OLD AND NEW RESULTS ON SNAS ON THE REAL LINE

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ABSTRACT. Classical and new results concerning the topological structure of skew-products semiflows, coming from non-autonomous maps and differential equations, are combined in order to establish rigorous conditions giving rise to the occurrence of strange nonchaotic attractors. A special attention is paid to the relation of these sets with the almost automorphic extensions of the base flow. The scope of the results is clarified by applying them to the Harper map, although they are valid in a much wider context.

1. INTRODUCTION

In this work we focus on invariant sets of non-autonomous dynamical systems. More concretely, we will consider a specific family of such systems and we will study the existence of attracting sets, specially when they are "strange", i.e, they are not regular (or piecewise regular) manifolds. These attracting sets are called chaotic when the dynamics on them has a strictly positive Lyapunov exponent. Here we concentrate on strange attractors with negative Lyapunov exponents; they are called strange nonchaotic attractors, or SNA. SNAs usually appear in quasi periodically forced systems (see [46] for specific examples), either discrete or continuous.

As far as we know, the first constructions of flows containing SNAs can be found in [39, 40, 61]. It is remarkable that those papers were written before the term SNA was coined so they do not use this terminology. The term SNA is introduced in [18], where some rigorous results for discrete maps can be found. Relevant rigorous examples of SNA for maps are also given in [3, 31].

Here we will focus on the existence of SNAs for certain families of maps and flows. The considered flows will be induced by Dirac systems (that includes the Schrödinger equation as a particular case), while the maps considered can be reduced to the previous flows by means of a suspension. Our exposition surveys several classical results on the occurrence of almost automorphic dynamics for almost periodic differential equations, reinterpreting them in the context of SNAs. We combine the above results with some spectral properties in order to prove rigorously some new results about SNAs on the real line.

In this study we will relate the invariant sets of certain nonlinear systems to those of some conservative non-autonomous two-dimensional linear systems. To explain some of the ideas behind this procedure, let us focus first on an autonomous linear two-dimensional system of saddle type. As almost all the orbits tend, when $t \to +\infty$, to the line corresponding to the unstable manifold, we can look at this line as an attracting set (we will not call it attractor because it is not compact).

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Therefore, if we look at this system in polar coordinates and we discard the radial coordinate, we have a dynamical system defined on $\mathbb{S}^1$ that has two attracting (and two repelling) points which are the intersection of the unstable (stable) manifold with the unit circle. In this paper, instead of using the usual angles of the polar coordinates we will use the *directions*, i.e. the projective line $\mathbb{P}^1$: the previous example induces a flow on $\mathbb{P}^1$ with one attracting and one repelling point. This setting can be generalized to non-autonomous two-dimensional linear differential systems in a very natural way: consider, for instance, a two-dimensional linear system that depends quasi periodically on time with d basic frequencies, that is, of the form $x' = A(\omega)\,x$, $\omega' = \alpha$, for $\omega \in \mathbb{T}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$, where $\alpha_1, \dots, \alpha_d$ are rationally independent. Assume that this system has a stable and an unstable manifold. In this setting, if we discard the radial coordinate, the space of directions is $\mathbb{T}^d \times \mathbb{P}^1$. This means that the induced flow in $\mathbb{T}^d \times \mathbb{P}^1$ has an attracting and a repelling torus. The same can be asserted in the case of a map on $\mathbb{T}^d \times \mathbb{R}^2$.

The examples considered in this paper are quasi periodically forced projective one-dimensional flows and maps which are induced, by a suitable transformation, from quasi periodically time dependent two-dimensional linear differential and difference systems. Then, the attracting and repelling tori contained in the space of directions $\mathbb{T}^d \times \mathbb{P}^1$ correspond to attracting and repelling tori of the one-dimensional flow. We will also show how the tori in the space of directions collide and break down, becoming a "strange" set. This corresponds to a similar phenomenon in the phase space of the one-dimensional model, that creates an SNA.

In this paper classic results concerning these models in a very general context are reinterpreted and combined with new ones. The presentation and the proofs are very technical and they require a good degree of familiarity with ergodic theory, spectral theory and topological dynamics. For this reason, in the introduction we have included a short (and not very technical) presentation of the main results: before proving them, we apply these results to the well-known Harper map. We also compare the structure of the SNAs in the Harper map with those given in [31]. The interested reader will find, in the remaining sections, full details and extensions of the results mentioned in the introduction, as well as precise references for the results previously known. A brief description of the contents and structure of the paper is included in Subsection 1.3.

We stress the fact that, for us, SNAs will be always associated to global discrete semiflows (or flows) on $\mathbb{T}^d \times \mathbb{R}$, given by the iterations of a continuous map (or homeomorphism) on this space. That is, we will be talking of SNAs *on the real line*. There are other possible definitions of this kind of sets: for instance, some authors include the graphics of non-continuous invariant curves contained in the projective bundle over the torus, or talk about SNAs for real flows or semiflows, given not by a map but by the solutions of a quasi periodic differential equations.

To conclude this general introduction, we point out that the existence of non-continuous invariant sets is a phenomenon deeply analyzed in the case of flows of higher dimension, finite or infinite: the condition usually required in these flows is their *monotonicity*. The interested reader is referred to [52], [42], and references therein.

1.1. **The Harper map.** The Harper map is defined on $\mathbb{T}^1 \times \mathbb{R}$ by

$$\begin{aligned} \bar{\omega} &= \omega + \alpha\,, \\ \bar{x} &= f(x,\omega,\lambda) = -\frac{1}{x} - \lambda - 2b\cos 2\pi\omega\,, \end{aligned} \tag{1.1}$$

where $\omega \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\alpha \notin \mathbb{Q}$. Here b represents a real parameter which will be fixed in the analysis, and λ represents a real parameter which will vary. This map can be seen as a discrete version of the Schrödinger equation with a quasi periodic potential. It has been studied in several places, for instance see [32, 19].

As f sends points with $x = 0$ to infinity (and, when x goes to infinity, f goes to a finite value), there are points $(\omega, x) \in \mathbb{T}^1 \times \mathbb{R}$ whose orbits blow up in finite time. In order to cope with the infinity point, we take projective coordinate $\varphi = \cot^{-1} x \in [0,\pi)$ on $\mathbb{R}$. This takes the Harper map to

$$\begin{aligned} \bar{\omega} &= \omega + \alpha\,, \\ \bar{\varphi} &= \cot^{-1}\left(-\frac{1}{\cot\varphi} - \lambda - 2b\cos 2\pi\omega\right)\,, \end{aligned} \tag{1.2}$$

which define a map in $\mathbb{T}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1 \equiv \mathbb{R}/\pi\mathbb{Z}$. Note that the orbit of any initial point (ω, φ) is defined for every (positive or negative) time.

It is important for the following description of results to note that both maps (1.1) and (1.2) are obtained from the map on $\mathbb{T}^1 \times \mathbb{R}^2$ given by

$$\begin{aligned} \bar{\omega} &= \omega + \alpha, \\ \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -\lambda - 2b\cos 2\pi\omega \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \tag{1.3}$$

by taking $x = z_2/z_1$ and $\varphi = \cot^{-1} z_2/z_1$ respectively. Note also that this last one is a conservative linear map. Using that the equation for the angle variable ω can be solved as $\omega(n) = \omega_0 + n\alpha$, this linear map can be rewritten as

$$\begin{bmatrix} z(n) \\ z(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda - 2b\cos 2\pi(\omega_0 + n\alpha) \end{bmatrix} \begin{bmatrix} z(n-1) \\ z(n) \end{bmatrix}. \tag{1.4}$$

This map (1.3) is in fact a second order operator, and we can put it into the more convenient form

$$-z(n+1) - z(n-1) - 2b\cos 2\pi(\omega_0 + n\alpha)\, z(n) = \lambda\, z(n)\,. \tag{1.5}$$

The left-hand side of this equation is the so-called almost Mathieu operator,

$$z(n) \mapsto -z(n+1) - z(n-1) - 2b\cos 2\pi(\omega_0 + n\alpha)\, z(n)\,. \tag{1.6}$$

Therefore, we can look at (1.5) as a spectral problem for (1.6).

