Hip–Hop solutions of the $2N$–Body problem

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Abstract. Hip–Hop solutions of the $2N$–body problem with equal masses are shown to exist using an analytic continuation argument. These solutions are close to planar regular $2N$–gon relative equilibria with small vertical oscillations. For fixed N, an infinity of these solutions are three–dimensional choreographies, with all the bodies moving along the same closed curve in the inertial frame.

Keywords: N-body problem, analytic continuation, Hip–Hop, Choreographies

1. Introduction

The equal–mass n–body problem has recently attracted much attention thanks to the work of Chenciner and other authors on the type of orbits called hip-hop solutions, and on the solutions that have eventually been called choreographies.

In a hip-hop solution, $2N$ bodies of equal mass stay for all time in the vertices of a regular rotating anti-prism whose basis, i.e. the regular polygons that define it, perform an oscillatory motion separating, reaching a maximum distance, approaching, crossing each other, and so on, as sketched in Figure 1 for $N = 3$. The $2N$ bodies can be arranged in two groups of N, each group moving on its plane on a rotating regular N–gon configuration (homographic), while the planes are always perpendicular to the z-axis, oscillate along this axis, and coincide (with opposite velocities) at regular intervals when they cross the origin. The orthogonal projection of both N–gons on the $z = 0$ plane is always a regular rotating $2N$–gon.

On the other hand, a choreography is a solution in which n bodies move along the same closed line in the inertial frame, chasing each other at equi–spaced intervals of time. It is well known the 'figure–eight' choreography in the three–body problem, shown by Chenciner and Montgomery (2002) in a most celebrated paper. A great many choreographies with $n > 3$ have been shown numerically to exist by Simo (2001).

The above results were obtained mostly by means of variational methods, which make it possible to find solutions that do not depend on a small parameter, i.e. far from solutions of an integrable problem. See (Chenciner and Venturelli, 2000), (Chenciner et al., 2002), (Chenciner and Féjoz, 2005), and references therein for details. In the case of hip–hop solutions, the question arises whether in some simple cases they could be obtained through the traditional analytic continuation method of Poincaré, which would give families (differentiable with respect to a parameter) of periodic solutions, at least in a rotating frame. In this respect, mention should be made of a result by Meyer and Schmidt (1993) on a similar solution with a large central mass and $2n$ very small, equal masses around it which was suggested as a model for the braided structure of some of Saturn's rings.

In this paper we show that Poincare's argument of analytic continuation can be used to add vertical oscillations to the circular motion of $2N$ bodies of equal mass occupying the vertices of a regular $2N$-gon. In this way, a family of three-dimensional orbits, periodic in a rotating frame, can be shown to exist. This is a Lyapunov family of orbits whose periods tend to the period of the vertical oscillations of the linearized system around the relative equilibrium solution. These solutions were found numerically by Davies et al. 1983. Infinitely many of this solutions are periodic in the inertial frame, provided that the quantity $H(N)$ given by (27) does not vanish, and are three–dimensional choreographies, in the sense that all bodies move at equi–spaced time intervals along a closed twisted curve in the inertial frame.

Some solutions found in our article may coincide with the generalized hip–hop solutions obtained by Chenciner in (2002). Terracini and Venturelli (2005) recently showed the existence of hip–hop solutions in the same problem using variational methods, adding vertical variations to the planar relative equilibrium in order to reduce the value of the action functional. The variational approach does not depend on any small parameter and yields global existence, while continuation methods give explicit approximations to solutions in a small neighbourhood of the relative equilibrium. A precise comparison of both methods from a purely analytic point of view would involve either estimating the distance from the variational solutions to the relative equilibrium or estimating the size of the neighbourhood in which the family can be continued, but both questions seem far from easy.

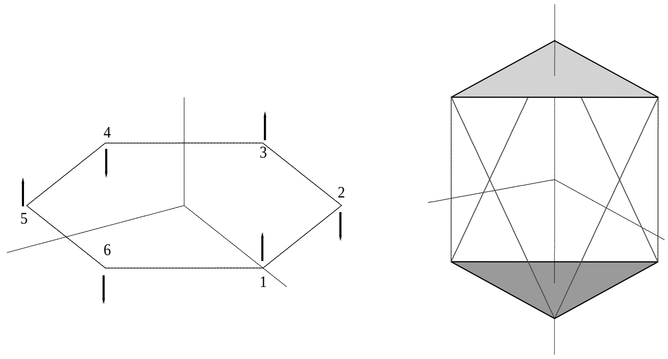

Figure 1. Qualitative representation of a Hip-Hop motion in the case of 6 bodies

2. Equations of motion

Consider $2N$ bodies with equal mass m moving under their mutual gravitational attraction and let $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$, $i = 1, \cdots, 2N$, be their positions and velocities. The equations of motion of the $2N$–body problem are

$$\ddot{\mathbf{r}}_i = Gm \sum_{k=1, k\neq i}^{2N} \frac{\mathbf{r}_k - \mathbf{r}_i}{r_{ki}^3}, \tag{1}$$

where $r_{ki} = |\mathbf{r}_k - \mathbf{r}_i|$. Scaling the time t by $\sqrt{Gm}\, t$ the Lagrangian function associated to the problem becomes

$$\mathcal{L} = \sum_{i=1}^{2N} \frac{1}{2}|\dot{\mathbf{r}}_i|^2 + \sum_{1\leq i<j\leq 2N} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

