THE POSITIVE ENTROPY KERNEL FOR SOME FAMILIES OF TREES

ESTHER BARRABÉS AND DAVID JUHER

Abstract. The fact that a continuous self-map of a tree has positive topological entropy is related to the amount of different *gods* (greatest odd divisors) exhibited by its set of periods. Llibre & Misiurewicz [11] and Blokh [9] give generic upper bounds for the maximum number of gods that a zero entropy tree map $f: T \longrightarrow T$ can exhibit, in terms of the number of endpoints and edges of T. In this paper we compute exactly the minimum of the positive integers n such that the entropy of each tree map $f: T \longrightarrow T$ exhibiting more than n gods is necessarily positive, for the family of trees which have a subinterval containing all the branching points (this family includes the interval and the stars). We also compute which gods are admissible for such maps.

1. Introduction

In the framework of the discrete dynamical systems, the study of the set of periods for continuous self-maps of one dimensional spaces has centered the attention in the last decades. In particular, we will focus on the study of the set of periods of maps $f: T \longrightarrow T$, where T is a tree (a graph without circles). The first and most famous result in this direction is the Sharkovsky's theorem ([12]), which gives a complete characterization of the set of periods of f when T is a closed interval. Later on, a similar characterization has been also given for n-stars (trees consisting of n edges attached at a unique central point) by Baldwin (in [6]). Recently, Alsedà, Juher and Mumbrú (see [2]) have characterized the set of periods of f when T is any generic tree, in terms of the topological structure of T.

One way to study the dynamical complexity of a continuous map $f: X \longrightarrow X$ of a compact metric space is computing its *topological entropy*, a nonnegative constant which measures how the iterates of f mix the points of X (see [1]). For example, a map with positive topological entropy is *chaotic* in the sense of Li and Yorke (see [10] and [8]). The entropy of f is closely related to the periods of the periodic orbits exhibited by f. There are some results that describe partially the set of periods of f depending on the fact that f has positive or zero entropy. In this paper we focus our attention on zero entropy continuous maps defined on trees and the problem of describing the admissible set of periods for this sort of maps.

When T is an interval, it is well known that a map $f: T \longrightarrow T$ has zero entropy if and only if the period of each periodic orbit of f is a power of 2 (see [3] for a historical survey on the proof of this result). For a generic tree, the zero entropy maps have been characterized by Alsedà and Ye (see [5]). The authors give the

Date: May 4, 2007.

2000 *Mathematics Subject Classification.* Primary 37E25, 37B40.

The authors have been partially supported by MEC grants MTM2006-05849/Consolider (including a FEDER contribution) and MTM2005-02139.

characterization in terms of the notion of *division* of a periodic orbit (see Section 3 for a definition and Theorem 3.1). They also give a maximal set of periods for zero entropy maps in terms of the number of endpoints of the tree. Another result in the same way is due to Blokh (Corollary 7 of [9]), which gives a better upper bound for the set of periods of zero entropy maps:

Theorem 1.1. *Let T be a tree with n endpoints and s edges, and let $f\colon T \longrightarrow T$ be continuous. Then the topological entropy of f is zero if and only if* $\mathrm{Per}(f) \subset \{k \cdot 2^l : l \geq 0, k \text{ odd }, k \leq s, \text{ each prime divisor of } k \text{ is not larger than } n\}$.

As we can see, the fact that a map has positive entropy is related to the odd factors exhibited by its set of periods. This is the main motivation for the notions of *god* of a nonnegative integer and *pantheon* of a subset of $\mathbb{N}$, first introduced by Llibre and Misiurewicz in [11]. Given any $n \in \mathbb{N}$, the *god* of n is simply the greatest odd divisor of n. It is denoted by $\mathrm{god}(n)$. Note that each $n \in \mathbb{N}$ can be written uniquely as $n = \mathrm{god}(n) \cdot 2^l$ for some $l \in \mathbb{N} \cup \{0\}$. For any $A \subset \mathbb{N}$, the set $\{\mathrm{god}(n) : n \in A\}$ is called the *pantheon* of A. Given a tree map f, the pantheon of $\mathrm{Per}(f)$ is called the *pantheon* of f, and is denoted by $\mathrm{Pan}(f)$.

In Section 2 we will show that Theorem 1.1 can be reformulated in terms of gods as follows: a map $f\colon T \longrightarrow T$ of a tree with n endpoints and s edges has zero entropy if and only if $\mathrm{Pan}(f) \subset A_T \cup B_T$ (see Theorem 2.2), where A_T is the set of odd numbers less or equal than n, and B_T is the set of odd non-prime numbers between $n+1$ and s.

On another hand, Llibre and Misiurewicz ([11]) showed that if a continuous map $f\colon G \longrightarrow G$ is defined on a graph G with s edges and the cardinality of the pantheon of f is greater than a constant $\Gamma(s)$, then the map has positive entropy. As the authors remark, the estimate $\Gamma(s)$ is not the best possible and they have not tried to optimize it. In the case of the interval and the circle, it is known that the best estimate of the minimum number of gods which forces positive entropy is two. But for a generic graph the problem of determining how many gods are permitted for a zero entropy map remains open. For any tree T, let us denote this number by N_T. That is, N_T is the minimum of the positive integers n such that the entropy of each tree map $f\colon T \longrightarrow T$ with $|\mathrm{Pan}(f)| > n$ is necessarily positive. As we have seen, $\Gamma(s)$, where s is the number of edges of T, is an upper bound of N_T. Furthermore, Theorem 1.1 gives another upper bound, $|A_T \cup B_T|$, much better than $\Gamma(s)$ in general. Nevertheless, the exact computation of this number in general is a difficult problem in which we are interested in.

A different problem consists of describing, for a fixed tree T, the set of gods k such that there exists a zero entropy map $f\colon T \longrightarrow T$ with $k \in \mathrm{Pan}(f)$. We call such a god an *admissible god* for T. The set of all admissible gods for T is called the *pantheon* of T, denoted by $\mathrm{Pan}(T)$, which, clearly, coincides with the union of the pantheons of all zero entropy maps defined on T. We call $\mathrm{Pan}(T)$ the *positive entropy kernel* by analogy with a well known notion, the *full periodicity kernel* (see, for example, [3]). The positive entropy kernel $\mathrm{Pan}(T)$ gives the maximum set of admissible gods for a zero entropy map, because any tree map $f\colon T \longrightarrow T$ exhibiting a periodic orbit P such that $\mathrm{god}(|P|) \notin \mathrm{Pan}(T)$ has positive entropy.

We would like to describe the set $\mathrm{Pan}(T)$ and the constant N_T depending on the particular geometry of the tree T, which is a difficult task in general. For example, if T is an interval it is well known that $f\colon T \longrightarrow T$ has zero entropy if and only if $\mathrm{Per}(f) \subset \{2^k : k \in \mathbb{N} \cup \{0\}\}$. Thus, in this case $\mathrm{Pan}(T) = \{1\}$ and $N_T = 1$.