For further reference, we define

$$D_\lambda(\omega(n)) = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda - 2b\cos 2\pi(\omega_0 + n\alpha) \end{bmatrix}.$$

and the propagation matrix of (1.4) as

$$Z_\lambda(n,\omega_0) = D_\lambda(\omega(n-1))\cdots D_\lambda(\omega(1))D_\lambda(\omega_0).$$

Any $\bar{\varphi}_0 \in \mathbb{P}^1$ admits a unique representative $\varphi_0 \in [0,\pi)$. For $\varphi_0 \in [0,\pi)$, we define $\mathbf{z}_{\varphi_0} = (\sin\varphi_0, \cos\varphi_0)^T$ and $\mathbf{z}_\lambda(n,\omega_0,\varphi_0) = Z_\lambda(n,\omega_0)\left[\begin{smallmatrix}\sin\varphi_0 \\ \cos\varphi_0\end{smallmatrix}\right]$ (the orbit of (1.4) with initial data $\left[\begin{smallmatrix}z(-1) \\ z(0)\end{smallmatrix}\right] = \left[\begin{smallmatrix}\sin\varphi_0 \\ \cos\varphi_0\end{smallmatrix}\right]$).

1.1.1. *Lyapunov exponents and invariant sets.* A key concept in our setting is exponential dichotomy. Roughly speaking, it is said that the system (1.4) admits an exponential dichotomy when any solution can be written as the sum of two solutions, one of them approaching the origin and the other one going to infinity, both in an exponential way.

Now let us consider the almost Mathieu operator (1.6) acting on $L^2(\mathbb{Z})$, the space of sequences $\{z_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |z_n|^2 < +\infty$. It is not difficult to see that this operator is self-adjoint and, therefore, its spectrum is real. Applying a well known result of Johnson [25], included here as Theorem 5.2, we have that

a) the spectrum of the Jacobi operator does not depend on ω_0,
b) the value λ belongs to the spectrum of the Jacobi operator (1.6) if and only if (1.4) does not admit an exponential dichotomy.

The interested reader can find in [2] and references therein a good survey on the description of this spectral problem. In particular, the spectrum is always a compact subset of $\mathbb{R}$ and, if $b \neq 0$, it is a Cantor set for any irrational frequency α.

The next step is to relate this spectrum with the so-called Lyapunov exponents. The Oseledets multiplicative ergodic theorem (see [45, 30] and Remark 4.2) implies that there exists a set $\Omega_\lambda \subset \mathbb{T}^1$, invariant by rotations of angle α and of full Lebesgue measure, such that for any $\omega_0 \in \Omega_\lambda$ the following limit exists and takes the same value:

$$\lim_{|n|\to\infty} \frac{1}{n} \ln \|Z_\lambda(n, \omega_0)\| = \beta(\lambda) \geq 0.$$

Moreover, if $\beta(\lambda) > 0$, there exists a measurable decomposition $\mathbb{T}^1 \times \mathbb{R}^2 = W_\lambda^+ \oplus W_\lambda^-$ in two one-dimensional subbundles with

$$\lim_{|n|\to\infty} \frac{1}{n} \ln |Z_\lambda(n, \omega_0)\, \mathbf{z}_0| = \mp\beta(\lambda) \quad \text{for } \omega_0 \in \Omega_\lambda \text{ and } (\omega_0, \mathbf{z}_0) \in W_\lambda^\pm \text{ with } \mathbf{z}_0 \neq 0\,.$$

In other words, the dynamics on W_λ^- is expansive with an average ratio of $\exp(\beta(\lambda))$, and the dynamics on W_λ^+ is contractive with average ratio of $\exp(-\beta(\lambda))$. The values $\pm\beta(\lambda)$ are the Lyapunov exponents of (1.4).

Sacker-Sell spectral theorem [50] provides a description of the possibilities for the dynamics on $\mathbb{T}^1 \times \mathbb{R}^2$. In our situation, there are only three different cases.

i) Elliptic case: $\beta(\lambda) = 0$, so that (1.4) does not admit an exponential dichotomy.
ii) Uniformly hyperbolic case: (1.4) admits an exponential dichotomy, so that $\beta(\lambda) > 0$.
iii) Non-uniformly hyperbolic case: $\beta(\lambda) > 0$ and (1.4) does not admit an exponential dichotomy.

In particular, if $\beta(\lambda) = 0$, then λ belongs to the spectrum of the almost Mathieu operator, whereas if λ belongs to the resolvent then $\beta(\lambda) > 0$. Non-uniformly hyperbolic situations also arise: in fact, Herman [20] proves that $\beta(\lambda) \geq \max(0, \ln|b|)$ (and in fact the equality holds in the spectrum, as proved later by Bourgain and Jitormirskaya [8]); so that in the case that $|b| > 1$, any value of the spectrum provides non-uniformly hyperbolic dynamics.

In what follows we will assume $\beta(\lambda) > 0$, and we will consider the dynamics induced by (1.2) on $\mathbb{T}^1 \times \mathbb{P}^1$. Assume first that λ belongs to the resolvent of the Jacobi operator (1.6), so that we are in the uniformly hyperbolic situation. In this case (see Proposition (4.9)), due to the exponential dichotomy, there exist two regular invariant curves on $\mathbb{T}^1 \times \mathbb{P}^1$, given by the graphs of two maps $\mathbb{T}^1 \to \mathbb{P}^1$, $\omega \mapsto$

$\bar{\varphi}^{\pm}_{\lambda}(\omega)$. One of these curves is uniformly attracting and the other one is uniformly repelling: in fact,

$$\mp\beta(\lambda) = \lim_{|n|\to\infty} \frac{1}{n} \ln |\mathbf{z}_{\lambda}(n, \omega_0, \varphi^{\pm}_{\lambda}(\omega_0))| \quad \text{for all } \omega_0 \in \mathbb{T}^1,$$

where $\mathbf{z}_{\lambda}$ was defined at the end of Section 1.1 and $\varphi^{\pm}_{\lambda}(\omega_0)$ is the representative of $\bar{\varphi}^{\pm}_{\lambda}(\omega_0)$ in $[0, \pi)$; and all the orbits of $\mathbb{T}^d \times \mathbb{P}^1$ starting outside the graphs of $\bar{\varphi}^{\pm}_{\lambda}$ are heteroclinic orbits, going from the graph of $\bar{\varphi}^{-}_{\lambda}$ to the graph of $\bar{\varphi}^{+}_{\lambda}$.

The dynamical behavior is quite more complicated in the non-uniformly hyperbolic situation. In this case (see Theorem 4.10), there is a unique minimal invariant set $M_{\lambda} \subset \mathbb{T}^d \times \mathbb{P}^1$ which is not a smooth curve. More concretely, this set contains exactly two non-closed invariant subsets that are the graphs of two non-continuous measurable functions $\bar{\varphi}^{\pm}_{\lambda} : \mathbb{T}^d \to \mathbb{P}^1$. Moreover, there exists a full measure set $\Omega_{\lambda} \subsetneq \mathbb{T}^1$ such that

$$\mp\beta(\lambda) = \lim_{|n|\to\infty} \frac{1}{n} \ln |\mathbf{z}_{\lambda}(n, \omega_0, \varphi^{\pm}_{\lambda}(\omega_0))| \quad \text{for all } \omega_0 \in \Omega_{\lambda}\,.$$

In addition, for $\omega_0 \in \Omega_{\lambda}$ the trajectories corresponding to $(\omega_0, \bar{\varphi}_0)$ with $\bar{\varphi}_0 \neq \bar{\varphi}^{\pm}_{\lambda}(\omega_0)$ are heteroclinic, as before. This is because $\bar{\varphi}^{+}_{\lambda}$ and $\bar{\varphi}^{-}_{\lambda}$ are the projections on $\mathbb{P}^1$ of the Oseledets subbundles W^{+}_{λ} and W^{-}_{λ}, and the trajectories of $\mathbb{T}^1 \times \mathbb{R}^2$ (for the map (1.3)) starting on an initial condition outside W^{+}_{λ} tend to W^{-}_{λ} at the same time that they go to infinity. We can also prove that there exists a residual subset $R_{\lambda} \subset M_{\lambda}$ (projecting on $\mathbb{T}^1$ outside Ω_{λ}) of points giving rise to orbits on $\mathbb{T}^1 \times \mathbb{R}^2$ which oscillate exponentially for $|n| \to \infty$.