As we are looking for solutions of the $2N$–body problem such that all the bodies stay for all time on the vertices of an anti-prism, it will suffice to know the position and velocity of one of the $2N$ bodies. Given $\mathbf{r}_1 = \mathbf{r}_1(t)$, we define

$$\mathbf{r}_i = R^{i-1}\mathbf{r}_1, \qquad \dot{\mathbf{r}}_i = R^{i-1}\dot{\mathbf{r}}_1,$$

for $i = 2, \cdots, 2N$, where R is a rotation plus a reflection in such a way that all the bodies are on the vertices of an anti-prism. Since in this configuration N of the bodies on a plane and the other N on a parallel plane we can assume, without loss of generality, that both planes are perpendicular to the z axis. In this case, the matrix R can be written as

$$R = \begin{pmatrix} \cos(\frac{\pi}{N}) & -\sin(\frac{\pi}{N}) & 0 \\ \sin(\frac{\pi}{N}) & \cos(\frac{\pi}{N}) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

PROPOSITION 1. $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2N}) = (\mathbf{r}, R\mathbf{r}, \ldots, R^{2N-1}\mathbf{r})$ *is a solution of the* $2N$*–body problem given by (1) if and only if* $\mathbf{r}(t)$ *satisfies the equation*

$$\ddot{\mathbf{r}} = \sum_{k=1}^{2N-1} \frac{(R^k - I)\mathbf{r}}{|(R^k - I)\mathbf{r}|^3}. \qquad (2)$$

Proof. Substituting $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2N}) = (\mathbf{r}_1, R\mathbf{r}_1, \ldots, R^{2N-1}\mathbf{r}_1)$ in (1) we have

$$R^{i-1}\ddot{\mathbf{r}}_1 = \sum_{j=1, j\neq i}^{2N} \frac{(R^{j-1} - R^{i-1})\mathbf{r}_1}{|(R^{j-1} - R^{i-1})\mathbf{r}_1|^3} = \sum_{j=1, j\neq i}^{2N} \frac{R^{i-1}(R^{j-i} - I)\mathbf{r}_1}{|(R^{j-i} - I)\mathbf{r}_1|^3},$$

for $i = 1, \ldots, 2N$. Using the fact that $R^{-l} = R^{2N-l}$, for $l = 1, \ldots, 2N$, we get

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \sum_{j=1}^{i-1} \frac{(R^{j-i} - I)\mathbf{r}_1}{|(R^{j-i} - I)\mathbf{r}_1|^3} + \sum_{j=i+1}^{2N} \frac{(R^{j-i} - I)\mathbf{r}_1}{|(R^{j-i} - I)\mathbf{r}_1|^3} \\ &= \sum_{l=1}^{i-1} \frac{(R^{2N-l} - I)\mathbf{r}_1}{|(R^{2N-l} - I)\mathbf{r}_1|^3} + \sum_{l=1}^{2N-i} \frac{(R^{l} - I)\mathbf{r}_1}{|(R^{l} - I)\mathbf{r}_1|^3} \\ &= \sum_{k=1}^{2N-1} \frac{(R^{k} - I)\mathbf{r}_1}{|(R^{k} - I)\mathbf{r}_1|^3}. \end{aligned}$$

That is, we get the same equation from the initial $2N$ equations (1). □

If $\mathbf{r} = (x, y, z)$ is the position of the first body, then the equations of motion (2) can be written as the following differential system of order two

$$\ddot{x} = \frac{\partial U(x,y,z)}{\partial x}, \quad \ddot{y} = \frac{\partial U(x,y,z)}{\partial y}, \quad \ddot{z} = \frac{\partial U(x,y,z)}{\partial z}, \qquad (3)$$

where $U(x, y, z)$ is the potential function

$$U(x,y,z) = \frac{1}{2} \sum_{k=1}^{2N-1} \frac{1}{\sqrt{4(x^2+y^2)\sin^2\left(\frac{k\pi}{2N}\right) + ((-1)^k - 1)^2 z^2}}.$$