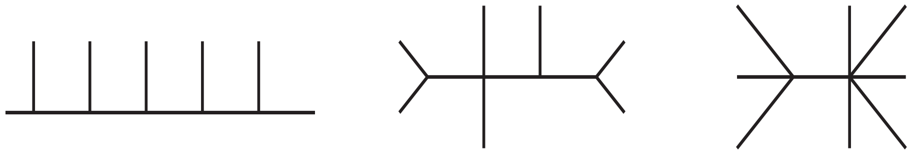

Figure 1. Three examples of 7-combs

Observe that both questions are not equivalent, in the sense that it is not true that $N_T = |\operatorname{Pan}(T)|$. For example, one may obtain that $\operatorname{Pan}(T) = \{1, 3, 5, 7\}$, while proving that any map on T exhibiting more than two of these gods at the same time necessarily has positive entropy, so in this case, $N_T = 2$. In general, we can only assure that $N_T \le |\operatorname{Pan}(T)|$.

As we have noticed, Theorem 1.1 gives a generic upper bound for $\operatorname{Pan}(T)$. It turns out that the results obtained by Barrabés and Juher in [7] allow us to improve this bound. See Theorem 2.6.

This paper contains two main results, labelled as Theorems A and B. Theorem A describes $\operatorname{Pan}(T)$ and N_T for a family of trees which we have termed *combs*. A *comb* is a tree which has a subinterval containing all the branching points (i.e. points x such that $T \setminus \{x\}$ has at least 3 connected components). See Figure 1 for some examples of combs. Of course, in particular the interval and the stars are combs. Theorem A states that if T is a comb then $\operatorname{Pan}(T) = A_T$ and $N_T = |A_T|$.

On another hand, Theorem B states that when a tree T can be obtained from another tree S by collapsing finitely many subtrees of S to points then $\operatorname{Pan}(T) \subset \operatorname{Pan}(S)$. This result can be a crucial tool to tackle the general problem of characterizing the pantheon of any tree T, since it allows to compute lower or upper bounds of $\operatorname{Pan}(T)$ by comparing different sorts of trees.

This paper is organized as follows. In Section 2 we introduce the basic definitions, discuss some generic bounds for $\operatorname{Pan}(T)$ and N_T and finally state Theorems A and B. To prove them, we will use several classical results and well known topics. We will recall all these notions in Sections 3 and 4. In the same sections we will also define the notion of *simplified model* and will use it to prove some useful technical results. Sections 5 and 6 are devoted to the proofs of Theorem A and B respectively.

2. Basic definitions and statement of the main results

Given any subset X of a topological space, we will denote by $\operatorname{Int}(X)$ and $\operatorname{Cl}(X)$ the interior and the closure of X, respectively. For a finite set A we will denote its cardinality by $|A|$.

By an *interval* we mean any space homeomorphic to $[0, 1] \subset \mathbb{R}$. A *tree* is a uniquely arcwise connected space that is either a point or a union of finitely many intervals. Any continuous map from a tree into itself will be called a *tree map*. The set of periods of all periodic orbits of a tree map f will be called the *set of periods of* f and will be denoted by $\operatorname{Per}(f)$. A triplet (T, P, f) such that $f : T \longrightarrow T$ is a tree map and $P \subset T$ is a finite f-invariant set will be called a *model*. If in addition P is a periodic orbit then the model (T, P, f) will be called *periodic*.

If T is a tree and $x \in T$, the *valence* of x is the number of connected components of $T \setminus \{x\}$. Each point of valence 1 will be called an *endpoint* of T and the set of such points will be denoted by $\operatorname{En}(T)$. A point of valence different from 2 will be called a *vertex* of T, and the set of vertices of T will be denoted by $V(T)$. The closure of each connected component of $T \setminus V(T)$ will be called an *edge* of T. An edge containing an endpoint will be called *external*, otherwise it will be called *internal*. The number of endpoints and the number of edges of T will be denoted, respectively, by $\mathrm{en}(T)$ and $\mathrm{ed}(T)$. It is clear that for any tree T with at least 3 endpoints the number of external and internal edges are $\mathrm{en}(T)$ and $\mathrm{ed}(T) - \mathrm{en}(T)$ respectively.

Given a tree T and $P \subset T$, we will define the *convex hull* of P, denoted by $\langle P \rangle_T$ or simply by $\langle P \rangle$, as the smallest closed connected subset of T containing P. When $P = \{x, y\}$ we will write $\langle x, y \rangle$ or $[x, y]$ to denote $\langle P \rangle$. The notations (x, y), $(x, y]$ and $[x, y)$ will be understood in the natural way.

The notion of *topological entropy*, introduced in [1], is defined for continuous maps on compact metric spaces and is a quantitative measure of the dynamical complexity of the map. It is an important topological invariant. The topological entropy of a map f will be denoted by $h(f)$.

As we have noticed in Section 1, the fact that a map has zero entropy is closely related to the gods exhibited by $\mathrm{Per}(f)$. Theorem 1.1 gives a maximal set of periods for zero entropy maps. This Theorem can be reworded using the following lemma (its proof is simple and it is left to the reader) and the subsequent notation.

Lemma 2.1. *Let T be a tree with* $\mathrm{en}(T) \geq 3$. *Then* $\mathrm{en}(T) \leq \mathrm{ed}(T) \leq 2\,\mathrm{en}(T) - 3$.

For each tree T we define the sets

$$\begin{aligned} A_T &= \{n \in \mathbb{N} : n \text{ odd}, \ n \leq \mathrm{en}(T)\}, \\ B_T &= \{n \in \mathbb{N} : n \text{ non-prime odd}, \ \mathrm{en}(T) < n \leq \mathrm{ed}(T)\}. \end{aligned}$$

Theorem 2.2. *For each tree map $f \colon T \longrightarrow T$, $h(f) = 0$ if and only if* $\mathrm{Pan}(f) \subset A_T \cup B_T$.

Proof. From Lemma 2.1 it follows that if $n \in \mathbb{N}$ is not prime and satisfies $n \leq \mathrm{ed}(T)$ then each prime divisor of n is not larger than $\mathrm{en}(T)$. Thus the theorem follows from Theorem 1.1. □

We are interested in calculating $\mathrm{Pan}(T)$ and N_T depending on the particular geometry (number and arrangement of vertices, edges and endpoints) of the tree T. We recall that $\mathrm{Pan}(T)$ is defined to be the union of the pantheons of all zero entropy maps on T, while N_T is the minimum of the positive integers n such that each tree map $f \colon T \longrightarrow T$ with $|\,\mathrm{Pan}(f)| > n$ necessarily satisfies $h(f) > 0$.

Theorem 2.2 gives upper bounds for $\mathrm{Pan}(T)$ and N_T, which are the set $A_T \cup B_T$ and its cardinality respectively. Furthermore, the following lemma and corollary give lower bounds. These bounds are general for any tree T, that is, they do not depend on the particular geometry of T.

Lemma 2.3. *Let $f \colon T \longrightarrow T$ be a tree map and let $k \in \{1, 2, \ldots, \mathrm{en}(T)\}$. Then there exists a tree map $g \colon T \longrightarrow T$ such that $h(g) = h(f)$ and* $\mathrm{Per}(g) = \mathrm{Per}(f) \cup \{k\}$.