In particular, we have shown the existence of a non-closed invariant set –with a "strange" structure– with negative Lyapunov exponent. This is one of the objects we are interested in and it will be discussed in the next section.

1.1.2. *Strange attractors in the Harper map.* As said before, the maximal Lyapunov exponent $\beta(\lambda)$ of (1.4) is positive for any value of λ if $|b| > 1$. This is the situation assumed along this subsection.

We have already mentioned that if λ belongs to the resolvent of (1.5), then $W^{\pm}_{\lambda}$ are vector bundles with a smooth dependence of ω, which implies the existence of two smooth curves on $\mathbb{T}^1 \times \mathbb{P}^1$, invariant for the map (1.2), given by the graph of smooth functions $\bar{\varphi}^{\pm}_{\lambda} : \mathbb{T}^1 \to \mathbb{P}^1$, one one attracting and one repelling. We stress that, if the range of $\bar{\varphi}^{\pm}_{\lambda}$ is $\mathbb{P}^1$, there is no way of going back to real coordinates and to get continuous curves because these curves will always "pass through" infinity.

If λ belongs to the spectrum of (1.5), then (1.3) does not admit an exponential dichotomy, and we are in the non-uniformly hyperbolic case. This implies that the vector bundles W^{+}_{λ} and W^{-}_{λ} do not depend continuously on ω and, therefore, the corresponding attracting and repelling sets for the map (1.2) in $\mathbb{T}^1 \times \mathbb{P}^1$ are not continuous curves (see Theorem 4.10). Moreover, it can be proved that the closure of the attracting and repelling sets is the same set M_{λ}, which is minimal (see Remark 5.7). Hence, this set contains attracting and repelling points. The projection of the set of attracting points on the basis $\mathbb{T}^1$ has total Lebesgue measure, while the projection of the points that are not attracted by the attractor is dense and has zero measure.

Let us now assume that we select a value $\lambda = \lambda_0$ in the resolvent of (1.5), and let J_{λ_0} be the largest open interval containing λ_0 and contained in the resolvent (this is called "spectral gap"). Let λ^* be a (finite) boundary point of J_{λ_0} (we note that J_{λ_0} cannot be the real line since the spectrum cannot be empty). We focus on the behavior of the invariant curves that exist for $\lambda = \lambda_0$ when λ approaches λ^*. In this case, we prove the following (see Theorem 5.3): the attracting and repelling curves approach each other in a monotone and non-uniform way; when λ reaches λ^*, the two curves "collide" on a dense set of points and they stop being continuous; and, for $\lambda = \lambda^*$, the two curves are continuous and coincident on a residual set $R \subset \mathbb{T}^1$, invariant by $\omega \mapsto \omega + \alpha$, whereas they are different and discontinuous in a full measure set of points of $\mathbb{T}^1$.

To discuss the structure of these curves for $\lambda = \lambda^*$ we introduce the concept of almost automorphic extension of the basis $\mathbb{T}^1$: a minimal invariant set $M \subset \mathbb{T}^1 \times \mathbb{P}^1$ is an almost automorphic extension of the basis $\mathbb{T}^1$ if there exists an element $\hat{\omega} \in \mathbb{T}^1$ such that the set $M_{\hat{\omega}} = M \cap (\{\hat{\omega}\} \times \mathbb{P}^1)$ reduces to a single point. It is clear that, due to the invariance of M and that the dynamics on $\mathbb{T}^1$ is an irrational rotation, the set R of values $\hat{\omega}$ such that $M_{\hat{\omega}}$ is a singleton is dense in $\mathbb{T}^1$. The simplest example of an almost automorphic extension is a regular invariant curve, (for which $R = \mathbb{T}^1$). However, in our situation ($\lambda = \lambda^*$), the closure M_{λ^*} of the attracting and repelling curves is an almost automorphic extension of $\mathbb{T}^1$ for which the set of values R is a residual set of zero measure (see Theorem 5.3).

Note that, as recalled before, the spectrum is a Cantor set. In particular there exists an infinite countable number of spectral gaps, and hence an infinite countable number of points λ^* to which the previous arguments apply.

As pointed out before, we are interested in SNAs on $\mathbb{R}$. So let us try to embed the set $M_{\lambda^*} \subset \mathbb{T}^1 \times \mathbb{P}^1$ into $\mathbb{T}^1 \times \mathbb{R}$. This can be easily done if the set M_{λ^*} is contained in $\mathbb{T}^1 \times [\delta, \pi - \delta]$ for a positive δ. In this case, the graph of the non-continuous invariant curve $\bar{\varphi}^- : \mathbb{T}^1 \to (0, \pi) \subset \mathbb{R}$ is what we call an SNA. This is the situation arising if λ^* is the first point of the spectrum (see Remark 6.5). If the projection of M on $\mathbb{P}^1$ contains the point 0, the situation is a bit more difficult: the attractor may be winding on $\mathbb{P}^1$ so that it cannot be naturally included in $\mathbb{R}$. To "unwind" the attractor, we proceed in the following way. Let $\bar{\varphi}^-_\lambda : \mathbb{T}^1 \to \mathbb{P}^1$ be the attracting invariant curve for any λ. We prove (see Theorem 5.3) that the graph of $\bar{\varphi}^-_\lambda - \bar{\varphi}^-_{\lambda_0}$ excludes the infinity point for all $\lambda \in \overline{J_{\lambda_0}} - \{\lambda_0\}$. Therefore, it is possible to unwind the invariant curve $\bar{\varphi}^-_\lambda$ for all $\lambda \in \overline{J_{\lambda_0}} - \{\lambda_0\}$ by means of a fixed smooth translation given by $\bar{\varphi}^-_{\lambda_0}$ (see Theorems 6.3 and 6.4). In particular, the translated curve $\bar{\varphi}^-_{\lambda^*} - \bar{\varphi}^-_{\lambda_0}$ is not continuous and attracting for the translated flow, and its closure on $\Omega \times \mathbb{P}^1$ is an almost automorphic extension of the basis that excludes the infinity point.

The process of unwinding the attractor in the Harper case is a smooth transformation (a translation) on $\mathbb{P}^1$ that allows to embed the attractor in $\mathbb{R}$, used to show the existence of SNAs on the real line.