The problem stated by system (3) can be formulated in Hamiltonian terms by the Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - U(x,y,z), \qquad (4)$$

where p_x, p_y and p_z are the momenta associated to the x, y, z coordinates. Introducing cylindrical coordinates by means of the canonical change

$$\begin{array}{rclrcl} x & = & r\cos\theta, & p_x & = & p_r\cos\theta - \dfrac{p_\theta}{r}\sin\theta, \\ y & = & r\sin\theta, & p_y & = & p_r\sin\theta + \dfrac{p_\theta}{r}\cos\theta, \\ z & = & d, & p_z & = & p_d, \end{array}$$

the Hamiltonian (4) becomes

$$\mathcal{H} = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2} + p_d^2) - \frac{1}{2}\sum_{k=1}^{2N-1} \frac{1}{\sqrt{4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2}}. \quad (5)$$

Then the equations of motion (2) for the first body are

$$\begin{array}{rcl} \dot{r} & = & p_r, \\ \dot{\theta} & = & \dfrac{p_\theta}{r^2}, \\ \dot{d} & = & p_d, \\ \dot{p_r} & = & \dfrac{p_\theta^2}{r^3} - 2r\displaystyle\sum_{k=1}^{2N-1} \frac{\sin^2\left(\frac{k\pi}{2N}\right)}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}, \\ \dot{p_\theta} & = & 0, \\ \dot{p_d} & = & -\dfrac{d}{2}\displaystyle\sum_{k=1}^{2N-1} \frac{\left((-1)^k - 1\right)^2}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}. \end{array} \quad (6)$$

Since $\dot{p_\theta} = 0$, the angular momentum $p_\theta = \Theta$ is constant and can be calculated from the initial conditions. Then, once r is obtained, we will get θ from the second equation in (6).

3. Symmetric periodic solutions of the reduced problem

We call *reduced problem* the problem given by considering in (6) only the equations for the variables r and d, and *complete problem* the whole set of equations (6). Our aim is to find periodic solutions of this reduced problem. This solutions will be, in general, quasi–periodic solutions of the complete problem.

Consider the problem posed by the Hamiltonian (5) for a fixed value of the angular momentum $p_\theta = \Theta$. The equations of motion of the reduced problem are

$$
\begin{aligned}
\dot r &= p_r,\\
\dot d &= p_d,\\
\dot{p_r} &= \frac{\Theta^2}{r^3} - 2r\sum_{k=1}^{2N-1}\frac{\sin^2\left(\frac{k\pi}{2N}\right)}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right)+\left((-1)^k-1\right)^2 d^2\right)^{3/2}},\\
\dot{p_d} &= -\frac{d}{2}\sum_{k=1}^{2N-1}\frac{\left((-1)^k-1\right)^2}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right)+\left((-1)^k-1\right)^2 d^2\right)^{3/2}},
\end{aligned}
\tag{7}
$$

which has a unique equilibrium point $(r, d, p_r, p_d) = (a, 0, 0, 0)$, where $a = \Theta^2/K_N^2$ and

$$K_N^2 = \frac{1}{4}\sum_{k=1}^{2N-1}\frac{1}{\sin\left(\frac{k\pi}{2N}\right)}. \tag{8}$$

The matrix of the linearized equations around this equilibrium point is

$$M = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\lambda_1^2 & 0 & 0 & 0\\ 0 & -\lambda_2^2 & 0 & 0 \end{pmatrix},$$

where

$$
\begin{aligned}
-\lambda_1^2 &= \frac{-3\Theta^2}{a^4} + \frac{2K_N^2}{a^3} = -\frac{K_N^8}{\Theta^6},\\
-\lambda_2^2 &= -\frac{1}{16a^3}\sum_{k=1}^{2N-1}\frac{((-1)^k-1)^2}{\sin^3\left(\frac{k\pi}{2N}\right)} = -\frac{S_N^2K_N^6}{\Theta^6},
\end{aligned}
$$

and

$$S_N^2 = \frac{1}{16}\sum_{k=1}^{2N-1}\frac{((-1)^k-1)^2}{\sin^3\left(\frac{k\pi}{2N}\right)}. \tag{9}$$

The matrix M has two pairs of imaginary eigenvalues $\pm i\lambda_1$ and $\pm i\lambda_2$. By Lyapunov's center theorem, (Meyer and Hall, 1992), there exist two one–parameter families of periodic solutions, emanating from the equilibrium point provided that λ_2/λ_1 is not a rational number. That is, it suffices to ensure that

$$\left(\frac{\lambda_2}{\lambda_1}\right)^2 = \frac{1}{4}\frac{\displaystyle\sum_{k=1}^{2N-1}\frac{((-1)^k-1)^2}{\sin^3\left(\frac{k\pi}{2N}\right)}}{\displaystyle\sum_{k=1}^{2N-1}\frac{1}{\sin\left(\frac{k\pi}{2N}\right)}} \tag{10}$$