Proof. Let $\{e_0, e_1, \ldots, e_{n-1}\}$, for $n = \mathrm{en}(T)$, be the set of endpoints of T. The result trivially holds when T reduces to a point, so we assume $n \geq 2$. For each $0 \leq i < n$, take a point $e_i' \in T \setminus \mathrm{En}(T)$ such that $(e_i, e_i') \cap V(T) = \emptyset$ (when $n = 2$, in addition we take e_0' and e_1' such that $e_1' \in (e_0', e_1)$). Let S be the convex hull of $\{e_i'\}_{i=0}^{n-1}$. Observe that S is a subtree of T which is homeomorphic to T. Let $h \colon S \longrightarrow T$ be a homeomorphism such that $h(e_i') = e_i$ for each $0 \leq i < n$.

For any $x \in S$, we define $g(x) = h^{-1} \circ f \circ h(x)$. Since g and f are conjugate, $h(f) = h(g\big|_S)$ and there is a bijection between the respective sets of periodic orbits. In particular, $\text{Per}(g\big|_S) = \text{Per}(f)$. Now we extend g to the whole T as follows. For $k \leq i < n$, let us define $g(x) = g(e'_i) \in S$ for each $x \in (e'_i, e_i]$. It easily follows that $g^j(x) \in S$ for each $j \geq 1$ and, consequently, since $x \in T \setminus S$, x is not periodic. Therefore, there are no periodic points of g in $(e'_i, e_i]$ for $k \leq i < n$. Finally we have to define g on $(e'_i, e_i]$ for each $0 \leq i < k$. Take points $m_i \in (e'_i, e_i)$. We define $g(m_i) = m_i$ and $g(e_i) = e_{i+1 \bmod k}$, and extend g to be monotone on $[e'_i, m_i]$ and $[m_i, e_i]$. Let us consider $x \in [e'_i, m_i]$. If $g(e'_i) = e'_i$, then $g(x) = x$; if not, due to the monotonicity of g, there exists some j such that $g^j(x) \in S$. In any case, x cannot belong to a periodic orbit. The same argument applies for $x \in [m_i, e_i]$, so $\{e_0, e_1, \ldots, e_{k-1}\}$ is the only non-trivial periodic orbit of g in $T \setminus S$. Therefore, $\text{Per}(g) = \text{Per}(f) \cup \{k\}$. Finally, we have to prove that the entropies of f and g are equal. Let

$$X = \{e_0, e_1, \ldots, e_{k-1}\} \bigcup \big(\text{Fix}(g) \bigcap \bigcup_{i=0}^{k-1} [m_i, e'_i]\big),$$

where $\text{Fix}(g)$ is the set of all fixed points of g. Observe that X is closed and invariant by g. Let $\Omega(g)$ be the set of nonwandering points of g. It is well known (see, for instance, Lemma 4.1.5 of [3]) that $\Omega(g)$ is closed and invariant and the entropies of g and $g\big|_{\Omega(g)}$ are equal. By the definition of g, we have that $\Omega(g)$ is the union of the invariant and closed sets X and $\Omega(g) \cap S$. Note also that $h(g\big|_X) = 0$, since X contains only a k-periodic orbit and fixed points of g. Therefore,

$$h(g) = h(g\big|_{\Omega(g)}) = \max\{h(g\big|_X), h(g\big|_{\Omega(g)\cap S})\} = h(g\big|_{\Omega(g)\cap S}) = h(g\big|_S) = h(f).$$

□

Corollary 2.4. *Given any tree T, there exists a map $g\colon T \longrightarrow T$ such that $h(g) = 0$ and $\text{Pan}(g) = A_T$.*

Proof. Consider the identity map f on T, which satisfies $h(f) = 0$ and $\text{Per}(f) = \{1\}$. Since each element in A_T is not greater than $\text{en}(T)$, we can use inductively Lemma 2.3 to construct the prescribed map. □

From the above results we get the following bounds for $\text{Pan}(T)$ and N_T.

Theorem 2.5. *Let T be a tree. Then $A_T \subset \text{Pan}(T) \subset A_T \cup B_T$ and $|A_T| \leq N_T \leq |A_T| + |B_T|$.*

Proof. Statements $A_T \subset \text{Pan}(T)$ and $|A_T| \leq N_T$ follow from Corollary 2.4. By Theorem 2.2, $\text{Pan}(T) \subset A_T \cup B_T$, so we have that $N_T \leq |\text{Pan}(T)| \leq |A_T| + |B_T|$. □

The upper bound of $\text{Pan}(T)$ (and, thus, of N_T) can be improved by using some recent results of the authors. In [7], for a fixed $p \in \mathbb{N}$, the minimum number of endpoints e_p of a tree admitting a zero entropy map f with a periodic orbit of period p is given. This minimum of endpoints can be computed from p as

$$e_p = s_1 s_2 \cdots s_k - \sum_{i=2}^{k} s_i s_{i+1} \cdots s_k,$$

where $p = s_1 s_2 \cdots s_k$ is the decomposition of p into a product of primes such that $s_i \leq s_{i+1}$ for $1 \leq i < k$, and it is easy to see that $e_p = e_{\text{god}(p)}$. The result implies that if T is a tree such that $\mathrm{en}(T) < e_p$, then $\mathrm{god}(p) \notin \mathrm{Pan}(T)$. Then, Corollary 1.3 of [7] gives the following result.

Theorem 2.6. *Let T be a tree and $\overline{B}_T = \{p \in B_T : e_p \leq \mathrm{en}(T)\}$. Then $A_T \subset \mathrm{Pan}(T) \subset A_T \cup \overline{B}_T$ and $|A_T| \leq N_T \leq |A_T| + |\overline{B}_T|$.*

For example, let us consider a tree T with $\mathrm{en}(T) = 19$ and the maximum possible number of edges, which is 35. In this case, $B_T = \{21, 25, 27, 33, 35\}$, while $\overline{B}_T = \{21, 27\}$ because $e_{25} = 20$, $e_{33} = 22$ and $e_{35} = 28$. Then, by Theorem 2.6, $\mathrm{Pan}(T) \subset A_T \cup \{21, 27\}$ and $10 \leq N_T \leq 12$. For another example, let T be any tree with 9 endpoints and 15 edges. In this case $B_T = \{15\}$ but, since $e_{15} = 10$, $\overline{B}_T = \emptyset$. Thus, from Theorem 2.6, we get that $\mathrm{Pan}(T) = A_T$ and $N_T = |A_T|$.

The problem of computing generically N_T and $\mathrm{Pan}(T)$ is not easy. Thus, it seems convenient to restrict our attention to any particular sort of trees. In the literature, the classical results on interval maps are usually first extended to *star* maps. For any $n \geq 2$, an *n-star* is a tree which is a union of n intervals whose intersection is a unique point x of valence n (which is called the *central point* of the star). Observe that if $n \geq 3$ any n-star has n endpoints and n edges. The following immediate consequence of Theorem 2.5 solves our problem for this sort of trees.

