{(j)}$. For (λ_1, λ_2) in these limit regions, the system is unstable.

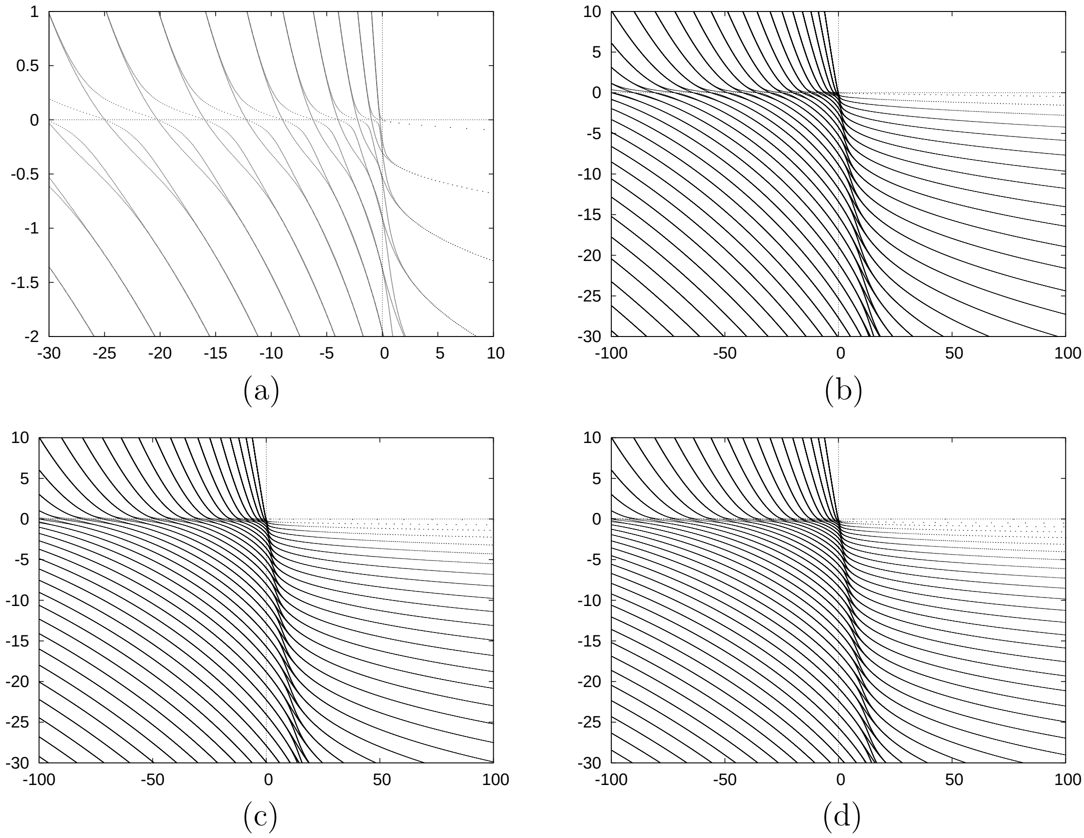


Figure 7: Stability and instability regions in the $(\lambda_1, \lambda_2)-$plane for $\alpha = 1$. (a) $e = 1-2^{-9}$, (b) $1 - 2^{-9}$ but in a larger domain, (c) $1 - 2^{-12}$, (d) $1 - 2^{-14}$. Note that the changes when $e \to 1$ are hard to distinguish at present scale.

Figure 9 (a), (b), (c), (d), shows magnifications of figure 8 for $e = 1 - 2^{-10}, 1 - 2^{-20}, 1 - 2^{-30}, 1 - 2^{-40}$, respectively, in a neighbourhood of $\lambda_2 = -1/8$. Moreover, in these plots we display also the curves $\lambda_2 = \lambda_1^{(j)}(\lambda_1)$ for $j = 0, 1, 2, 3$ (see figure 5). In figures 9 (a) and (b), one can distinguish these curves as the ones having an end point on $\lambda_2 = -1/8$. We can see that the stability tongues in the region $\lambda_1 < 0$, $\lambda_2 > 0$ become thinner and quickly tend to the curves $\lambda_2 = \lambda_1^{(j)}(\lambda_1)$ as e goes to 1.