Table I. Values of $(\lambda_2/\lambda_1)^2$ for different values of N

N	$(\lambda_2/\lambda_1)^2$
2	$\frac{4(4-\sqrt{2})}{7}$
3	$\frac{51}{15+4\sqrt{3}}$
5	$\frac{160+\sqrt{5}}{20+\sqrt{5}+4\sqrt{2+\sqrt{5}}}$
6	$\frac{12(\sqrt{2}+12\sqrt{6})}{15+6\sqrt{2}+4\sqrt{3}+12\sqrt{6}}$

is not the square of a rational number. A number of values of this expression for different values of N are shown on Table I.

The $d = p_d = 0$ plane is invariant and the r-mode solutions lie in this plane with periods approaching $2\pi/\lambda_1$. These are actually the homographic solutions near the relative equilibrium. The d-mode solutions are three–dimensional orbits whose periods tend to $2\pi/\lambda_2$ and this is the family we are interested in.

The equations of motion of the reduced problem (7) are invariant by the symmetries

$$\begin{aligned}\mathcal{S}_1 : (t, r, d, p_r, p_d) &\longrightarrow (-t, r, -d, -p_r, p_d),\\ \mathcal{S}_2 : (t, r, d, p_r, p_d) &\longrightarrow (-t, r, d, -p_r, -p_d),\end{aligned}$$

and we have the following well known proposition.

PROPOSITION 2. *Let $\mathbf{q}(t) = (r(t), d(t), p_r(t), p_d(t))$ be a solution of the equations (7). If $\mathbf{q}(t)$ satisfies that $(d(0), p_r(0)) = (0, 0)$ and $(p_r(T), p_d(T)) = (0, 0)$, then $\mathbf{q}(t)$ is a doubly symmetric periodic solution of period $4T$.*

We will show the existence of doubly symmetric periodic orbits in system (7).

Let $\mathbf{q}_0 = (r_0, d_0, p_{r0}, p_{d0}) = (r_0, 0, p_{r0}, 0)$. The solution of system (7) with these initial conditions is given by $d(t) = 0$, for all t, together with any solution $r(t)$ of the Kepler problem

$$\ddot{r} = \frac{\Theta^2}{r^3} - \frac{K_N^2}{r^2},$$

with K_N^2 given by (8). As is well known, its solutions can be written as

$$r(t) = a(1 - e\cos E(t)),$$

where a is the semimajor axis, e the eccentricity and E the eccentric anomaly. The function $r(t)$ is periodic of period $T = 2\pi a^{3/2}/K_N = 2\pi/\lambda_1$ and $a(1-e^2)K_N^2 = \Theta^2$. These solutions will be called 'planar' as opposed to the 'spatial' or 'three–dimensional' solutions when $d(t)$ (the distance from the first body $\mathbf{r}_1$ to the $z=0$ plane) is not identically zero.

In order to obtain periodic solutions of the reduced problem, we add perturbations to periodic planar orbits in the vertical direction. If the perturbation is small enough, the motion can be decoupled into a planar plus a vertical motion in a first approximation. Substituting d by εd on the equations (7), and keeping terms in ϵ^2, we obtain

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{d} &= p_d, \\ \dot{p}_r &= \frac{\Theta^2}{r^3} - \frac{K_N^2}{r^2} + \frac{3S_N^2}{2r^4}\, d^2\epsilon^2 + O(\epsilon^4), \\ \dot{p}_d &= -\frac{S_N^2}{r^3}\, d + \frac{3W_N^2}{2r^5}\, d^3\epsilon^2 + O(\epsilon^4), \end{aligned} \tag{11}$$

where

$$W_N^2 = \frac{1}{64} \sum_{k=1}^{2N-1} \frac{((-1)^k - 1)^4}{\sin^5\left(\frac{k\pi}{2N}\right)}. \tag{12}$$

System (11) can then be written as

$$\dot{\mathbf{q}} = \mathcal{F}(\mathbf{q},\epsilon) = \mathcal{F}_0(\mathbf{q}) + \epsilon^2 \mathcal{F}_1(\mathbf{q}) + \mathcal{O}(\epsilon^4) \tag{13}$$

where

$$\mathcal{F}_0(\mathbf{q}) = (p_r, p_d, \frac{\Theta^2}{r^3} - \frac{K_N^2}{r^2}, -\frac{S_N^2}{r^3}\, d), \qquad \mathcal{F}_1(\mathbf{q}) = (0, 0, \frac{3S_N^2}{2r^4}\, d^2, \frac{3W_N^2}{2r^5}\, d^3).$$