Figures 1-4 show some graphics corresponding to the attracting and repelling sets for the equation (1.2) with $b = 1.1$ and different values of λ. The horizontal and vertical axes represent the one-dimensional torus $\mathbb{T}^1$ and the real projective line, respectively, so that the graphics must be interpreted like sets of points over the two-dimensional torus. Figures 1 and 2 show the situation near the first point of the spectrum, which we will call λ^*. The four graphics in Figure 1 correspond to values of λ in the resolvent, approaching λ^* from the left: -3, -2.8, -2.75 and -2.783 respectively. The green and red curves represent the attractor and repeller,

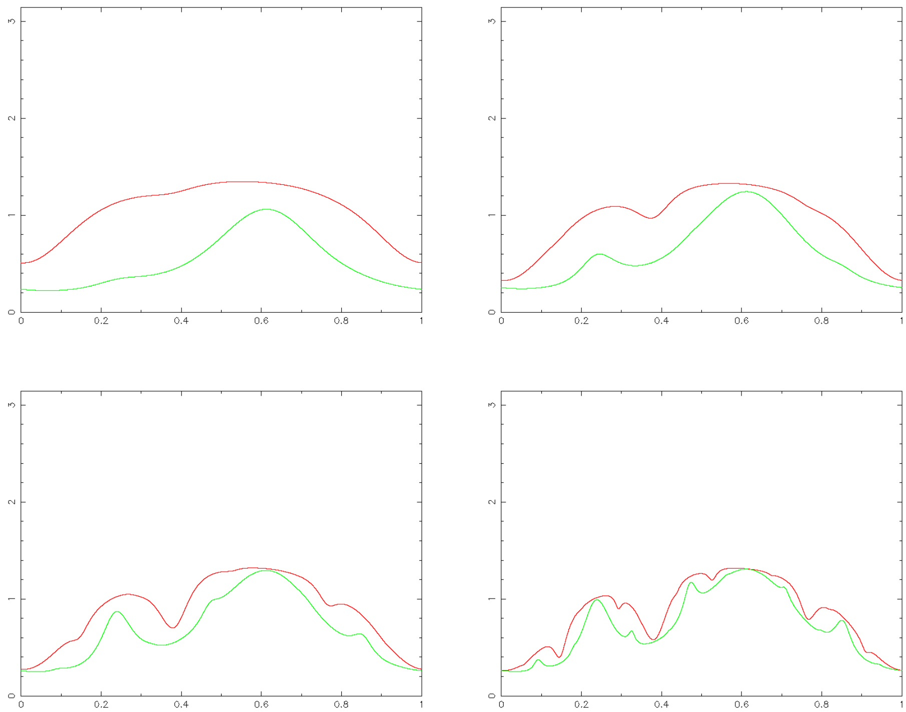

FIGURE 1. Approximation of the attracting and repelling invariant curves as λ approaches the first point of the spectrum

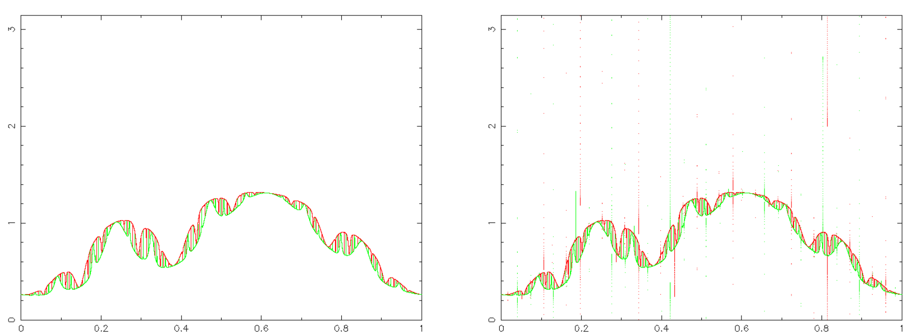

FIGURE 2. Behavior near the first point of the spectrum

given by copies of the base $\mathbb{T}^1$. The evolution of these curves is monotone and the distance between them decreases as λ increases. As mentioned before, these properties are rigorously proved in Theorem 5.3. Numerical computation gives a value close to 2×10^{-3} for the distance between both curves in the fourth figure.

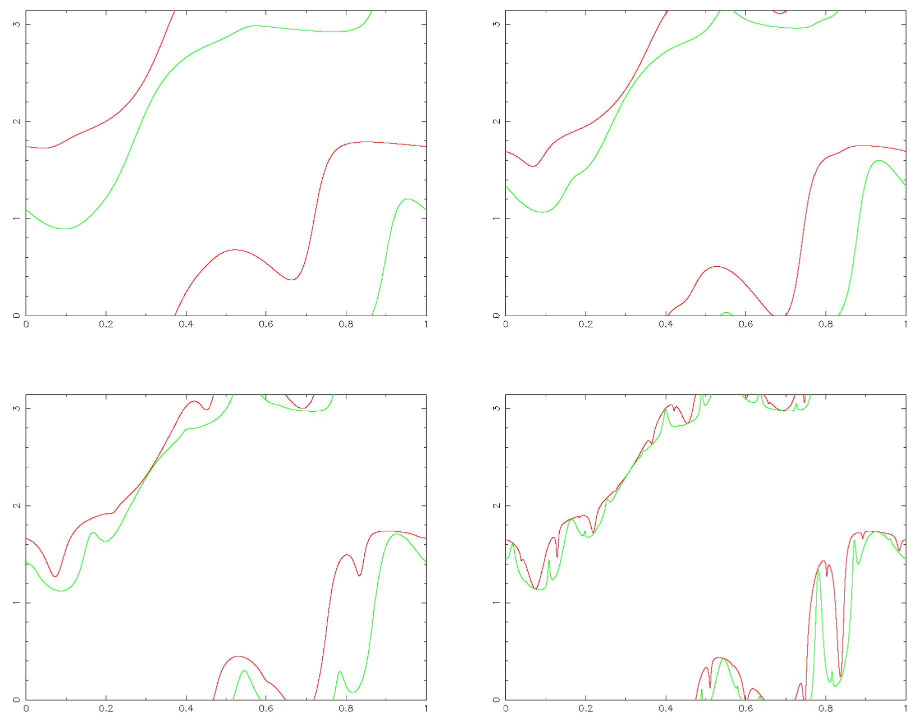

FIGURE 3. Approximation of the attracting and repelling invariant curves as λ approaches the left extreme point of another spectral interval

In the case $\lambda = -2.7363764521 49$, represented in the first graphic of Figure 2, the attracting and repelling sets seem to be continuous curves at a distance of the order of 10^{-12}. The second graphic corresponds to $\lambda = -2.736376452148$. In this case the attracting (green) and repelling (red) sets do not seem to be continuous curves. But this cannot be ensured by their representations: recall that in fact they are continuous outside the spectrum, which is a Cantor set, so that to find numerically a value of λ for which the curves are not continuous is highly improbable; and since we have a finite number of pixels to represent any set, it is easy to obtain a discontinuous set of points when computing a continuous curve with a very strange shape. This last fact can easily be a source of errors in the interpretation of the plots. Nevertheless, if we assume that the numerical computation is sharp enough to guarantee that the points plot to draw Figure 2 are correct, the results rigorously proved in this paper allow us to say something more: one observes that there is not a monotone way to pass from the attracting/repelling pair in the first graphic of Figure 2 to the one of the second, and this lack of monotonicity allows us to ensure that this small interval (of length 10^{-12}) contains the point λ^*, for which an SNA on $\mathbb{R}$ occurs. This point λ^* corresponds to the value of the parameter for which the two curves have their first contact. Hence the graphic of the SNA must look like the green set in the first plot of Figure 2.