Theorem 2.7. *Let T be a star. Then $\mathrm{Pan}(T) = A_T$ and $N_T = |A_T|$.*

Proof. The result is well known when T is an interval. If T has at least 3 endpoints, the theorem follows from Theorem 2.5 and the fact that $B_T = \emptyset$, because in this case $\mathrm{en}(T) = \mathrm{ed}(T)$. □

Theorem A below states that the same holds for another family of trees: the *combs*. For any $n \geq 2$, an *n-comb* is defined to be any tree T such that $\mathrm{en}(T) = n$ and there exists a subinterval of T containing $V(T) \setminus \mathrm{En}(T)$. In other words, each point with valence at least 3 (these points are usually called *branching points*) belongs to an external edge. Observe that the interval and the stars are particular cases of combs. In Figure 1 one can find three examples of 7-combs.

Remark 2.8. Any proper subtree of an n-comb is a t-comb for some $t \leq n$.

Theorem A. *Let T be a comb. Then $\mathrm{Pan}(T) = A_T$ and $N_T = |A_T|$.*

To state Theorem B we need to introduce a relation $\preceq$ among trees. We say that $T \preceq S$ if T is homeomorphic to a tree obtained by choosing finitely many disjoint subtrees of S and collapsing each one to a point. It is easy to check that $\preceq$ is not a total ordering.

Theorem B. *Let T and S be trees such that $T \preceq S$. Then,* $\mathrm{Pan}(T) \subset \mathrm{Pan}(S)$.

Theorem B can be used to adjust the upper and lower bounds discussed above for the pantheon of a generic tree T. In some cases, it is possible to compute exactly the pantheon of a collection of trees by $\preceq$-comparing them with another regular enough family of trees. As an example, consider a $(3, 5)$-star T (see [7] for the definition of a generic $(s_1, \ldots, s_k)$-star). It is easy to construct a zero entropy map $f : T \longrightarrow T$ such that $15 \in \mathrm{Per}(f)$ (see [7] for details). Then, from Theorem B we have that

$$15 \in \mathrm{Pan}(S) \quad \text{for each tree } S \text{ such that } \quad T \preceq S. \tag{2.1}$$

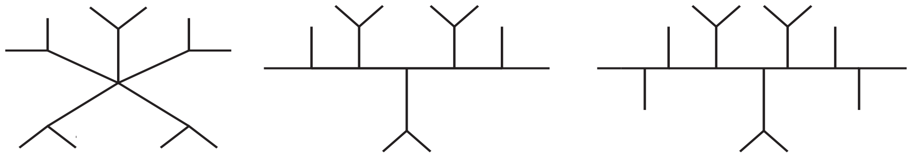

FIGURE 2. An example of a $(3,5)$-star T (on the left) and two trees greater than T in the ordering $\preceq$.

Consider now the set of all trees S such that $T \preceq S$ and $\operatorname{en}(S) \leq 13$ (see Figure 2). Each tree S in this family satisfies $\overline{B}_S = \{15\}$ and, hence, from (2.1) and Theorem 2.6 it follows that $\operatorname{Pan}(S) = A_S \cup \{15\}$.

3. Zero entropy tree maps and simplified models

The zero entropy orbits on trees are characterized in terms of the notion of *division*, first introduced by Alsedà and Ye in [5]. Next we recall this notion.

Let $f\colon T \longrightarrow T$ be a tree map and let P be a periodic orbit of f of period larger than 1. The map $f_P\colon \langle P\rangle \longrightarrow \langle P\rangle$ defined by $f_P = r \circ f$, where $r\colon T \longrightarrow \langle P\rangle$ is the natural retraction, will be called the *natural restriction of f to $\langle P\rangle$*. Let y be a fixed point of f_P. Let Z be the connected component of $\langle P\rangle \setminus P$ containing y and $Z_1, Z_2, \ldots, Z_l$ the connected components of $\langle P\rangle \setminus Z$. These sets will be called the *components of P*. We say that *P has a division of type (l,m)* if there exist $\{M_1, M_2, \ldots, M_m\}$ with $m \geq 2$, a partition of $\langle P\rangle \setminus Z$, such that each M_i consists of a union of components Z_i of P, $f(M_i \cap P) = M_{i+1} \cap P$ for $1 \leq i < m$ and $f(M_m \cap P) = M_1 \cap P$. The sets M_i will be called the *branches of P*. Observe that each branch contains $|P|/m$ points of P.

The following result, which is a part of Corollary C of [5], characterizes the zero entropy tree maps in terms of their orbits.

Theorem 3.1. *Let $f\colon T \longrightarrow T$ be a tree map. Then $h(f) = 0$ if and only if for every $n \in \mathbb{N}$ each periodic orbit of f^n of period larger than 1 has a division.*

Given a periodic orbit of a zero entropy tree map, sometimes it is convenient to *reduce* it (in a sense given by the proof of Lemma 3.4) in order to get another periodic orbit, which we will call *simplified*, satisfying a number of useful properties. Let us define this notion. Let (T,P,f) be a periodic model. We say that (T,P,f) is a *simplified model* if either $|P| = 1$ or the following conditions hold:

(S1) $|P| > 1$.
(S2) $h(f) = 0$.
(S3) $\operatorname{En}(T) \subset P$.
(S4) If $M_1, M_2, \ldots, M_m$ are the branches of P defined by any division of P, then $\operatorname{Int}(\langle M_i\rangle_T) \cap P = \emptyset$ for each $1 \leq i \leq m$.

Lemma 3.2. *Let $f\colon T \longrightarrow T$ be a tree map with $h(f) = 0$, X a subtree of T and let $r\colon T \longrightarrow X$ be the natural retraction from T onto X. Then, $h(r \circ f\big|_X) = 0$.*

Proof. Since $h(f) = 0$, for every $n \in \mathbb{N}$ each periodic orbit of f^n of period larger than 1 has a division by Theorem 3.1. On the other hand, by Lemma 2.5 of [5], this is also true for $r \circ f\big|_X$. Hence, again by Theorem 3.1, $h(r \circ f\big|_X) = 0$. $\square$

Lemma 3.3. *Let $f\colon T \longrightarrow T$ be a zero entropy tree map exhibiting a periodic orbit of period $2^l s$ with s odd. Then, there exists a zero entropy tree map $g\colon T \longrightarrow T$ exhibiting a periodic orbit of period s.*

Proof. It is enough to take $g = f^{2^l}$. Then, $h(g) = 2^l h(f) = 0$. Moreover, from Lemma 2.1.10 of [3], each periodic orbit of f of period $2^l s$ is a periodic orbit of g of period $2^l s/\mathrm{gcd}(2^l s, 2^l) = s$. □

The previous lemmas allow us to prove the following one, which is the main result of this Section.

Lemma 3.4. *Let (T, P, f) be a periodic model such that $|P| > 1$ and $h(f) = 0$. Then, there exists a simplified model (S, Q, g) such that $S \subset T$. Moreover, if $\mathrm{god}(|P|) \in B_T$ then $|Q| \in B_S$.*

Proof. Set $s = \mathrm{god}(|P|)$. By Lemma 3.3 there is a tree map $f'\colon T \longrightarrow T$ with $h(f') = 0$ exhibiting a periodic orbit P' such that $|P'| = s$. If $s = 1$, the model is simplified by definition. Let us suppose that $s > 1$. Note that the model (T, P', f') satisfies (S1) and that $\mathrm{god}(|P'|) = |P'| = s$.