Let $\mathbf{q}_0$ be a vector of initial conditions. The solution of (13) with initial value $\mathbf{q}_0$ at $t=0$ can be expanded as a power series in ϵ^2 as

$$\mathbf{q}(t, \mathbf{q}_0, \epsilon) = \mathbf{q}^{(0)}(t, \mathbf{q}_0) + \epsilon^2 \mathbf{q}^{(1)}(t, \mathbf{q}_0) + \mathcal{O}(\epsilon^4),$$

where $\mathbf{q}^{(0)}(t, \mathbf{q}_0)$ is the solution of the unperturbed problem

$$\dot{\mathbf{q}}^{(0)}(t, \mathbf{q}_0) = \mathcal{F}_0(\mathbf{q}^{(0)}) \tag{14}$$

with initial conditions $\mathbf{q}_0$, and $\mathbf{q}^{(1)}(t, \mathbf{q}_0)$ is the solution of

$$\dot{\mathbf{q}}^{(1)}(t, \mathbf{q}_0) = \mathcal{F}_1(\mathbf{q}^{(0)}(t, \mathbf{q}_0)) + \mathcal{D}\mathcal{F}_0(\mathbf{q}^{(0)}(t, \mathbf{q}_0))\mathbf{q}^{(1)}(t, \mathbf{q}_0)$$

with initial conditions $\mathbf{q}^{(1)}(0, \mathbf{q}_0) = 0$. The entries of the matrix $\mathcal{DF}$ are the partial derivatives of $\mathcal{F}$ with respect to the $\mathbf{q}$ variable, and by the formula of Lagrange we have

$$\mathbf{q}^{(1)}(t,\mathbf{q}_0) = \mathcal{Q}(t,\mathbf{q}_0)\int_0^t \mathcal{Q}^{-1}(\tau,\mathbf{q}_0)\mathcal{F}_1(\mathbf{q}^{(0)}(\tau,\mathbf{q}_0))d\tau, \tag{15}$$

where

$$\mathcal{Q}(t,\mathbf{q}_0) = \left.\frac{\partial \mathbf{q}^{(0)}(t,\xi)}{\partial \xi}\right|_{\xi=\mathbf{q}_0} \tag{16}$$

THEOREM 1. *Let* $T_0^* = \frac{2k+1}{2}\frac{\pi}{\lambda_2}$, $a = \frac{\Theta^2}{K_N^2}$ *and* $\mathbf{q}_0^* = (a,0,0,p_{d0})$. *Assume that* λ_2/λ_1 *given by (10) is not a rational number. Then there exist* Δr_0, ΔT *such that the solution* $\mathbf{q}(t,\mathbf{q}_0,\epsilon)$ *of system (11) with initial conditions* $\mathbf{q}_0 = (a+\Delta r_0,0,0,p_{d0})$ *is a doubly symmetric periodic solution of period* $4(T_0^*+\Delta T)$.

The functions Δr_0 *and* ΔT *are given by*

$$\Delta r_0 = -\epsilon^2 \frac{3\, p_{d0}^2\, S_n^2}{a^4\,\lambda_1\,(\lambda_1^2 - 4\,\lambda_2^2)} + O(\epsilon^4)$$

$$\Delta T = \epsilon^2\, \frac{9\, p_{d0}^2}{a^5\,{\lambda_1}^3\left({\lambda_1}^2 - 4\,{\lambda_2}^2\right)^2}\left(B_1(k,N) + \frac{1}{32\,{\lambda_2}^5} B_2(k,N)\right) + O(\epsilon^4)$$

where

$$\begin{aligned} B_1(k,N) &= -\,{S_N}^2\left({\lambda_1}^2 - 2\,{\lambda_2}^2\right)\sin(\tfrac{(1+2\,k)\,\pi\,\lambda_1}{2\,\lambda_2}),\\ B_2(k,N) &= (1+2\,k)\,\pi\,{W_N}^2\,{\lambda_1}^3\left({\lambda_1}^2 - 4\,{\lambda_2}^2\right)^2 + {S_N}^2\,{\lambda_2}^2\,B_{22}, \end{aligned} \tag{17}$$

and

$$B_{22}(k,N) = \lambda_1\pi(1+2k)(\lambda_1^2-4\lambda_2^2)(3\lambda_1^2-8\lambda_2^2)+32\lambda_2^3(\lambda_1^2-2\lambda_2^2)\sin\left(\frac{(1+2k)\pi\,\lambda_1}{2\lambda_2}\right).$$

Note that for $\epsilon \to 0$ the periods of the solutions given by the theorem tend to $2\pi/\lambda_2$ and they belong to a symmetric Lyapunov family.