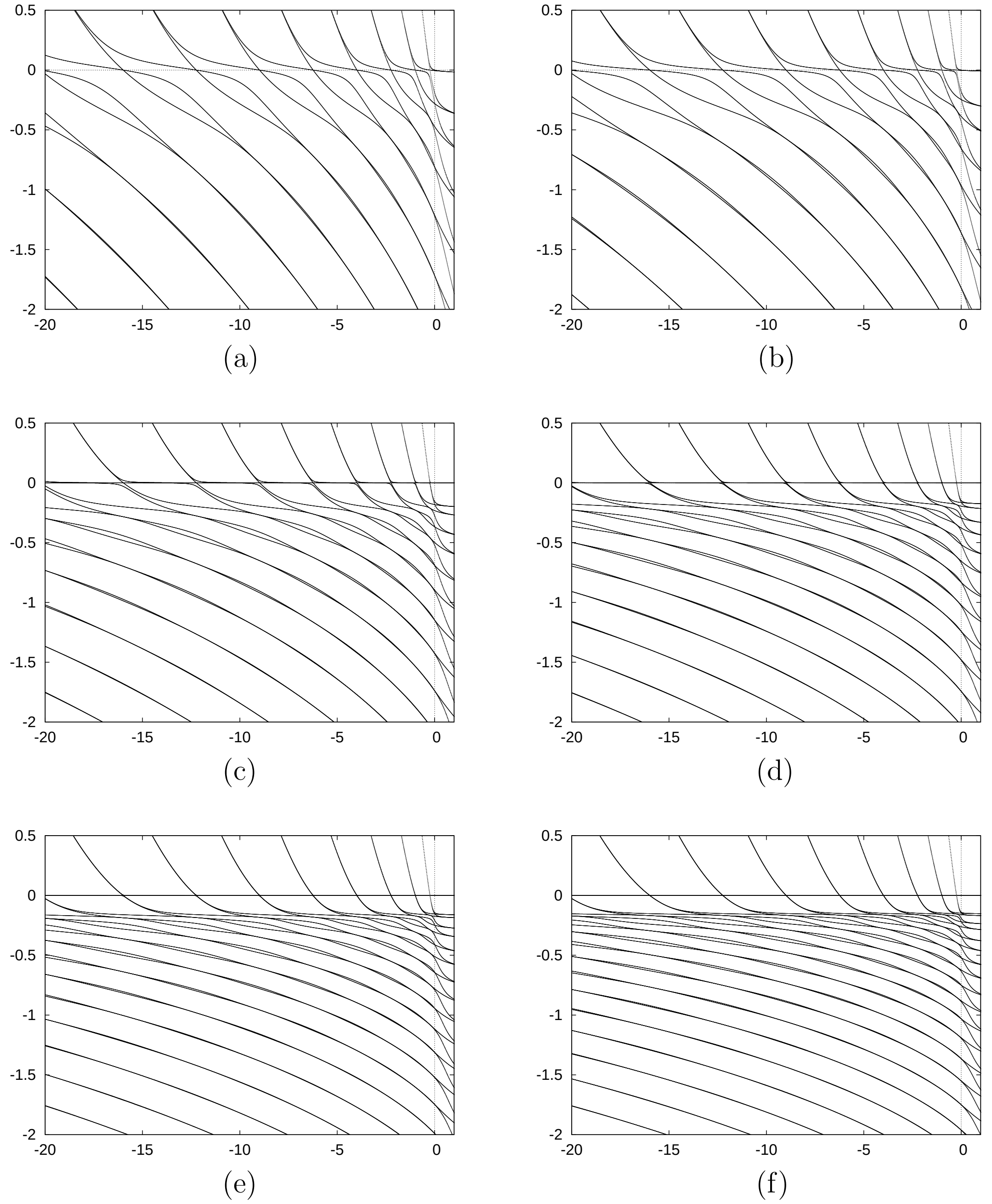


Figure 8: Stability and instability regions in a small domain of the $(\lambda_1, \lambda_2)-$plane for $\alpha = 1$. (a) $e = 1 - 2^{-10}$, (b) $1 - 2^{-12}$, (c) $1 - 2^{-20}$, (d) $1 - 2^{-25}$, (e) $1 - 2^{-30}$, (f) $1 - 2^{-35}$.