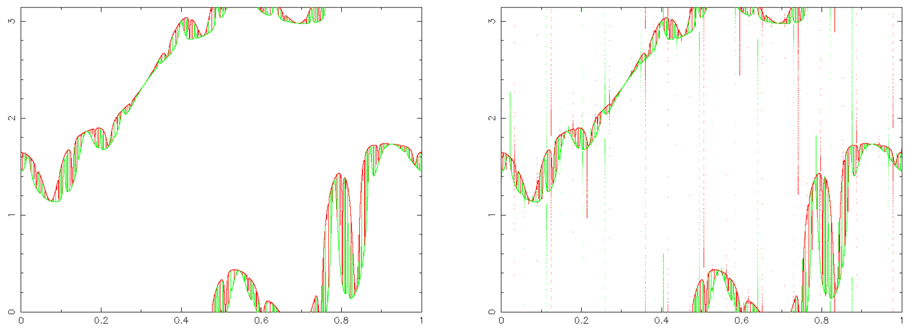

FIGURE 4. Behavior near the left extreme point of another spectral interval

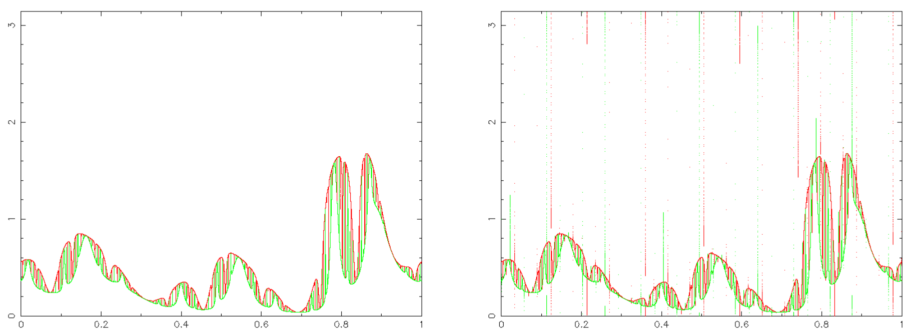

FIGURE 5. Unwound attracting and repelling sets

This structure is basically reproduced at any other extreme point of a spectral gap. The four graphics in Figure 3, corresponding to the values 1.7, 1.9, 1.95 and 1.96 of λ, are a new sample of the collision phenomenon; and the two graphics of Figure 4 correspond to two close values of λ (1.960148407660 and 1.960148407661) between one can reasonable expect the existence of a left extreme point of a spectral interval. The difference with the previous figures is that in these cases, the attracting sets cross the infinity point of $\mathbb{P}^1$. Hence, in order to obtain an SNA on $\mathbb{R}$, an unwinding process is required. The graphics of Figure 5 show the attracting and repelling sets obtained after the unwinding process before described: more precisely, they correspond to the map obtained from (1.2) by $\bar{\varphi} \mapsto \bar{\varphi} - \bar{\varphi}^-_{\lambda_0}$ for $\lambda_0 = 1.7$. (We point out that this new map is not necessarily a Harper map.)

The structure described in the case that λ is taken in the boundary of a spectral gap seems to be lost when λ is not one of these countable values but is still inside the spectrum. For instance, in [4] it is proved that, if the parameter b is positive and large enough, there exists a subset of the spectrum (with positive Lebesgue measure) of values of λ for which the only minimal subset is the whole phase space $\mathbb{T}^1 \times \mathbb{P}^1$. In this case, the functions $\bar{\varphi}^{\pm}_{\lambda} : \mathbb{T}^d \to \mathbb{P}^1$ corresponding to the projective coordinates of the Oseledets subbundles $W^{\pm}_{\lambda}$ (i.e. the attracting/repelling sets in the Harper map) cannot have any point of continuity: otherwise the closure of their graphs would be almost automorphic extensions of the base, impossible since the only minimal set is the whole space. So that the closure of *any* orbit of the attracting set is the whole space. For these reasons, we do not consider that, in this case, the system has a strange nonchaotic attractor. However, some authors consider the graph of the curve $\bar{\varphi}_\lambda^-$ as an (unbounded) SNA on $\mathbb{P}^1$ ([19]).

1.2. **An example by G. Keller.** The previously described properties suggest a strong similarity between the structure of the SNA found in the Harper map and the well-known example given by G. Keller in [31]. We think it is very interesting to insists in this similitude.

Keller's example uses the map $T : \mathbb{T}^1 \times [0, \infty) \to \mathbb{T}^1 \times [0, \infty)$ defined by $T(\omega, x) = (\omega+\alpha, f(x)g_\mu(\omega))$. It is assumed that f is positive, bounded C^1, increasing, strictly concave and $f(0) = 0$. Moreover, it is supposed that g_μ is positive, C^0 and there exists a value μ_0 such that if $\mu < \mu_0$ g_μ never vanishes, and if $\mu \geq \mu_0$ g_μ has at least a zero. It is clear that the Lyapunov exponent of the invariant curve $x = 0$ is given by

$$\lambda_\mu = \ln f'(0) + \int_{\mathbb{T}^1} \ln g_\mu(\omega)\, d\omega.$$

It is also assumed that $\lambda_\mu > 0$ on a neighborhood of μ_0. Note that this implies that the curve $x = 0$ is repelling. Under these conditions, it can be proved ([31]) that

1. if $\mu < \mu_0$ there exists a unique attracting invariant curve $\varphi_\mu : \mathbb{T}^1 \to [0, \infty)$. The curve is as smooth as the map (see also [56]).
2. If $\mu \geq \mu_0$ there exists an upper semicontinuous (and Lebesgue-a.e. discontinuous!) function $\varphi_\mu : \mathbb{T}^1 \to [0, \infty)$ with an invariant graph with a negative Lyapunov exponent. The repelling invariant curve $x = 0$ is contained in the closure of the graph of φ.

The Lyapunov exponent of the invariant graph of item 2 is defined as

$$\beta_s(\varphi) = \int_{\mathbb{T}^1} \ln |f'(\varphi(\omega))g_\mu(\omega)|\, d\omega.$$

The Birkhoff ergodic theorem (see, for example, [30]) ensures that $\beta_s(\varphi)$ coincides a.e. with the Lyapunov exponent of single orbits on the graph of φ. On the other hand, there exists a zero measure set of points on this graph giving a different Lyapunov exponent. This is also clear if we note that φ intersects the repelling curve $x = 0$ for a dense set of values of ω.

At this point we want to note the very particular nature of this attracting set:

3. The graph of φ is only attracting for initial conditions (ω, x) such that ω belongs to a suitable full measure proper subset of $\mathbb{T}^1$ (that is, there is a dense zero measure set of values of ω for which the initial conditions (ω, x), $x > 0$, are not attracted to the graph of φ). In particular, the basin of attraction has empty interior.
4. The attractor is not minimal (it properly contains the invariant set $x = 0$). We remark that, in this context, repelling curves are not persistent (see [28, 29] for details). Therefore, having $x = 0$ as invariant curve for all the values of the parameters is highly nongeneric.

In the previous section we have discussed the existence of an SNA for the Harper map when λ is in the boundary of a spectral gap and the dynamics is non-uniformly hyperbolic, possibly after an unwinding process. As said before, we want to make some remarks comparing the structures of the closure of the attractor for the Harper map (we will call it M) and the one given by Keller (we will call it Γ).

Note that assertions 1 and 2 above show that Γ appears as a consequence of the collision as $\mu \to \mu_0^-$ of the invariant curve $x = 0$ and the continuous attracting invariant curve φ_μ. The set Γ is given by the closure of φ_{μ_0} and contains the graph of $x = 0$. Similarly, the set M appears as a consequence of the collision as the spectral parameter in the resolvent tends to the extreme point of a spectral gap of two hyperbolic curves $\bar{\varphi}$, and in fact it agrees with the closure of the graph of any of them.

On the other hand, as asserted in point 4, the set Γ is not minimal, since it contains a proper subset ($x = 0$) which is invariant. Consequently, it is not an almost automorphic extension of the basis $\mathbb{T}^1$, as M is. However, we note that Γ is still a so-called *pinched* set: its sections reduce to a point for a residual subset of base points. In addition, the sections of M and Γ do no reduce to a point for a full measure set of elements of the torus. In this sense, the two sets have a quite similar "shape", with the difference that Γ is forced, by construction, to contain the invariant subset $x = 0$. Finally, as a consequence of this pinched structure, also the dynamics inside both sets (partially described for Γ in the point 3) is similar.

1.3. **Contents of the paper.** Section 2 contains some basic definitions and results on topological dynamics and ergodic theory which will be fundamental in the rest of the paper. In Section 3 we define the type of SNAs we will work in this paper: non-closed and bounded invariant curves for a continuous map on $\mathbb{T}^d \times \mathbb{R}$ with negative Lyapunov exponent. Some monotonicity conditions required for the map are fundamental in the dynamical description of the skew-product flow that it induces. The connection between the closure of one of these objects and the almost automorphic extensions of the quasi periodic base flow will be also established.