Proof. Notice that the solution of the unperturbed problem (14) with initial condition $\mathbf{q}_0^*$ is

$$\mathbf{q}^{(0)}(t,\mathbf{q}_0^*) = (a, p_{d0}\frac{1}{\lambda_2}\sin(\lambda_2 t), 0, p_{d0}\cos(\lambda_2 t)). \tag{18}$$

This solution satisfies

$$\mathbf{q}^{(0)}(T_0^*,\mathbf{q}_0^*) = (a, p_{d0}\frac{(-1)^k}{\lambda_2}, 0, 0), \tag{19}$$

and $\mathbf{q}^{(0)}(t,\mathbf{q}_0^*)$ is a doubly symmetric periodic solution of the unperturbed system of period $4T_0^*$.

We must find initial conditions $\mathbf{q}_0 = (a + \Delta r_0, 0, 0, p_{d0})$ and $T = T_0^* + \Delta T$ such that the solution $\mathbf{q}(t, \mathbf{q}_0, \epsilon)$ of system (11) satisfies

$$\begin{cases} p_r(T_0^* + \Delta T, \mathbf{q}_0, \epsilon) = p_r^{(0)}(T_0^* + \Delta T, \mathbf{q}_0) + \epsilon^2 p_r^{(1)}(T_0^* + \Delta T, \mathbf{q}_0) + O(\epsilon^4) = 0 \\ p_d(T_0^* + \Delta T, \mathbf{q}_0, \epsilon) = p_d^{(0)}(T_0^* + \Delta T, \mathbf{q}_0) + \epsilon^2 p_d^{(1)}(T_0^* + \Delta T, \mathbf{q}_0) + O(\epsilon^4) = 0 \end{cases} \tag{20}$$

By Proposition 2, $\mathbf{q}(t, \mathbf{q}_0, \epsilon)$ will be a doubly symmetric periodic solution of period $4(T_0^* + \Delta T)$.

For a fixed value p_{d0}, and $k = 0, 1$, we consider $p_r^{(k)}(T_0^* + \Delta T, \mathbf{q}_0)$ and $p_d^{(k)}(T_0^* + \Delta T, \mathbf{q}_0)$ as functions of $(\Delta T, \Delta r_0)$. Expanding (20) as power series in the Δ's we get

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = & \begin{pmatrix} p_r^{(0)}(T_0^*, \mathbf{q}_0^*) \\ p_d^{(0)}(T_0^*, \mathbf{q}_0^*) \end{pmatrix} + \begin{pmatrix} \frac{\partial p_r^{(0)}}{\partial t} & \frac{\partial p_r^{(0)}}{\partial r} \\ \frac{\partial p_d^{(0)}}{\partial t} & \frac{\partial p_d^{(0)}}{\partial r} \end{pmatrix}_{(T_0^*, \mathbf{q}_0^*)} \begin{pmatrix} \Delta T \\ \Delta r_0 \end{pmatrix} + O_2(\Delta T, \Delta r_0) + \\ & + \epsilon^2 \left[\begin{pmatrix} p_r^{(1)}(T_0^*, \mathbf{q}_0^*) \\ p_d^{(1)}(T_0^*, \mathbf{q}_0^*) \end{pmatrix} + \begin{pmatrix} \frac{\partial p_r^{(1)}}{\partial t} & \frac{\partial p_r^{(1)}}{\partial r} \\ \frac{\partial p_d^{(1)}}{\partial t} & \frac{\partial p_d^{(1)}}{\partial r} \end{pmatrix}_{(T_0^*, \mathbf{q}_0^*)} \begin{pmatrix} \Delta T \\ \Delta r_0 \end{pmatrix} + O_2(\Delta T, \Delta r_0) \right] \\ & + O(\epsilon^4) \end{aligned}$$

Now we have from (19) that

$$\begin{pmatrix} p_r^{(0)}(T_0^*, \mathbf{q}_0^*) \\ p_d^{(0)}(T_0^*, \mathbf{q}_0^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that if $\begin{vmatrix} \frac{\partial p_r^{(0)}}{\partial t} & \frac{\partial p_r^{(0)}}{\partial r} \\ \frac{\partial p_d^{(0)}}{\partial t} & \frac{\partial p_d^{(0)}}{\partial r} \end{vmatrix}_{(T_0^*, \mathbf{q}_0^*)} \neq 0$, the system (20) can be solved for $(\Delta T, \Delta r_0)$ in a neigbourhood of $(0, 0)$ by means of the implicit function theorem. An approximation to $(\Delta T, \Delta r_0)$ can be easily computed from

$$\begin{pmatrix} \Delta T \\ \Delta r_0 \end{pmatrix} = -\epsilon^2 \left[\begin{pmatrix} \frac{\partial p_r^{(0)}}{\partial t} & \frac{\partial p_r^{(0)}}{\partial r} \\ \frac{\partial p_d^{(0)}}{\partial t} & \frac{\partial p_d^{(0)}}{\partial r} \end{pmatrix}^{-1}_{(T_0^*, \mathbf{q}_0^*)} \begin{pmatrix} p_r^{(1)}(T_0^*, \mathbf{q}_0^*) \\ p_d^{(1)}(T_0^*, \mathbf{q}_0^*) \end{pmatrix} + O_2(\Delta T, \Delta r_0) \right] + O(\epsilon^4) \tag{21}$$