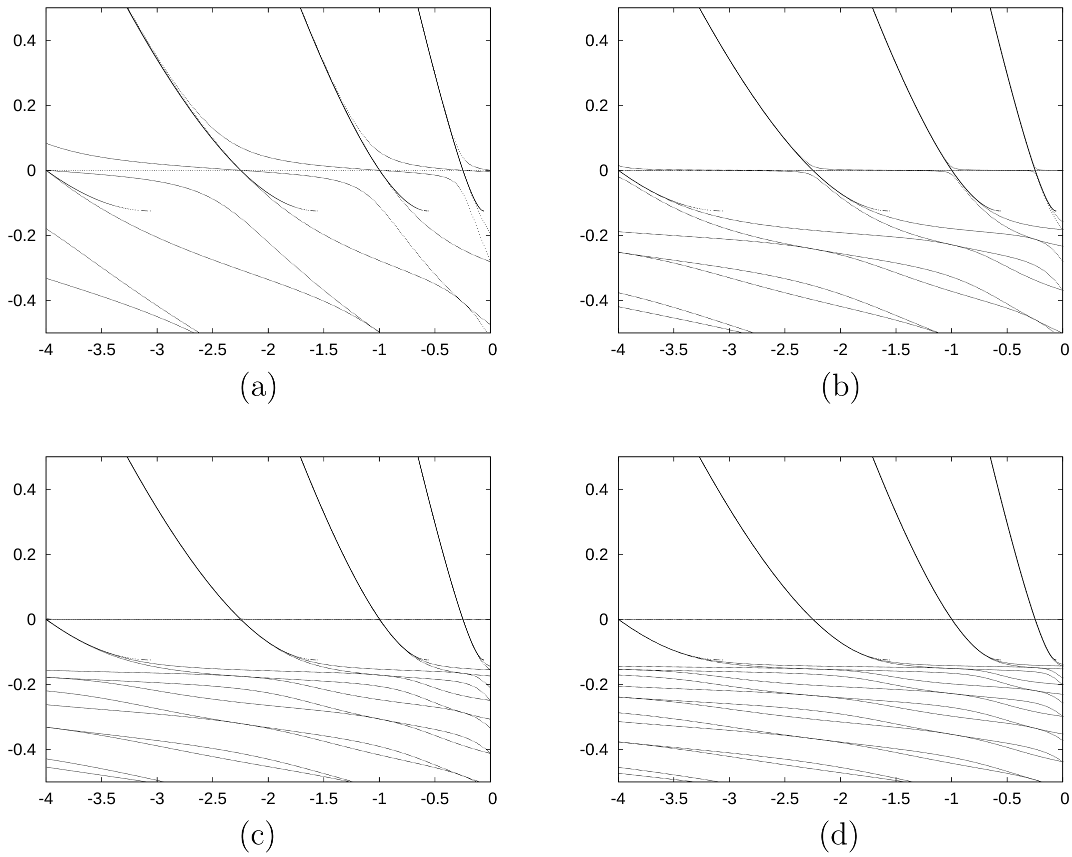


Figure 9: Stability and instability regions in the $(\lambda_1, \lambda_2)-$plane for $\alpha = 1$. (a) $e = 1 - 2^{-10}$, (b) $1 - 2^{-20}$, (c) $1 - 2^{-30}$, (d) $1 - 2^{-40}$.

Figures 10 and 11 show the stability/instability regions for some values of $\alpha < 1$. Like in the case $\alpha = 1$, we can distinguish the limit curves $\lambda_2 = \lambda_1^{(j)}$, $j = 0, 1, ...$ However after lemma 3, we know that for some values of $\alpha < 1$, there are λ_1, λ_2 values such that (28) is not satisfied. So, additional limit curves are expected in these cases. To explain the results, from now on, for α_k in lemma 3 we shall use the values given by the conjecture 1, that is we shall assume that the conjecture is true. Let us take $\alpha = 0.5$. Then $\hat{\lambda} = -0.5625...$, and $2/5 = \alpha_2 < \alpha < \alpha_1 = 2/3$. Therefore, there are exactly two values of λ_2, with $\hat{\lambda} < \lambda_{2,1} < \lambda_{2,0} = 0$ such that for any λ_1, (28) is not satisfied. In figure 10 we see the evolution of the regions for $\alpha = 0.5$ and different values of e tending to 1. If $\lambda_2 < \hat{\lambda}$, the instability pockets accumulate from below to the line $\lambda_2 = \hat{\lambda}$. However, for $\hat{\lambda} < \lambda_2 < 0$, they tend to some limit regions along two horizontal strips which are limited by the lines $\lambda_2 = 0$, $\lambda_2 = \lambda_{2,1}$ and $\lambda_2 = \hat{\lambda}$ (see figures 10 (c) and (d)). In the figure 11 (a) and (b) we take $\alpha = \alpha_2 = 2/5$. In this case, there exist $\hat{\lambda} = \lambda_{2,2} < \lambda_{2,1} < \lambda_{2,0} = 0$, such that for any λ_1, (28) does not hold. Therefore, we get two strips of limit instability regions for $\hat{\lambda} < \lambda_2 < 0$. For $\alpha = 0.2$ we have $\hat{\lambda} = -2.025...$ and $\alpha_5 < \alpha < \alpha_4$. In this case there exist $\hat{\lambda} < \lambda_{2,4} < \lambda_{2,3} < \lambda_{2,2} < \lambda_{2,1} < \lambda_{2,0} = 0$. So, there are five strips of limit regions for $\hat{\lambda} < \lambda_2 < 0$ (see figures 11 (c) and (d)).