In the approach we present, the SNAs will be, roughly speaking, associated to the projective flow induced on $\mathbb{T}^d \times \mathbb{P}^1$ by a quasi periodic family or differential equations (of Dirac or Schrödinger type) or difference equations (of Jacobi type). All these cases admit a simultaneous analysis: the Schrödinger equation can be understood as a particular type of Dirac system, and the Jacobi equation can also be included in the same setting by means of the suspension of the discrete semiflow that it induces; i.e. of a new real flow, associated to a Dirac-type differential equation, such that the restriction of the orbits to the integer values of the time provides the orbits of the discrete initial flow. (Appendix B summarizes the most basic facts concerning this suspension process.) Section 4 presents some facts concerning the dynamical description of these projective flows, paying special attention to the non-uniformly hyperbolic case. Some classical results of one-dimensional dynamics are needed in this description. For the reader's convenience, the detailed proofs of these results are included in Appendix A.

This type of dynamics is the suitable one for us to look for strange nonchaotic attractors. However, in order to find SNAs on $\mathbb{R}$, we need something more. In Section 5 we include our initial equation in a one-parametric family of equations of the same kind, defining a spectral problem for a linear operator in a suitable space. Under the assumptions that our equation corresponds to a value of the parameter given by the finite extreme point of a spectral gap and that the corresponding dynamics is non-uniformly hyperbolic, we show the existence of a non-continuous invariant curve on $\mathbb{T}^d \times \mathbb{P}^1$ with negative Lyapunov exponent. In addition, we show that this curve appears as a result of the collision of two hyperbolic continuous curves as the parameter approaches that extreme point.

Finally, all these results will be glued up in Section 6 in order to show how to obtain an SNA for a flow of the type considered in Section 3, taking as starting point the non-continuous invariant curve with negative Lyapunov exponent obtained in the previous section. In the case of a differential equation: if the closure of this curve on $\mathbb{T}^d \times \mathbb{P}^1$ does not cross the infinity point, the SNA is obtained by means of a Poincaré section; whereas in the other case a previous unwinding procedure is required. The situation is similar in the case of an initial difference equation, for which our our analysis is based on the construction of the suspension mentioned before. A brief summary of situations in which our conditions are satisfied completes the paper.

2. Basic notions and results

We recall in this section some definitions and results which are standard in topological dynamics and ergodic theory, which will be used throughout the paper.

Let Ω be a complete metric space. A continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t,\omega) \mapsto \sigma(t,\omega)$ satisfying

(1) $\sigma_0 = \mathrm{Id}$,
(2) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s,\, t \in \mathbb{R}$,

where $\sigma_t(\omega) = \sigma(t,\omega)$, is a *real continuous flow* on Ω, while a continuous map $\sigma : \mathbb{R}^+ \times \Omega \to \Omega,\ (t,\omega) \mapsto \sigma(t,\omega)$ satisfying (1) and (2) for $s,\, t \in \mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$ is a *real continuous semiflow* on Ω.

By replacing $\mathbb{R}$ by $\mathbb{Z}$, we obtain the definitions of *discrete continuous flow* and *semiflow*. Note that a discrete semiflow is then given by the iterations of the map $\sigma_1(\omega) = \sigma(1,\omega)$: by defining, as usual, $\sigma_1^0(\omega) = \omega$, we obtain $\sigma_1^n(\omega) = \sigma(n,\omega)$ for $n \in \mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$. In addition, σ is a flow if and only if the map σ_1 is a homeomorphism with inverse $\sigma_1^{-1}(\omega) = \sigma(-1,\omega)$, in which case $\sigma_1^n(\omega) = \sigma(n,\omega)$ for $n \in \mathbb{Z}$.

In order to unify notations, $\mathbb{Y}$ will represent either $\mathbb{R}$ or $\mathbb{Z}$. So that $t \in \mathbb{Y}$ will be either a real or an integer number, depending on the type of flow considered.

Let $(\Omega,\sigma,\mathbb{Y})$ be a continuous flow. The *orbit* or *trajectory* of the point ω is the set $\{\sigma_t(\omega) \mid t \in \mathbb{Y}\}$. A subset $\Omega_1 \subset \Omega$ is *σ-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{Y}$. A σ-invariant subset $M \subset \Omega$ is *minimal* (or *σ-minimal*) if it is compact and does not contain properly any other compact σ-invariant set. Clearly, a compact σ-invariant subset is minimal if and only if every orbit is dense on it, and every compact σ-invariant set contains a minimal subset. The continuous flow $(\Omega,\sigma,\mathbb{Y})$ is *recurrent* or *minimal* if Ω itself is minimal.

Let m be a normalized Borel measure on Ω; i.e. a finite regular measure defined on the Borel subsets of Ω and with $m(\Omega) = 1$. The measure m is *σ-invariant* if $m(\sigma_t(\Omega_1)) = m(\Omega_1)$ for every Borel subset $\Omega_1 \subset \Omega$ and every $t \in \mathbb{Y}$. If, in addition, $m(\Omega_1) = 0$ or $m(\Omega_1) = 1$ for every σ-invariant subset $\Omega_1 \subset \Omega$, then the measure m is *σ-ergodic*. The measure m is *concentrated* on $\Omega_1 \subset \Omega$ if $m(\Omega_1) = 1$.

In the case of a continuous semiflow $(\Omega,\sigma,\mathbb{Y}^+)$, the *forward orbit* of the point ω is the set $\{\sigma_t(\omega) \mid t \in \mathbb{Y}^+\}$; a subset Ω_1 of Ω is *positively σ-invariant* if $\Omega_1 \subset \sigma_t^{-1}(\Omega_1)$ for all $t \in \mathbb{Y}^+$, and it is *σ-invariant* if $\Omega_1 = \sigma_t^{-1}(\Omega_1)$; a positively σ-invariant subset $M \subset \Omega$ is *minimal* (or *σ-minimal*) if it is compact and it does not contain properly any closed, positively σ-invariant subset; the semiflow $(\Omega,\sigma,\mathbb{Y}^+)$

is *minimal* if Ω itself is minimal. A normalized Borel measure m is *σ-invariant* if $m(\Omega_1) = \mu(\sigma_t^{-1}(\Omega_1))$ for every Borel subset $\Omega_1 \subset \Omega$ and every $t \in \mathbb{Y}^+$, and it is *σ-ergodic* if every ϕ-invariant subset has measure 0 or 1. A point $\omega_0 \in \Omega$ admits a *backward orbit* if for every $t \in \mathbb{Y}$ there exists a point in Ω, represented by $\sigma(t, \omega_0)$, such that $\sigma(s, \sigma(t, \omega_0)) = \sigma(t + s, \omega_0)$ for every $s \in \mathbb{Y}^+$. Every point in a minimal subset M admits at least a backward orbit contained in M. And, in the case that Ω is locally compact, the semiflow σ admits a flow extension (that is, there exists a flow on $\Omega \times \mathbb{Y}^+$ agreeing with σ) if and only if each point in Ω admits a unique backward orbit.

Let us represent by $\mathbb{T}^d$ the d-dimensional torus, identified as usual with $(\mathbb{R}/\mathbb{Z})^d$. Throughout the paper, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ will be a fixed vector with rationally independent components in the real case $\mathbb{Y} = \mathbb{R}$ and such that $1, \alpha_1, \ldots, \alpha_d$ are rationally independent in the discrete case $\mathbb{Y} = \mathbb{Z}$. In both cases we will refer to α as a rationally independent *frequency vector*. Then the map $\sigma_\alpha : \mathbb{Y} \times \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto \omega{\cdot}t = \omega + \alpha t$ defines a flow over the torus, which is minimal and *uniquely ergodic*, the induced Lebesgue measure on $\mathbb{T}^d$ being the unique σ_α-ergodic measure. In addition, this is an *almost periodic* flow: given any $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\mathsf{d}(\omega_1, \omega_2) < \delta$ for a pair of points $\omega_1, \omega_2 \in \mathbb{T}^d$, then $\mathsf{d}(\omega_1{\cdot}t, \omega_2{\cdot}t) < \varepsilon$ for every $t \in \mathbb{Y}$. The flow $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Y})$ is usually called a *Kronecker flow* with frequency vector α.