The functions $\frac{\partial p_r^{(0)}}{\partial t}, \frac{\partial p_d^{(0)}}{\partial t}$ can be computed from (18). In order to get the terms $\frac{\partial p_r^{(0)}}{\partial r}, \frac{\partial p_d^{(0)}}{\partial r}$ we must compute (16). Then

$$\begin{pmatrix} \frac{\partial p_r^{(0)}}{\partial t} & \frac{\partial p_r^{(0)}}{\partial r} \\ \frac{\partial p_d^{(0)}}{\partial t} & \frac{\partial p_d^{(0)}}{\partial r} \end{pmatrix}_{(T_0^*, \mathbf{q}_0^*)} = \begin{pmatrix} 0 & -\lambda_1 \sin(\frac{2k+1}{2}\frac{\lambda_1}{\lambda_2}\pi) \\ -p_{d0}\lambda_2(-1)^k & \frac{3\, p_{d0}\, \lambda_2\, (-1)^k\, \sin\left(\frac{(1+2k)\,\pi\,\lambda_1}{2\,\lambda_2}\right)(\lambda_1^2 - 2\lambda_2^2)}{a\,\lambda_1\left(\lambda_1{}^2 - 4\,\lambda_2{}^2\right)} \end{pmatrix} \tag{22}$$

and $p_r^{(1)}(T_0^*, \mathbf{q}_0^*)$, $p_d^{(1)}(T_0^*, \mathbf{q}_0^*)$ are the last two components of

$$\mathbf{q}^{(1)}(T_0^*, \mathbf{q}_0^*) = \mathcal{Q}(T_0^*, \mathbf{q}_0^*) \int_0^{T_0^*} \mathcal{Q}^{-1}(\tau, \mathbf{q}_0)\mathcal{F}_1(\mathbf{q}^{(0)}(\tau, \mathbf{q}_0))d\tau,$$

Table II. Numerical values of $(\Delta T, \Delta r_0)$ for $\Theta = 1$, $N = 3$ and $k = 0$.

	$p_{d0} = 2$	$p_{d0} = 1$	$p_{d0} = 0.5$
$\epsilon = 0.03$	$(0.000660, 0.000496)$	$(0.000164, 0.000123)$	$(0.000041, 0.000030)$
$\epsilon = 0.06$	$(0.002661, 0.001997)$	$(0.000660, 0.000496)$	$(0.000164, 0.000123)$
$\epsilon = 0.09$	$(0.006061, 0.004534)$	$(0.001490, 0.001119)$	$(0.000371, 0.000279)$
$\epsilon = 0.12$	$(0.010963, 0.008166)$	$(0.002661, 0.001997)$	$(0.000660, 0.000496)$

which can be easily computed and are given by

$$\begin{aligned} p_r^{(1)}(z_0^*) &= \frac{-3\, p_{d0}^2 S_N^2}{a^4\, \lambda_1\, \left(\lambda_1^2 - 4\,\lambda_2^2\right)} \sin\left(\frac{(1+2k)\pi}{2}\, \frac{\lambda_1}{\lambda_2}\right), \qquad (23) \\ p_d^{(1)}(z_0^*) &= \frac{9\,(-1)^k\, p_{d0}^3}{32\, a^5\, {\lambda_1}^3\, {\lambda_2}^4 \left({\lambda_1}^2 - 4\,{\lambda_2}^2\right)^2}\, B_2, (k, N) \end{aligned}$$

where $B_2(k, N)$ is given by (17). Finally we can substitute (23) and (22) in (21) and we obtain the approximation to $(\Delta T, \Delta r_0)$. ■

Theorem 1 gives an approximation to initial conditions for doubly symmetric periodic orbits for ϵ sufficiently small. This results have been checked numerically and a good agreement has been obtained.

For fixed values of N, k, Θ, p_{d0} and ϵ, we compute numerically the values (T, r) near to (T_0^*, a) such that the orbit with initial conditions $q = (r, 0, 0, p_{d0})$ is a doubly symmetric periodic orbit of period $4T$. The integration of the differential equations has been done by means of a Runge-Kutta RK78 algorithm. Then we compute $\Delta T = T - T_0^*$ and $\Delta r_0 = r - a$. Table II shows the numerically computed values of $(\Delta T, \Delta r_0)$, for $\Theta = 1$ and different values of ϵ.