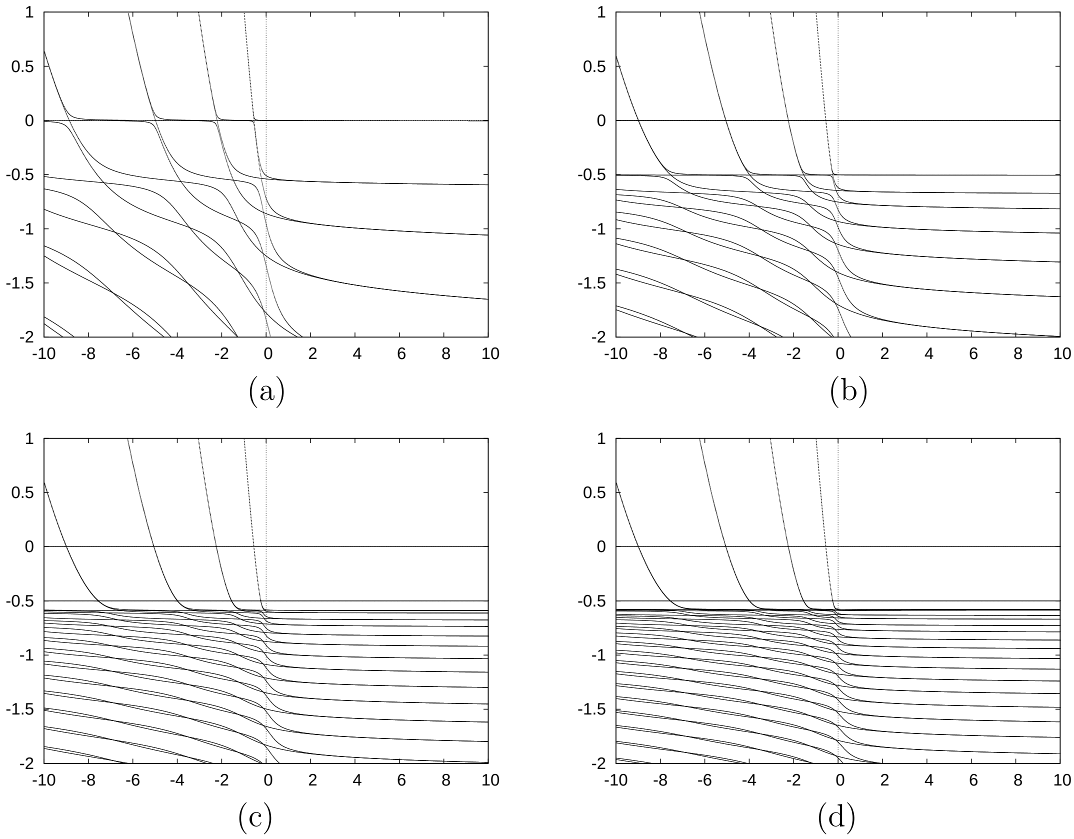


Figure 10: Stability and instability regions in the $(\lambda_1, \lambda_2)-$plane for $\alpha = 0.5$. (a) $e = 1 - 10^{-2}$, (b) $1 - 10^{-4}$, (c) $1 - 10^{-8}$, (d) $1 - 10^{-10}$.

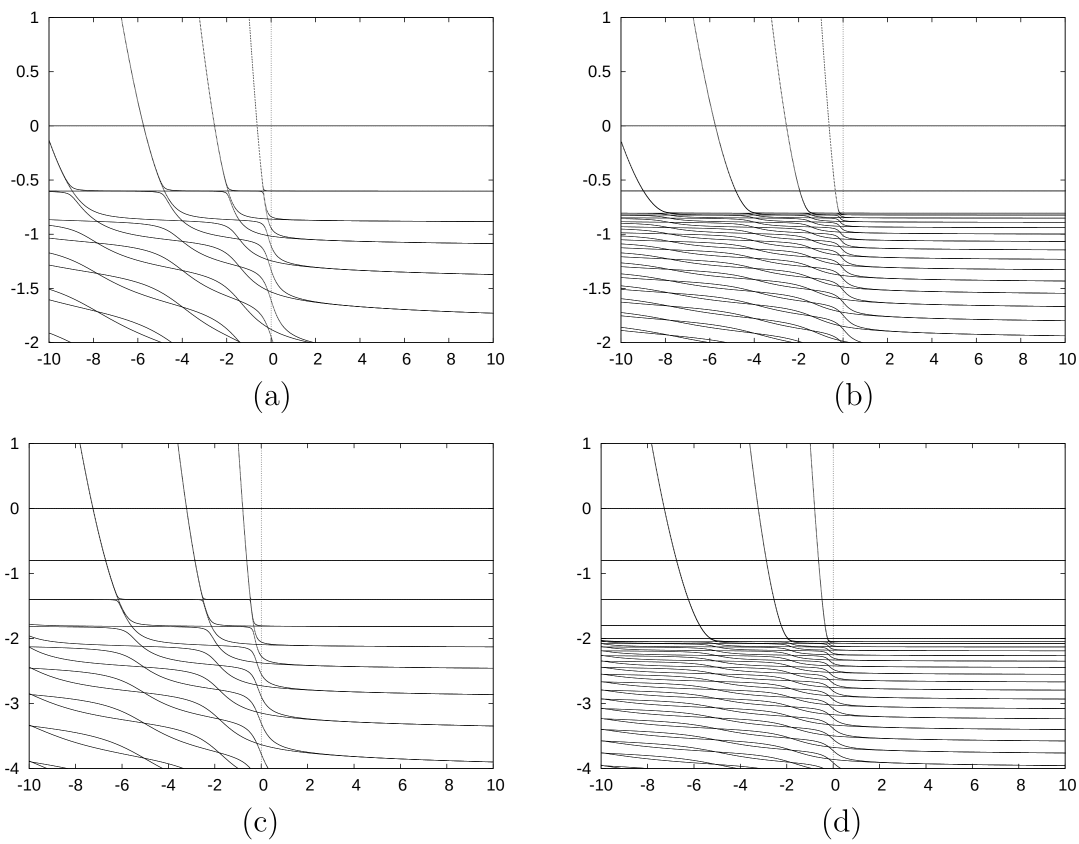


Figure 11: Stability and instability regions in the $(\lambda_1, \lambda_2)-$plane (a) $(\alpha, e) = (0.4, 1 - 10^{-3})$, (b) $(\alpha, e) = (0.4, 1 - 10^{-9})$, (c) $(\alpha, e) = (0.2, 1 - 10^{-2})$, (d) $(\alpha, e) = (0.2, 1 - 10^{-6})$.