We proceed by induction. Set $P_0 = P'$ and $T_0 = \langle P'\rangle_T$. Let $r\colon T \longrightarrow T_0$ be the natural retraction and set $f_0 = r \circ f'\big|_{T_0}$. By Lemma 3.2, $h(f_0) = 0$. Moreover, $\mathrm{En}(T_0) \subset P_0$. Thus, the model (T_0, P_0, f_0) satisfies (S1–3). Now we claim that if $s \in B_T$ then $|P_0| \in B_{T_0}$. Indeed, otherwise $\mathrm{god}(|P_0|) = |P_0| = s \in A_{T_0}$ by Theorem 2.2, but $A_{T_0} \subset A_T$ because $\mathrm{en}(T_0) \leq \mathrm{en}(T)$. Thus, $s \in A_T \cap B_T = \emptyset$, a contradiction which proves the claim.

Now assume that we have constructed a sequence of models $\{(T_i, P_i, f_i)\}_{i=0}^k$ such that, for each $0 \leq i \leq k$,

(i) $T_i \subset T$
(ii) (T_i, P_i, f_i) satisfies (S1–3)
(iii) If $\mathrm{god}(|P|) \in B_T$ then $|P_i| \in B_{T_i}$

and, for each $0 \leq i < k$,

(iv) $|P_{k+1}| = |P_k|/m_k$ for some $m_k \geq 2$.

If in addition (T_k, P_k, f_k) satisfies (S4), then we are done by setting $S = T_k$, $g = f_k$ and $Q = P_k$.

Assume that (T_k, P_k, f_k) does not satisfy (S4). That is, P_k has a division with branches $M_1, M_2, \dots, M_m$ with $m \geq 2$ and, for some $i \in \{1, 2, \dots, m\}$,

$$\mathrm{Int}(\langle M_i\rangle_{T_k}) \cap P_k \neq \emptyset. \tag{3.1}$$

Let $\phi\colon T_k \longrightarrow \langle M_i\rangle_{T_k}$ be the natural retraction. Set $T_{k+1} = \langle M_i\rangle_{T_k}$, $f_{k+1} = \phi \circ (f_k)^m\big|_{T_{k+1}}$ and $P_{k+1} = P_k \cap T_{k+1}$. Since $T_{k+1} \subset T_k$, $(T_{k+1}, P_{k+1}, f_{k+1})$ satisfies (i). Let us see that it satisfies (ii). Observe that $\mathrm{En}(T_{k+1}) \subset P_{k+1}$. From (3.1) it follows that

$$|P_{k+1}| > \mathrm{en}(T_{k+1}). \tag{3.2}$$

In particular, $|P_{k+1}| > 1$. On the other hand, $h(f_{k+1}) = 0$ using Lemma 3.2. Summarising, $(T_{k+1}, P_{k+1}, f_{k+1})$ satisfies (ii). By the definition of division, P_{k+1} is a periodic orbit of f_{k+1} with period $|P_k|/m$. Then, $(T_{k+1}, P_{k+1}, f_{k+1})$ satisfies (iv) by taking $m_k = m$. Finally we claim that it satisfies (iii). Assume that $\mathrm{god}(|P|) \in B_T$. Since (T_k, P_k, f_k) satisfies (iii), $|P_k| \in B_{T_k}$. Note that $|P_{k+1}|$ is odd because $|P_{k+1}| = |P_k|/m$ and $|P_k|$ is odd. Hence, $\mathrm{god}(|P_{k+1}|) = |P_{k+1}|$. By

Theorem 2.2, $|P_{k+1}| \in A_{T_{k+1}} \cup B_{T_{k+1}}$. Thus, from (3.2) and the definition of $B_{T_{k+1}}$ we have that $|P_{k+1}| \in B_{T_{k+1}}$ and the claim follows.

Since each model in the sequence satisfies (S1) and (iv), it easily follows that this iterative procedure stops after finitely many steps. □

4. Horseshoes and simplicial models

It is also well known that positive topological entropy is due to the existence of *horseshoes*. For example, the next result, which is a particular instance of Lemma 6.1 of [11], is a useful tool to prove that a map has positive entropy.

Lemma 4.1. *Let $f\colon T \longrightarrow T$ be a tree map. Let $I, J_1, J_2 \subset T$ be closed intervals containing no points of $V(T)$ in their interiors such that $\operatorname{Int}(J_1) \cap \operatorname{Int}(J_2) = \emptyset$. If there exist positive integers r, s, t such that $f^r(I) \supset J_1 \cup J_2$, $f^s(J_1) \supset I$ and $f^t(J_2) \supset I$, then $h(f) > 0$.*

Corollary 4.2. *Let $f\colon T \longrightarrow T$ be a tree map. Let $K, L \subset T$ be closed intervals containing no points of $V(T)$ in their interiors such that $\operatorname{Int}(K) \cap \operatorname{Int}(L) = \emptyset$. If there exist positive integers i, j such that $f^i(K) \supset K \cup L$ and $f^j(L) \supset K$ then $h(f) > 0$.*

Proof. Use Lemma 4.1 with $I = J_1 = K$, $J_2 = L$, $s = r = i$ and $t = j$. □

A model (T, P, f) will be called *simplicial* if $\operatorname{En}(T) \subset P$, $f(V(T)) \subset P \cup V(T)$ and f is monotone on each connected component of $T \setminus (P \cup V(T))$. In this case, the closure of each connected component of $T \setminus (P \cup V(T))$ will be called a *basic interval*. The following result is a particular instance of the main result of [4]:

Theorem 4.3. *Let $f\colon T \longrightarrow T$ be a tree map such that $h(f) = 0$. Let P be a periodic orbit of f. Then, there exists a tree map $g\colon \langle P\rangle_T \longrightarrow \langle P\rangle_T$ such that $g\big|_P = f\big|_P$, $h(g) = 0$ and $(\langle P\rangle_T, P, g)$ is a simplicial model.*

The following is a technical result which we will use to prove the main result of this paper:

Proposition 4.4. *Let (T, P, f) be a periodic simplicial model such that $|P|$ is odd. If there is a basic interval $[v, x]$ such that $x \in \operatorname{En}(T)$ and $|f^k([v, x]) \cap P| > 1$ for some $k \geq 0$ then $h(f) > 0$.*

Proof. Set $n = |P|$ and $P = \{x_i\}_{i=0}^{n-1}$, in such a way that $x_0 = x$, $f(x_i) = x_{i+1}$ for $0 \leq i < n-1$ and $f(x_{n-1}) = x_0$. By hypothesis, there exists $x_j \neq x_k$ such that $f^k([x_0, v]) \supset [x_k, x_j]$. Assume that $k > j$ (the argument for $k < j$ is similar). Set $K = [x_0, v]$. Then, $f^n(K) \supset f^{n-k}([x_k, x_j]) \supset [x_0, x_{n+j-k}]$. Since $(P \cup V(T)) \cap \operatorname{Int}(K) = \emptyset$ and $x_0 \in \operatorname{En}(T)$, it follows that

$$f^n(K) \supset [x_0, x_{n+j-k}] \supset K. \tag{4.1}$$

Set $\sigma = k - j$, so that $f^\sigma(x_{n+j-k}) = x_0$. Since (T, P, f) is a simplicial model, $f(P \cup V(T)) \subset P \cup V(T)$. Assume that there is $w \in (P \cup V(T)) \cap [v, x_{n+j-k})$ such that $f^\sigma(w) \neq x_0$. It easily follows that there is a basic interval $L \subset [v, x_{n+j-k})$ such that $f^\sigma(L) \supset K$. Since $f^n(K) \supset K \cup L$ by (4.1), from Corollary 4.2 we get that $h(f) > 0$ and we are done in this case.