4. Hip–Hop periodic orbits and choreographies

The question whether orbits which are periodic in the reduced system are periodic also in the inertial frame is of course only a question of commensurability between π and the angle rotated in the inertial system in a period. If this angle can be seen to change along the family of periodic solutions, then there will exist infinitely many periodic solutions in the inertial system (whenever its value is commensurable with π). It suffices then to see that its derivative with respect to ϵ^2 is different from zero for $\epsilon = 0$.

Now, a small variation on this simple argument shows that there exist infinitely many choreographies as well. Think of the orbit as having period $4T$, where T is the time spent from the initial planar position as a regular $2N$–gon to the maximum separation of the planes containing the N–gon configurations. After a time $2T$, the N bodies that at $t = 0$ where thrown upwards will hit the initial plane with a velocity symmetric to the initial one, which is exactly the initial velocity of the other N bodies, which were thrown downwards. If at $t = 2T$ the position of N bodies is the same as the position at $t = 0$ of the other N bodies, they will follow the same path, so we have the kind of motion that has been called a choreography.

We give an outline of the computation of the derivative. As we have seen in the previous Section, for small values of ϵ we can obtain periodic solutions $\mathbf{q}(t, \mathbf{q}_0, \epsilon)$ of the reduced problem for initial conditions $\mathbf{q}_0 = (a + \Delta r_0, 0, 0, p_{d0})$ and period $4T = 4T_0^* + 4\Delta T$. For a fixed value p_{d0}, the function $r(t)$ is given by

$$\begin{aligned} r(t, \mathbf{q}_0, \epsilon) &= \left[r^{(0)}(t, a) + \left(\frac{\partial r^{(0)}}{\partial r} \right)_{\mathbf{q}_0^*} \Delta r_0 + O_2(\Delta r_0) \right] + \\ &\quad + \epsilon^2 \left[r^{(1)}(t, a) + \left(\frac{\partial r^{(1)}}{\partial r} \right)_{\mathbf{q}_0^*} \Delta r_0 + O_2(\Delta r_0) \right] + O(\epsilon^4) \\ &= a + \cos(\lambda_1 t)\Delta r_0 + \epsilon^2 r^{(1)}(a, t) + O(\epsilon^4) \end{aligned} \tag{24}$$

where Δr_0 is given in Theorem 1, and

$$r^{(1)}(a, t) = \frac{3\, p_{d0}^2\, S_N{}^2 \left(\sin(t\,\lambda_2)^2\, \lambda_1{}^2 + 2\, \left(-1 + \cos(t\,\lambda_1)\right)\, \lambda_2{}^2 \right)}{2\, a^4\, \lambda_2{}^2 \left(\lambda_1{}^4 - 4\, \lambda_1{}^2\, \lambda_2{}^2 \right)}$$

is the first component of the vector $\mathbf{q}^{(1)}(t)$ as given by (15).

Thus, in order to find periodic solutions in the inertial reference system, for a fixed Θ we must find solutions of the equation

$$\dot{\theta} = \frac{\Theta}{r^2} \tag{25}$$

with $\theta(0) = 0$ at such that

$$\theta(4qT) = 2\pi p \tag{26}$$

for some integers p and q.

Substituting (24) in (25), we get that

$$\dot{\theta} = \frac{\Theta}{a^2}\left(1 - \frac{2\cos(\lambda_1 t)\Delta r_0 + \epsilon^2 r^{(1)}(a,t)}{a}\right) + O(\epsilon^4)$$

Integrating and remembering that ΔT is $O(\epsilon^2)$, we have

$$\theta(4T) = \frac{\Theta}{a^2}\, 4T_0^* + \ \epsilon^2 \frac{3\,(1+2k)\pi p_{d0}^2\, \Theta^{16}}{8\, a^7\, {K_N}^{17}\, {S_N}^5\, \left({K_N}^2 - 4\, {S_N}^2\right)} H(N) + O(\epsilon^4)$$

where

$$H(N) = \ 8\, {S_N}^6 + 3\, {K_N}^4 {W_N}^2 \ + {K_N}^2\, ({S_N}^4 - 12\, {S_N}^2\, {W_N}^2) \tag{27}$$

and K_N, S_N, W_N are defined , respectively, in (8), (9) and (12). Thus, it is enough to see that the term $H(N)$ is different from zero to guarantee that there exist infinitely many values of the parameter ϵ^2 such that (26) holds. This is indeed the case for $2 \le N \le 10$, and probably for infinitely many values of N, although we do not have a formal proof of this fact.

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