Finally, in order to check the goodness of the asymptotic formulae given in the main theorem we perform the following computations. For a fixed value of α and taking $\gamma = -1/\alpha$, we select a point (λ_1, λ_2) in the parameter plane. Then we use the least squares method to fit the numerically computed tr by a function

$$f(\delta) = \begin{cases} z_a + z_b \log \delta & \text{if} \quad \lambda_2 > \hat{\lambda}(\alpha) \\ z_2 + z_3 \cos(z_1 \log \delta + z_4) & \text{if} \quad \lambda_2 < \hat{\lambda}(\alpha) \end{cases} .$$

In this way we can compare the values z_b and z_1 with the theoretical ones predicted by the asymptotic formulae, that is,

$$z_b^t = -\frac{(2-\alpha)}{2\alpha}\beta = -\frac{(2-\alpha)}{2\alpha}\sqrt{1 - \frac{\lambda_2}{\hat{\lambda}(\alpha)}}, \qquad z_1^t = -\frac{(2-\alpha)}{2\alpha}\hat{\beta}.$$

Note that z_b^t and z_1^t do not depend on λ_1. The computations have been done for $\alpha = 1$ and $\alpha = 0.5$. In any case, the points (λ_1, λ_2) have been selected at different regions (see Tables 2 and 3). So, for $\alpha = 1$, points A_1, A_2, A_3 lie on the limit regions defined by $\lambda_1^{(0)}, \lambda_2^{(0)}$ and $\lambda_3^{(0)}$ with $\lambda_2 > 0$ (see Table 1). Other points are plotted in the Figure (12). The signs $+, -$ on this figure mean that tr goes to $+\infty$ or $-\infty$ as e goes to 1, at the limit region. At C_1, C_2 we have oscillatory behaviour of tr. In the case $\alpha = 1/2$, points (λ_1, λ_2) have been selected in a similar way according to the limit regions (see Figure 10).

The Table 2 shows the values of z_a and z_b for parameters λ_1, λ_2 corresponding to hyperbolic behaviour of tr. One can see a very good agreement between z_b and z_b^t. In the figure 13 (a) we plot $\log|\mathrm{tr}|$ as a function of $\log \delta$, using the numerical computation of tr, for some values of the parameters α, λ_1 and λ_2. The difference $\log|\mathrm{tr}| - (z_a + z_b \log \delta)$ is displayed in the figure 13 (b). Similar plots are obtained for different values of the parameters. Table 3 shows the values of z_i, $i = 1, \ldots 4$, in cases of oscillatory behaviour as well as z_1^t. Figures 14 (a) and (b) show the typical behaviour of tr and $\mathrm{tr} - (z_2 + z_3 \cos(z_1 \log \delta + z_4))$ respectively, as functions of $\log \delta$, in the elliptic case.

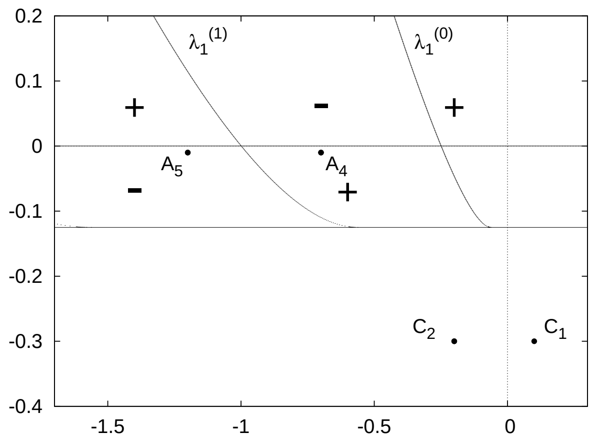


Figure 12: Some of the points used in the check of the goodness of the formulae.

point	α	λ_1	λ_2	z_a	z_b	z_b^t
A_1	1	-0.5	1	4.3439101933...	-1.5000000000...	$-3/2$
A_2	1	-2.0	1	1.0757003494...	-1.5000000000...	$-3/2$
A_3	1	-3.0	1	1.0815271087...	-1.5000000000...	$-3/2$
A_4	1	-0.7	-0.01	-2.7537028676...	-0.4795831523...	$-\sqrt{23}/10$
A_5	1	-1.2	-0.01	-3.2980826090...	-0.4795831523...	$-\sqrt{23}/10$
B_1	0.5	1.0	1	6.66605868...	-2.50000000...	$-5/2$
B_2	0.5	1.0	-0.3	3.17124093...	-1.02469507...	$-\sqrt{4.2}/2$
B_3	0.5	1.0	-0.55	4.99803332...	-0.22360679...	$-\sqrt{0.2}/2$
B_4	0.5	-1.8	-0.1	-1.90615984...	-1.36014705...	$-\sqrt{7.4}/2$
B_5	0.5	-2.0	0.5	1.10195805...	-2.06155281...	$-\sqrt{17}/2$
B_6	0.5	-4.0	-0.1	-1.80582582...	-1.36014705...	$-\sqrt{7.4}/2$
B_7	0.5	-4.0	0.5	0.82486302...	-2.06155281...	$-\sqrt{17}/2$