Now let X represent a metric space. We will work throughout the paper with a real *skew-product semiflow* $(\mathbb{T}^d \times X, \phi, \mathbb{Y}^+)$ *projecting onto* $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Y})$; i.e. with a continuous semiflow on the bundle $\mathbb{T}^d \times X$ of the form

$$\phi : \mathbb{Y}^+ \times \mathbb{T}^d \times X \to \mathbb{T}^d \times X\,, \quad (t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))\,. \tag{2.1}$$

That is, ϕ reproduces the flow on the base torus and it is given on the fiber by a map $u : \mathbb{Y}^+ \times \mathbb{T}^d \times X \to X$ satisfying

$$u(t + s, \omega, x) = u(t, \omega{\cdot}s, u(s, \omega, x)) \quad \text{for all } s, t \in \mathbb{Y}^+ \text{ and } (\omega, x) \in \mathbb{T}^d \times X\,. \tag{2.2}$$

Let M be a ϕ-minimal subset of $\mathbb{T}^d \times X$. From the minimality of the base flow it is easy to deduce that the restriction to M of the projection $\mathbb{T}^d \times X \to X,\ (\omega, x) \mapsto x$ is surjective. The restricted semiflow $(M, \phi, \mathbb{Y}^+)$ is an *almost automorphic extension* of $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Y})$ if there exists an element $\omega \in \mathbb{T}^d$ such that the section $M_\omega = \{x \in X \mid (\omega, x) \in M\}$ contains a unique element. And $(M, \phi, \mathbb{Y}^+)$ is a *copy* of the base flow if M_ω contains a unique element for every $\omega \in \mathbb{T}^d$. In this case, the restriction of the semiflow to M can be extended to a real flow which reproduces exactly the one on the base.

In fact, throughout the next pages, the metric space X will be $\mathbb{R}$, $\mathbb{R}^2$ or the real projective line $\mathbb{P}^1$, which we will identify with $\mathbb{R}/\pi\mathbb{Z}$. And we will almost always work with *skew-product flows* $(\mathbb{T}^d \times X, \phi, \mathbb{Y})$ and almost automorphic extensions $(M, \phi, \mathbb{Y})$, whose definitions just require to change $\mathbb{Y}^+$ by $\mathbb{Y}$ in the previous subsection.

Remark 2.1. In order to simplify the explanation, the results of the following sections will be referred to skew-product semiflows or flows over a Kronecker base; that is, we will restrict ourselves to the *quasi periodic* case. However all the results can be extended to skew-products over a base flow $(\Omega, \sigma, \mathbb{Y})$ if Ω is a compact metric space and the flow σ is minimal and almost periodic. These conditions ensure the ergodic uniqueness on this base flow, and the proofs of the analogous results just require to change the Lebesgue measure on $\mathbb{T}^d$ by the unique ergodic measure on Ω, the Haar measure.

The reader can find in Katok and Hasselblatt [30], Fink [14], Ellis [12], Sacker and Sell [47], Shen and Yi [52] and references therein the basic properties on Kronecker, minimal, almost periodic and almost automorphic flows, as well as several results of extensibility of semiflows.

3. Strange nonchaotic attractors

Our approach to the concept of strange nonchaotic attractor is described in this section. As in most of the examples appearing in the literature, they will be non-closed bounded invariant objects for a discrete monotone semiflow defined on a real one-dimensional bundle over $\mathbb{T}^d$. In addition we will show the connection between the SNAs of this type and the almost automorphic extensions of the base flow.

Let us consider a discrete Kronecker flow $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Z})$ and a discrete continuous skew-product semiflow $(\mathbb{T}^d \times \mathbb{R}, \phi, \mathbb{Z}^+)$ projecting onto $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Z})$, written as $\phi(n,\omega,x) = (\omega{\cdot}n, \varphi(n,\omega,x))$ (where $\omega{\cdot}n \equiv \sigma_\alpha(n,\omega) = \omega + n\,\alpha$). As explained in the previous section, the semiflow ϕ is given by the successive iterations of the continuous map $\phi_1 : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}\,,\ (\omega,x) \mapsto \phi(1,\omega,x)$, which is a homeomorphism if and only if ϕ is a flow. We assume in this section the following conditions on ϕ.

Hypotheses 3.1. *There exists the partial derivative $\varphi_x(1,\omega,x)$, the map $\mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, $(\omega,x) \mapsto \varphi_x(1,\omega,x)$ is continuous, and $\varphi_x(1,\omega,x) > 0$ for every $(\omega,x) \in \mathbb{T}^d \times \mathbb{R}$.*

Note that this guarantees the existence, continuity and positiveness on $\mathbb{T}^d \times \mathbb{R}$ of $\varphi_x(n,\omega,x)$ for every $n \in \mathbb{Z}^+$, since the chain rule applied to (2.2) gives

$$\varphi_x(n,\omega,x) = \varphi_x(1,\omega{\cdot}(n-1), \varphi(n-1,\omega,x))\,\varphi_x(n-1,\omega,x)\,. \tag{3.1}$$

In addition, under Hypotheses (3.1) the discrete semiflow ϕ satisfies the following fundamental *monotonicity* condition:

$$\varphi(1,\omega,x_1) < \varphi(1,\omega,x_2) \quad \text{whenever } \omega \in \mathbb{T}^d \text{ and } x_1, x_2 \in \mathbb{R} \text{ with } x_1 < x_2\,. \tag{3.2}$$

Definition 3.2. A *ϕ-invariant curve* is a (Lebesgue) measurable map $c : \mathbb{T}^d \to \mathbb{R}$ defined everywhere which is bounded and such that $\varphi(1,\omega,c(\omega)) = c(\omega{\cdot}1)$ for every $\omega \in \mathbb{T}^d$. The corresponding *ϕ-invariant graph* is the set $C = \{(\omega, c(\omega)) \mid \omega \in \mathbb{T}^d\} \subset \mathbb{T}^d \times \mathbb{R}$.

It is easy to check that the measure μ_C defined for the continuous functions $g \in C(\mathbb{T}^d \times \mathbb{R}, \mathbb{R})$ by

$$\int_{\mathbb{T}^d \times \mathbb{R}} g(\omega,x)\, d\mu_C = \int_{\mathbb{T}^d} g(\omega, c(\omega))\; d\omega \tag{3.3}$$

is a ϕ-ergodic measure concentrated on C. Note also that c is continuous if and only if C is closed, in which case this set is a copy of the base.

Definition 3.3. Let $c : \mathbb{T}^d \to \mathbb{R}$ be a ϕ-invariant curve and C the associated graph. The *Lyapunov exponent* of c is defined by

$$\beta_s(c) = \int_{\mathbb{T}^d} \ln \varphi_x(1,\omega,c(\omega))\, d\omega = \int_{\mathbb{T}^d \times \mathbb{R}} \ln \varphi_x(1,\omega,x)\, d\mu_C\,, \tag{3.4}$$

μ_C being the ϕ-ergodic measure defined by (3.3).

There is a strong connection between this measure-theoretic Lyapunov exponent and the usual Lyapunov index of a trajectory of the semiflow. Namely, we associate to each $(\omega_0, x_0) \in \mathbb{T}^d \times \mathbb{R}$ an index $\widetilde{\beta}_s$ by

$$\widetilde{\beta}_s(\omega_0, x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \varphi_x(n, \omega_0, x_0) . \tag{3.5}$$

Note that, as a consequence of (3.1),

$$\widetilde{\beta}_s(\omega_0, x_0) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \varphi_x(1, \phi(j, \omega_0, x_0)) .$$

Birkhoff ergodic theorem and the ϕ-invariance of c show that

$$\beta_s(c) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \varphi_x(1, \omega{\cdot}j, c(\omega{\cdot}j)) = \widetilde{\beta}_s(\omega, c(\omega))$$

for (Lebesgue) a.e. $\omega \in \mathbb{T}^d$.