Now assume that $f^\sigma(w) = x_0$ for all $w \in (P \cup V(T)) \cap [v, x_{n+j-k})$. In particular, $f^\sigma(v) = x_0$ and, since $f^\sigma(x_0) = x_\sigma$, as above it follows that

$$f^\sigma(K) \supset [x_0, x_\sigma] \supset K. \tag{4.2}$$

Finally, observe that $f^\sigma(x_\sigma) \neq x_0$ because n is odd. It easily follows that there is a basic interval $L \subset [v, x_\sigma]$ such that $f^\sigma(L) \supset K$. Since $f^\sigma(K) \supset K \cup L$ by (4.2), from Corollary 4.2 we get that $h(f) > 0$ and we are done. □

5. Proof of Theorem A

By Theorem 2.5 and the definition of $\text{Pan}(T)$ and N_T, it is enough to prove that any continuous self-map of T exhibiting a period whose god belongs to B_T has positive topological entropy.

Let $f\colon T \longrightarrow T$ be a comb map with a periodic orbit P such that $\text{god}(|P|) \in B_T$. To prove the theorem we assume that $h(f) = 0$, and we will get a contradiction.

By Lemma 3.4, there is a simplified model (S, Q, g) with $S \subset T$ and $|Q| \in B_S$. In particular, $h(g) = 0$ and $\text{En}(S) \subset Q$. In addition, by Theorem 4.3 we can assume that (S, Q, g) is a simplicial model. Thus, $g(V(S)) \subset V(S) \cup Q$ and g is monotone on each basic interval.

By Remark 2.8, S is a comb. Observe that S is neither an interval nor a star, because otherwise the fact that $|Q| \in B_S$ would contradict Theorem 2.7. Since $\text{En}(S) \subset Q$ and $|Q| > \text{en}(S)$, we have that $Q \setminus \text{En}(S) \neq \emptyset$. Note that

(5.1) any point in $Q \setminus \text{En}(S)$ does not belong to an external edge

because, otherwise, we would have a basic interval $[x, v]$ with $x \in \text{En}(S) \subset Q$ and $v \in Q$, obtaining, by applying Proposition 4.4, that $h(g) > 0$, a contradiction. Since S is a comb, any point of $V(S)$ belongs to an external edge. So from (5.1) we get

$$(Q \setminus \text{En}(S)) \cap V(S) = \emptyset. \tag{5.2}$$

Let y be a fixed point of g with respect to which the orbit Q has a division. Let (l, m) be the type of this division and denote by $Z_0, \ldots, Z_{l-1}$ and $M_0, \ldots, M_{m-1}$, respectively, the components and the branches of Q, in such a way that $g(M_i \cap P) = M_{i+1} \cap P$. Observe that, since $|Q|$ is odd, m is also odd.

We claim that $Q \setminus \text{En}(S)$ has at most two elements. Indeed: if $|Q \setminus \text{En}(S)| \geq 3$, then there is at least one component Z_i of Q such that $\text{Int}(\langle Z_i \rangle) \cap Q \neq \emptyset$. Since each branch of Q is a union of components of Q, this fact contradicts property (S4) of a simplified model. So the claim follows.

Now we claim that if $Q \setminus \text{En}(S)$ consists of two points x, z then $(x, z) \cap V(S) \neq \emptyset$. Let us prove it. First we note that $y \in (x, z)$ because otherwise there is a component (and, thus, a branch) of Q containing either x or z in its interior, in contradiction with property (S4) of a simplified model. Now assume that the claim is false, so that $(x, z) \cap V(T) = \emptyset$. It easily follows that Q has exactly two components and, thus, $m = l = 2$, a contradiction with the fact that m is odd.

Let x be a point of $Q \setminus \text{En}(S)$. By (5.2), $x \notin V(S)$ and therefore $S \setminus \{x\}$ has two connected components. Let Y be the connected component of $S \setminus \{x\}$ containing y. Then, $S \setminus Y$ coincides with a component Z_i of the orbit Q. Observe that the branch of Q containing Z_i (which we can assume to be M_0 without loss of generality) does not contain other components of Q, because otherwise $x \in \text{Int}(\langle M_0 \rangle)$, in contradiction with property (S4) of a simplified model. Hence, $M_0 = \langle M_0 \rangle = S \setminus Y$. Observe that $V(S) \cap \text{Int}(Y) \neq \emptyset$ (it follows from (5.1) when $|Q \setminus \text{En}(S)| = 1$ and from the previous claim when $|Q \setminus \text{En}(S)| = 2$). Take $v \in V(S) \cap \text{Int}(Y)$ such that $(x, v) \cap V(S) = \emptyset$. In other words, $v \in V(S) \setminus \text{En}(S)$ and $[x, v]$ is a basic interval contained in $\text{Cl}(Y)$. Set $K = [x, v]$.

Let $e \in \mathrm{En}(S)$ be such that $[v, e]$ is an external edge. By (5.1), $(v, e] \cap Q = \{e\}$. Since $e \in Y = S \setminus M_0$, there is $0 < i < m$ such that $e \in M_i$. Let us see that there exists $\alpha \in \mathbb{N}$ such that

$$g^{\alpha}(v) \in M_0. \tag{5.3}$$

Note that $g^{m-i}(e)$ and $g^{2m-i}(e)$ belong to M_0, since $e \in M_i$. Take $\alpha = m - i$ if $g^{m-i}(e) \neq x$ and $\alpha = 2m - i$ if $g^{m-i}(e) = x$. Note that $g^{\alpha}(e) \neq x$, because $Q \cap M_0$ is a periodic orbit of g^m and $|M_0 \cap Q| > 1$. Then (5.3) holds because, otherwise, $g^{\alpha}([v, e]) \cap Q \supset [g^{\alpha}(v), g^{\alpha}(e)] \cap Q \supset \{x, g^{\alpha}(e)\}$ and, consequently, $h(g) > 0$ by Proposition 4.4, a contradiction.

Observe that $\alpha \mod m = -i \mod m \neq 0$. Therefore, $g^{\alpha}(x) \in M_{\alpha \bmod m} \neq M_0$. From this fact and (5.3) it follows that

$$g^{\alpha}(K) \supset [x, g^{\alpha}(x)] \supset K. \tag{5.4}$$

Since (S, Q, g) is a simplicial model, $g(Q \cup V(S)) \subset Q \cup V(S)$. Finally, since $g^i(x) \in M_i$ and $g^i(g^{\alpha}(x)) \in M_0$, from the finiteness of $(Q \cup V(S)) \cap [x, g^{\alpha}(x)]$ it easily follows that there exists a basic interval $L \subset [x, g^{\alpha}(x)]$ such that $g^i(L) \supset K$. Since $g^{\alpha}(K) \supset K \cup L$ by (5.4), from Corollary 4.2 we get that $h(g) > 0$, a contradiction which proves the theorem.