Table 2. Least squares fit for some values of the parameters giving rise to hyperbolic behaviour.

point	α	λ_1	λ_2	z_1, z_2	z_3, z_4	z_1^t
C_1	1	0.1	-0.3	-0.591607978300025	-3.25904693119756	$-\sqrt{1.4}/2$
				-0.758707554534354	-0.99960080326998	
C_2	1	-0.2	-0.3	-0.591607978302245	-2.19337090928979	$-\sqrt{1.4}/2$
				0.193094342117449	0.979970152735149	
E_1	0.5	1	-0.8	-0.974679434482771	-54.9061553695576	$-\sqrt{3.8}/2$
				5.12366550763497	-0.0251374945219104	
E_2	0.5	-2	-1	-1.32287565553419	-2.23700973253491	$-\sqrt{7}/2$
				0.0313389295860508	-0.394244091090544	

Table 3. Least squares fit for some values of the parameters giving rise to oscillatory behaviour.

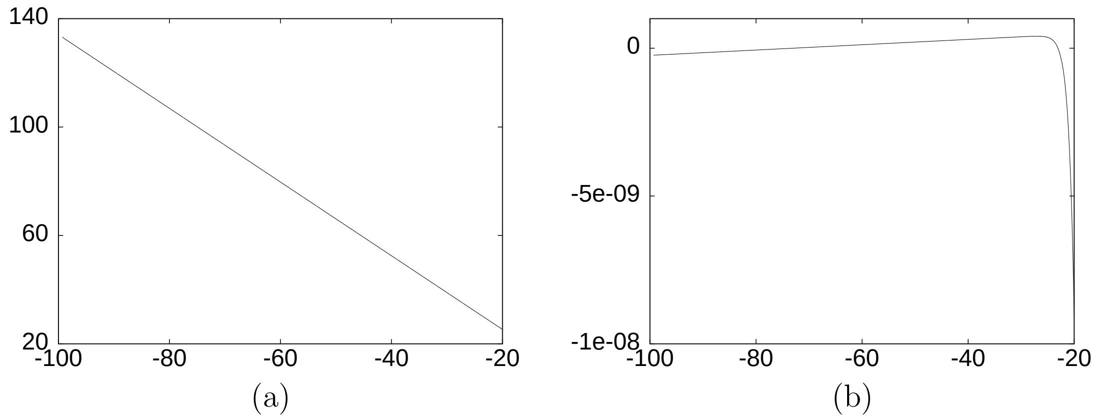
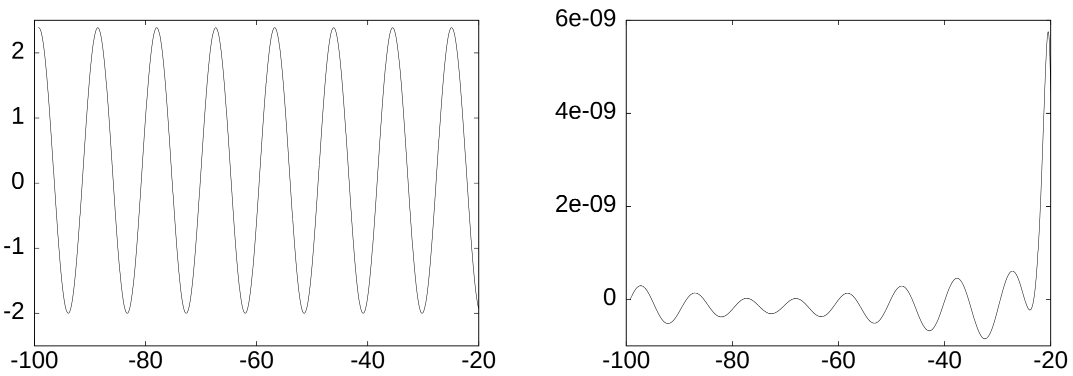


Figure 13: Plots for $\alpha = 0.5$, $\lambda_1 = -1.8$, $\lambda_2 = -0.1$ with $\log\delta$ in the horizontal axes (a) Graph of $\log|\mathrm{tr}|$ numerically computed. (b) Plot of $\log|\mathrm{tr}| - f(\delta)$ with $z_a = -1.90615984216061, z_b = -1.36014705088255$.



(a) (b)

Figure 14: Plots for $\alpha = 1$, $\lambda_1 = -0.2$, $\lambda_2 = -0.3$ with $\log\delta$ in the horizontal axes (a) Graph of tr numerically computed. (b) Plot of $\mathrm{tr} - f(\delta)$ with $z_1 = 0.193094342117449$, $z_2 = 2.19337090928979$, $z_3 = 0.591607978302245$, $z_4 = 2.16162250085465$.

5 On the collapsed gaps

In this section we apply the results of [11] concerning coexistence of periodic solutions to the equation (7). We recall that (7) includes the Newtonian case of homographic potential where a is minus the eccentricity. Coexistence is related to collapsed gaps which corresponds to the endpoints of instability pockets found in the (λ_1, λ_2)-plane. In this way we can give additional information about the instability pockets for the equation (7).

We shall write (7) as

$$(1 + a\cos t)\ddot{x} - (\lambda_1 + \lambda_2 + \lambda_1 a \cos t)x = 0. \tag{45}$$

If $a = 0$, the solutions can be obtained explicitly. In this case, if $\lambda_1 + \lambda_2 < 0$, the stability parameter tr=tr(λ_1, λ_2) oscillates with $|tr| \le 2$, and $|tr| = 2$ for some special values of (λ_1, λ_2). In the (λ_1, λ_2)-plane the lines defined by

$$\lambda_1 + \lambda_2 = -n^2, \qquad n \ge 0\,, \tag{46}$$

$$\lambda_1 + \lambda_2 = -\left(n + \frac{1}{2}\right)^2, \qquad n \ge 0 \tag{47}$$

give rise to tr$= 2$ and tr$= -2$ respectively.

For $a \neq 0$ small enough, one can expect that tr crosses the lines ± 2, for values of λ_1, λ_2 near the ones defined by (46) and (47), giving rise to some instability regions. So, if we fix $a \neq 0$, small enough, in a neighbourhood of each line (46) and (47), we should have two curves such that tr$= 2$ and tr$= -2$ respectively. Moreover it is well known from the general theory of Hill equation (see [5]) that $|\mathrm{tr}| = 2$ is related to the existence of 2π and 4π-periodic orbits. So, following the notation used in [11], we introduce α_{2n}, β_{2n}, α_{2n+1} and β_{2n+1} such that

- for $\lambda_2 = -\lambda_1 + \alpha_{2n}(a, \lambda_1)$, (45) has a non trivial even 2π-periodic solution,
- for $\lambda_2 = -\lambda_1 + \beta_{2n}(a, \lambda_1)$, (45) has a non trivial odd 2π-periodic solution,
- for $\lambda_2 = -\lambda_1 + \alpha_{2n+1}(a, \lambda_1)$, (45) has a non trivial even 4π-periodic solution,
- for $\lambda_2 = -\lambda_1 + \beta_{2n+1}(a, \lambda_1)$, (45) has a non trivial odd 4π-periodic solution.

It is clear that if $a = 0$ one has $\alpha_{2n} = \beta_{2n} = -n^2$ and $\alpha_{2n+1} = \beta_{2n+1} = -(n+1/2)^2$, for $n \geq 0$. Furthermore for any $a \neq 0$ and λ_1 such that $\alpha_m(a, \lambda_1) \neq \beta_m(a, \lambda_1)$, we get an instability interval for λ_2. However it may happen that $\alpha_m(a, \lambda_1) = \beta_m(a, \lambda_1)$ for some values of the parameters a, λ_1 and some positive integers m. In this case the instability interval for λ_2 degenerates to a point and we get the so called collapsed gaps. For a fixed value of $a \neq 0$, they can be seen in the (λ_1, λ_2)-plane as the endpoints of the instability pockets (see figures 8, 9).

We trivially get collapsed gaps for any a, $|a| < 1$, on the λ_1-axes. If we take $\lambda_2 = 0$, the equation (45) reduces to $\ddot{x} - \lambda_1 x = 0$. So, for $\lambda_1 < 0$ we get collapsed gaps if $\lambda_1 = -k^2$ and $\lambda_1 = -(k+1/2)^2$, $k \geq 0$, that is, $\alpha_{2k}(a, -k^2) = \beta_{2k}(a, -k^2) = -k^2$ and $\alpha_{2k+1}(a, -(k+1/2)^2) = \beta_{2k+1}(a, -(k+1/2)^2) = -(k+1/2)^2$.

Lemma 7. *Assume $0 < |a| < 1$ is fixed.*

1. *If $\lambda_1 = -k^2$ for some integer $k \geq 0$, then $\alpha_{2n}(a, \lambda_1) = \beta_{2n}(a, \lambda_1)$, for $n \geq k_0$ where, $k_0 = k$ if $k \geq 1$, and $k_0 = 1$ if $k = 0$.*
2. *If $\lambda_1 = -(k+1/2)^2$ for some integer $k \geq 0$, then $\alpha_{2n+1}(a, \lambda_1) = \beta_{2n+1}(a, \lambda_1)$, for $n \geq k$.*

Otherwise, $\alpha_m(a, \lambda_1) \neq \beta_m(a, \lambda_1)$ for $m \geq 1$, and there is an open instability interval for λ_2.