6. Proof of Theorem B

It is easy to see that the definition of $T \preceq S$ is equivalent to the following one: $T \preceq S$ if and only if T is homeomorphic to a tree obtained by choosing a set X which is a finite union of edges of S and collapsing each connected component of X to a point. If only internal edges are collapsed, then $\mathrm{en}(T) = \mathrm{en}(S)$ and $\mathrm{ed}(T) = \mathrm{ed}(S) - k$, where k is the number of collapsed edges. Clearly, if $k = \mathrm{ed}(S) - \mathrm{en}(S)$ (the total number of internal edges of S), the tree obtained is the $\mathrm{en}(S)$−star.

Theorem B will be easily obtained as a consequence of Proposition 6.4. To prove it, we need two technical results.

Lemma 6.1. *Let S and T be trees such that $T \preceq S$. Let X be a union of external edges of S and assume that T has been obtained by collapsing each connected component of X to a point. If there exists a periodic model (T, P, f) such that $h(f) = 0$ then there exists a periodic model (S, Q, g) such that $h(g) = 0$ and $|Q| = |P|$.*

Proof. The result follows easily if T is a point. So, from now on we assume that T is a proper tree. Let $X_1, X_2, \ldots, X_k$ be the connected components of X. Since X_i is a union of external edges for each $1 \leq i \leq k$, it follows that $X_i \setminus \mathrm{Int}(X_i)$ consists of a single point in S, which we call x_i. Consider the equivalence relation $\sim$ on S defined by: $x \sim y$ if and only if either $x = y$ or there is $1 \leq i \leq k$ such that $\{x, y\} \subset X_i$. Then, T can be identified with the quotient space $S/\sim$. Consider the standard projection $\pi\colon S \longrightarrow T$, which is a continuous map. It is not difficult to see that π is a homeomorphism between $S \setminus \mathrm{Int}(X)$ and T. Set $w_i := \pi(x_i)$ for each $1 \leq i \leq k$. Now, for each $x \in S$, we define

$$g(x) = \begin{cases} x_i & \text{if } f \circ \pi(x) = w_i \text{ for some } 1 \leq i \leq k \\ \pi^{-1} \circ f \circ \pi(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $g\colon S \longrightarrow S$ is well defined and continuous. Observe that there are no periodic points of g in $\mathrm{Int}(X)$, because $g(\mathrm{Int}(X)) \subset S \setminus \mathrm{Int}(X)$. On the other hand, since $\pi \circ g(x) = f \circ \pi(x)$ for all $x \in S$ and π is a homeomorphism between

$S \setminus \operatorname{Int}(X)$ and T, it follows that there is a period-preserving bijection between the sets of periodic orbits of f and g. Let Q be the image of the orbit P by this bijection. Then, (S, Q, g) is a periodic model with $|Q| = |P|$. Finally we have to show that $h(g) = 0$. Since $\operatorname{Per}(g) = \operatorname{Per}(f)$, in particular $\operatorname{Pan}(g) = \operatorname{Pan}(f)$, which is contained in $A_T \cup B_T$ by Theorem 2.2, because $h(f) = 0$. Since $T \preceq S$, then $\operatorname{en}(T) \leq \operatorname{en}(S)$ and $\operatorname{ed}(T) \leq \operatorname{ed}(S)$. It follows that $A_T \subset A_S$ and $B_T \subset A_S \cup B_S$. Summarizing, $\operatorname{Pan}(g) = \operatorname{Pan}(f) \subset A_T \cup B_T \subset A_S \cup B_S$. Thus, $h(g) = 0$ by Theorem 2.2. □

To prove the next result we need to recall briefly a standard notion. We will say that a model (T, Q, f) is a *Markov model* if $V(T) \subset Q$ and f is monotone on the closure of each connected component of $T \setminus Q$. These connected components will be called *Q-basic intervals*. As usual, we can consider the *(Markov) f-graph of Q*, whose vertices are the Q-basic intervals and there is an arrow from the vertex I to the vertex J if and only if $f(I) \supset J$. If $\{I_1, I_2, \ldots, I_k\}$ is the set of Q-basic intervals, then we can construct a $k \times k$ matrix $M = (m_{ij})$, called the *transition matrix of* (T, Q, f), defined by

$$m_{ij} = \begin{cases} 1 & \text{if } f(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.2. If (T, Q, f) is a Markov model, then $h(f) = \max(0, \log(\sigma))$, where σ is the spectral radius of the transition matrix of (T, Q, f). This follows from Theorem 4.4.5 of [3] with the obvious changes (see also [4]).

Lemma 6.3. *Let S and T be trees such that $T \preceq S$. Assume that T has been obtained by collapsing one internal edge of S to a point. If there exists a Markov model $(T, \overline{P}, f)$ such that $h(f) = 0$ and $\overline{P}$ contains a periodic orbit P, then there exists a Markov model $(S, \overline{Q}, g)$ such that $h(g) = 0$ and $\overline{Q}$ contains a periodic orbit Q with $|Q| = |P|$.*

Proof. Let $[a, b]$ be the internal edge of S which has been collapsed to obtain T. Then, T can be identified with the quotient space $S/\sim$, where $\sim$ is the equivalence relation defined by: $x \sim y$ if and only if either $x = y$ or $\{x, y\} \subset [a, b]$. Consider the standard projection $\pi \colon S \longrightarrow T$, which is a continuous map. Then, $\pi([a, b])$ reduces to a point in T. Set $x = \pi([a, b])$. Observe that $a, b \in V(S)$ and that $x \in V(T) \subset \overline{P}$. It is not difficult to see that π is a homeomorphism between $S \setminus [a, b]$ and $T \setminus \{x\}$. Set $\overline{Q} := \pi^{-1}(\overline{P} \setminus \{x\}) \cup \{a, b\}$. Then, $\overline{Q} \supset V(S)$.

Next we define the map $g \colon S \longrightarrow S$ satisfying the prescribed properties. We start by defining g on a and b. If $f(x) = x$, then we define $g(a) = a$ and $g(b) = b$. Otherwise, we define $g(a) = g(b) = \pi^{-1}(f(x))$, which belongs to $\overline{Q} \setminus \{a, b\}$ since $f(x) \in \overline{P} \setminus \{x\}$. Now we define $g(z)$ for each $z \in \overline{Q} \setminus \{a, b\}$. If $f(\pi(z)) \neq x$, then we set $g(z) = \pi^{-1}(f(\pi(z)))$, which belongs to $\overline{Q} \setminus \{a, b\}$. If $f(\pi(z)) = x$, then we define $g(z) = a$ if $b \notin (z, a)$, and $g(z) = b$ otherwise.