Proof Following [11] we introduce the polynomial $Q(\mu; a, \lambda_1) = 2a(\mu^2 + \lambda_1)$. Then the main result in [11] implies that

(a) sign $[\alpha_{2n}(a, \lambda_1) - \beta_{2n}(a, \lambda_1)] =$ sign $\Pi_{m=-n}^{n-1} Q(m; a, \lambda_1)$ for $n = 1, 2, 3, \ldots$

(b) sign $[\alpha_{2n+1}(a, \lambda_1) - \beta_{2n+1}(a, \lambda_1)] =$ sign $\Pi_{m=-n}^{n} Q(m - 1/2; a, \lambda_1)$ for $n = 0, 1, 2, 3, \ldots$

where the sign of a real number x is understood to be $-1, 0, 1$, according to whether $x < 0$, $x = 0$ or $x > 0$ respectively. Assume $\lambda_1 = -k^2$, $k \geq 0$. Then the right hand part of (a) vanishes, for $n \geq k$ if $k \geq 1$ and for $n \geq 1$ if $k = 0$. This proves first part of the lemma. In a similar way, part 2 holds after (b). Moreover, if $\lambda_1 \neq -k^2$ and $\lambda_1 \neq -(k+1/2)^2$ for any $k \geq 0$, then $Q(\mu, a, \lambda_1)$ and $Q(\mu - 1/2, a, \lambda_1)$ do not have integer roots, and the right hand part of (a) and (b) is different from zero for any n. $\square$

Assume a is fixed. The lemma above implies that the endpoints of instability pockets are on the vertical lines $\lambda_1 = -k^2$ and $\lambda_1 = -(k+1/2)^2$, with $k > 0$ integer. Moreover these points are in the halfplane $\lambda_2 < 0$. Figure 15 shows the instability pockets in the (λ_1, λ_2)-plane for three values of a. We plot also some vertical lines with constant λ_1 . Moreover three fat points on the line $\lambda_1 = -4$ are distinguished. They correspond to endpoints of one pocket for three different values of a.

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6 Appendix

To prove lemma 1 we use a very well-known result for the general Ince equation

$$(1 + a\cos t)\,\ddot{y} + b\sin t\,\dot{y} + (c + \tilde{d}\cos t)\,y = 0, \tag{48}$$

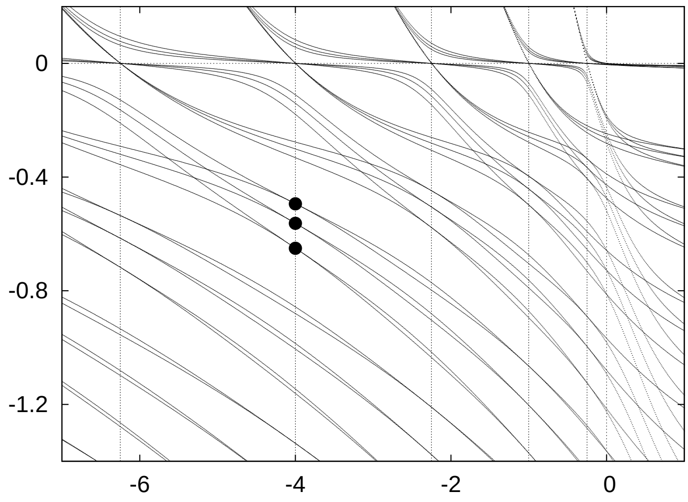

Figure 15: Some instability pockets in the (λ_1, λ_2)-plane for the equation (45) with $-a = 1 - 2^{-10}, 1 - 2^{-11}, 1 - 2^{-12}$, and the lines $\lambda_1 = -1/4, -1, -9/4, -2, -25/4$

where a, b, c and $\tilde{d}$ are real parameters with $|a| < 1$. In [5] it is shown that by performing the following change of variables $y = (1 + a\cos t)^{\frac{b}{2a}} x$ if $a \neq 0$, and $y = e^{\frac{b}{2}\cos t} x$ if $a = 0$, (48) becomes

$$\ddot{x} + \frac{1}{(1 + a\cos t)^2}\left[-\frac{ab}{2} - \frac{b^2}{4} + c + \left(\tilde{d} + ac - \frac{b}{2}\right)\cos t + \left(a\tilde{d} + \frac{b^2}{4}\right)\cos^2 t\right] x = 0. \quad (49)$$

Assume $a \neq 0$. Equation (49) can be written as (7) if and only if the following equalities are satisfied

$$\begin{aligned} \lambda_1 + \lambda_2 &= \frac{ab}{2} + \frac{b^2}{4} - c, \\ 2a\lambda_1 + a\lambda_2 &= -\tilde{d} - ac + \frac{b}{2}, \\ a^2\lambda_1 &= -a\tilde{d} - \frac{b^2}{4}. \end{aligned} \quad (50)$$

The two last equations in (50) give us $\lambda_1 = -\frac{1}{a^2}\left(a\tilde{d} + \frac{b^2}{4}\right)$, and $\lambda_2 = -c + \frac{\tilde{d}}{a} + \frac{b}{2a} + \frac{b^2}{2a^2}$. Using these expressions for λ_1 and λ_2 the first equation in (50) becomes

$$\left(1 - \frac{1}{a^2}\right)\left(\frac{b^2}{4} + \frac{ba}{2}\right) = 0.$$

We recall that $|a| \neq 1$, then the equation above is satisfied if and only if $b = 0$ or $b = -2a$. In the first case we get $\lambda_1 = -\tilde{d}/a$ and $\lambda_2 = \tilde{d}/a - c$. If $b = -2a$, $\lambda_1 = -\tilde{d}/a - 1$ and $\lambda_2 = \tilde{d}/a - c + 1$. We note that λ_1 and λ_2 are independent on a if and only if $\tilde{d} = ad$.

If $a = 0$, (50) implies $b = 0$ and the reduction is trivial.

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