So, we have that $g(\overline{Q}) \subset \overline{Q}$. Finally, we extend g to S by taking any piecewise monotone extension of $g\big|_{\overline{Q}}$. That is, g maps monotonically any $\overline{Q}$-basic interval $[v, w]$ onto $[g(v), g(w)]$. Then, $(S, \overline{Q}, g)$ is a Markov model, since $V(S) \subset \overline{Q}$.

Let us see that $\overline{Q}$ contains a periodic orbit Q with $|P| = |Q|$. We distinguish three cases. If $x \notin P$, then we take $Q = \pi^{-1}(P)$. If $P = \{x\}$, then $f(x) = x$ and any point in $[a, b]$ is a fixed point of g, so we take $Q = \{a\}$. In both cases, it is easy to check that Q is a periodic orbit of g and $|Q| = |P|$. Finally, let us consider the case $x \in P$ and $|P| > 1$. For $1 \leq i < |P|$, we take $z_i = \pi^{-1}(f^i(x))$. Then,

$z_i \in \overline{Q} \setminus \{a,b\}$ and $g(z_i) = z_{i+1}$ for $1 \le i < |P|-1$. If $b \in (z_{|P|-1}, a)$ then we set $z_0 = b$, otherwise we set $z_0 = a$. From the definition of g it is easy to check that $g(z_{|P|-1}) = z_0$ and $g(z_0) = z_1$, so that Q is a periodic orbit of g and $|Q| = |P|$.

Finally we have to show that $h(g) = 0$. Let $\mathcal{A}_T$ be the set of $\overline{P}$-basic intervals in T and let $\mathcal{A}_S$ be the set of $\overline{Q}$-basic intervals in S. Since π is a homeomorphism between $S \setminus [a,b]$ and $T \setminus \{x\}$, the map $\kappa\colon \mathcal{A}_S \setminus \{[a,b]\} \longrightarrow \mathcal{A}_T$ given by $\kappa([v,w]) = [\pi(v), \pi(w)]$ is a bijection. Observe that κ^{-1} acts as a bijection between the f-graph of $\overline{P}$ and a subgraph of the g-graph of $\overline{Q}$, sending each loop $I_0 \to I_1 \to \ldots \to I_n \to I_0$ in the f-graph of $\overline{P}$ to $\kappa^{-1}(I_0) \to \kappa^{-1}(I_1) \to \ldots \to \kappa^{-1}(I_n) \to \kappa^{-1}(I_0)$. On the other hand, since either g is the identity map on $[a,b]$ or $g([a,b])$ collapses to a point, there are no other loops in the g-graph of $\overline{Q}$ except, perhaps, the loop $[a,b] \to [a,b]$. Let M and N be the transition matrices of the f-graph of $\overline{P}$ and the g-graph of $\overline{Q}$, respectively. Denote by $\operatorname{tr}(\cdot)$ the trace of a matrix (that is, the sum of all the entries from its diagonal). From Lemma 4.4.1 of [3], it follows that, for each $n \ge 1$, $\operatorname{tr}(M^n) \le \operatorname{tr}(N^n) \le \operatorname{tr}(M^n) + 1$. Then, by Lemma 4.4.2 of [3], the spectral radius of N equals that of M. Then, by Remark 6.2, $h(g) = h(f) = 0$. □

Proposition 6.4. *Let (T, P, f) be a periodic simplicial model with $h(f) = 0$ and let S be a tree such that $T \preceq S$. Then there exists a periodic model (S, Q, g) such that $h(g) = 0$ and $|Q| = |P|$.*

Proof. The result follows easily if T is a point. So, from now on we assume that T is a proper tree. Let $k \ge 0$ be the number of edges of S which have been collapsed to obtain T. If $k = 0$ then S and T are homeomorphic and the proposition follows trivially, so we assume $k \ge 1$. We remark that if e is an external edge of a tree X and we obtain a new tree $X' \preceq X$ by collapsing one internal edge of X to a point, then e keeps being an external edge of X'. On the other hand, $(T, P \cup V(T), f)$ is a Markov model. Therefore, if l is the number of internal edges which have been collapsed to obtain T from S, with $0 \le l \le k$, the proposition follows by iteratively applying Lemma 6.3 l times and, finally, by applying Lemma 6.1. □

Now we are ready to prove Theorem B.

Proof of Theorem B. Let $p \in \operatorname{Pan}(T)$. Then, there exists a map $f\colon T \longrightarrow T$ and a periodic orbit P of f such that $h(f) = 0$ and $|P| = 2^k p$ for some $k \ge 0$. From Theorem 4.3, there exists a map $g\colon \langle P \rangle_T \longrightarrow \langle P \rangle_T$ such that $h(g) = 0$, $(\langle P \rangle_T, P, g)$ is a simplicial model and $g\big|_P = f\big|_P$. Since $\langle P \rangle_T \preceq T$, then $\langle P \rangle_T \preceq S$ and, by using Proposition 6.4, we get that $p \in \operatorname{Pan}(S)$. □

References

[1] R. Adler, A. Konheim and M. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.* **114**, pp. 309–319, 1965.
[2] L. Alsedà, D. Juher and P. Mumbrú. Periodic behavior on trees. *Ergod. Th. & Dynam. Sys.* **25**, pp. 1373–1400, 2005.
[3] L. Alsedà, J. Llibre and M. Misiurewicz. *Combinatorial dynamics and entropy in dimension one*, volume 5 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
[4] Ll. Alsedà, F. Mañosas, P. Mumbrú. Minimizing topological entropy for continuous maps on graphs. *Ergod. Th. & Dynam. Sys.* **20**(6), pp. 1559-1576, 2000.
[5] L. Alsedà and X. Ye. No division and the set of periods for tree maps. *Ergod. Th. & Dynam. Sys.* **15**, pp. 221–237, 1995.

[6] S. Baldwin. An extension of Šarkovskiĭ's theorem to the n-od. *Ergod. Th. & Dynam. Sys.* **11**(2), pp. 249–271, 1991.

[7] E. Barrabés, D. Juher. The minimum tree for a given zero entropy period. *Int. J. Math. Math. Sci.* **19**, pp. 3025–3033, 2005.

[8] F. Blanchard, E. Glasner, S. Kolyada and A Maas. On Li-Yorke pairs. *J. Reine Angew. Math.* **547**, pp. 51–68, 2002.

[9] A.M. Blokh. Periods implying almost all periods for tree maps. *Nonlinearity* **5**(6), pp. 1375–1382, 1992.

[10] T.-Y. Li and J. Yorke. Period three implies chaos. *American Mathematical Monthly* **82**, pp. 985–992, 1975.

[11] J. Llibre and M. Misiurewicz. Horseshoes, entropy and periods for graph maps. *Topology* **32**, pp. 649–664, 1993.

[12] A. N. Sharkovsky. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.* **16**, pp. 61–71, 1964. English translation: Proceedings of the Conference "Thirty Years after Sharkovsky's theorem: New perspectives" (Murcia, 1994). *Internat. J. Bifur. Chaos* **5**, pp. 1263–1273, 1995.

Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Lluís Santaló s/n, 17071 Girona, Spain
E-mail address: barrabes@ima.udg.es

Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Lluís Santaló s/n, 17071 Girona, Spain
E-mail address: juher@ima.udg.es