Following Jäger [21], we will call *strange nonchaotic attractor* (*SNA* for short) to the graph of a non-continuous ϕ-invariant curve with negative Lyapunov exponent. Note that these sets are in fact SNAs *on the real line*. Our goal in the next sections will be to establish conditions ensuring the existence of certain flows $(\mathbb{T}^d \times \mathbb{R}, \phi, \mathbb{Z})$ having an SNA. More precisely, we will determine for such flows a ϕ-invariant curve $c : \mathbb{T}^d \to \mathbb{R}$ such that

- c is continuous at the points of a σ_α-invariant residual subset of $\mathbb{T}^d$,
- c is discontinuous at the points of a σ_α-invariant subset of $\mathbb{T}^d$ with full measure,
- and the Lyapunov of c is $\beta_s(c) < 0$.

In addition, our conditions will ensure that the closure of such a set on $\mathbb{T}^d \times \mathbb{R}$ (which is compact, due to the boundedness of the curve) is a ϕ-minimal subset of the product space. In order to understand a bit better the structure of the SNA, we will describe in the next theorem the shape of the ϕ-minimal subsets: the conditions imposed on the flow ϕ mean a strong restriction on the structure of these sets. Before we associate a new index to the minimal subsets.

Definition 3.4. Let $M \subset \mathbb{T}^d \times \mathbb{R}$ be a positively ϕ-invariant compact subset. The *upper Lyapunov exponent* of M is defined by

$$\beta_s(M) = \sup_{(\omega,x)\in M} \widetilde{\beta}_s(\omega, x) . \tag{3.6}$$

Theorem 3.5. *Let $M \subset \mathbb{T}^d \times \mathbb{R}$ be a ϕ-minimal subset. Then,*

(i) *the restriction of ϕ to M admits a flow extension,*
(ii) *M is an almost automorphic extension of the base flow, and*
(iii) *if $\beta_s(M) < 0$, M is a copy of the base; i.e. M is the graph of a continuous ϕ-invariant curve.*

Proof. (i) Define $M_0 = \cap_{k=1}^{\infty} \phi_1^k M$ and note that each one of its points has a backward orbit, which is necessarily unique by the injectivity in x of $\varphi(n, \omega, x)$ for each (n, ω) (which follows from the monotonicity condition (3.2)). This means that the semiflow on M_0 admits a flow extension (see Section 2). In addition, M_0 is a compact positively ϕ-invariant subset of M, and hence, by minimality, $M = M_0$. This result appears in [52].

(ii) From the minimality of the base flow it follows easily that M projects onto the whole $\mathbb{T}^d$. We define $c_i(\omega) = \inf M_\omega$ and $c_s(\omega) = \sup M_\omega$ where, as usual, $M_\omega = \{x \in \mathbb{R} \mid (\omega, x) \in M\}$. Obviously, the sets $\{(\omega, c_i(\omega)) \mid \omega \in \mathbb{T}^d\}$ and $\{(\omega, c_s(\omega)) \mid \omega \in \mathbb{T}^d\}$ are contained in M. Let us check that they are ϕ-invariant graphs. Assume by contradiction that there exists $\omega_0 \in \mathbb{T}^d$ such that $c_i(\omega_0{\cdot}1) \neq \varphi(1, \omega_0, c_i(\omega_0))$. Then $c_i(\omega_0{\cdot}1) < \varphi(1, \omega_0, c_i(\omega_0))$ and therefore $\varphi(-1, \omega_0{\cdot}1, c_i(\omega_0{\cdot}1)) < c_i(\omega_0)$. Here we use again condition (3.2). But this contradicts the definition of $c_i(\omega_0)$, since $\varphi(-1, \omega_0{\cdot}1, c_i(\omega_0{\cdot}1)) \in M_{\omega_0}$. The proof is analogous in the case of c_s.

It is obvious that the maps c_i and c_s are semicontinuous. Consequently, they are continuous at the points of a (common) residual subset $R \subset \mathbb{T}^d$. Take now $\omega_1 \in R$. Then, by minimality of M, there exists a sequence $(n_k) \uparrow \infty$ such that

$$\begin{aligned}(\omega_1, c_i(\omega_1)) &= \lim_{k\to\infty} (\omega_1{\cdot}n_k, \varphi(n_k, \omega_1, c_s(\omega_1)) \\ &= \lim_{k\to\infty} (\omega_1{\cdot}n_k, c_s(\omega_1{\cdot}n_k)) = (\omega_1, c_s(\omega_1))\,;\end{aligned}$$

that is, $c_i(\omega_1) = c_s(\omega_1)$. Consequently, the section M_{ω_1} reduces to a point, which proves our assertion.

(iii) Our first objective is to show that, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ independent of ω such that $|x - c_i(\omega)| < \delta$ implies $|\varphi(n, \omega, x) - c_i(\omega{\cdot}n)| < \varepsilon$ for every $n \in \mathbb{Z}^+$. We define $r(\omega, y) = \varphi(1, \omega, c_i(\omega) + y) - c_i(\omega{\cdot}1) - \varphi_x(1, \omega, c_i(\omega))\, y$. It is easy to check the existence of $L(y) \leq 0$ such that $|r(\omega, y)| \leq L(y)|y|$ for every $\omega \in \mathbb{T}^d$ and $\lim_{y\to 0^+} L(y)/y = 0$. We fix $(\omega, x) \in \mathbb{T}^d \times \mathbb{R}$ and define $y(n) = \varphi(n, \omega, x) - c_i(\omega{\cdot}n)$. Then y is a solution of the linear difference equation

$$y(n+1) = \varphi_x(1, \omega{\cdot}n, c_i(\omega{\cdot}n))\, y(n) + r(\omega{\cdot}n, y(n))\,. \tag{3.7}$$

The condition $\beta_s(M) < 0$ provides an easy proof of the fact that the sequence $y(n) \equiv 0$ is an exponentially asymptotically stable solution of the associated homogeneous equation. The boundedness properties of $r(\omega, y)$ and the first approximation theorem imply that $y(n) \equiv 0$ is an exponentially asymptotically stable solution of the difference equation (3.7) (see e.g. Lakshmikantham and Trigiante [36]). In addition, these properties are uniform in $\omega \in \mathbb{T}^d$, from where our assertion follows.

Now we fix $\varepsilon > 0$ and take $(\omega, x) \in M$. Let ω_1 be a continuity point for c_i. We choose a sequence $(n_k) \uparrow \infty$ such that $\omega_1 = \lim_{k\to\infty} \omega{\cdot}(-n_k)$ and consider the sequences $(\omega{\cdot}(-n_k), \varphi(-n_k, \omega, x))$ and $(\omega{\cdot}(-n_k), \varphi(-n_k, \omega, c_i(\omega)))$. Then, for a common subsequence (n_j),

$$\lim_{j\to\infty} (\omega{\cdot}(-n_j), \varphi(-n_j, \omega, x)) = \lim_{j\to\infty} (\omega{\cdot}(-n_j), \varphi(-n_j, \omega, c_i(\omega))) = (\omega_1, c(\omega_1))\,,$$

since $M_{\omega_1} = \{c_i(\omega_1)\}$. Consequently, $|\varphi(-m, \omega, x) - \varphi(-m, \omega, c_i(\omega))| < \delta$ for a point $m \in \mathbb{Z}^+$, and going forward we find $|x - c_i(\omega)| < \varepsilon$. Since ε is arbitrary, we conclude that $M_\omega = \{c_i(\omega)\}$ for every $\omega \in \mathbb{T}^d$, which in its turn ensures the continuity of c_i. This completes the proof. A similar result for the one-dimensional torus can be found in [57]. □

In particular, if a ϕ-minimal set M contains an SNA (a non-closed graph), then $\beta_s(M) \geq 0$. This implies that M contains also a ϕ-invariant graph with non-negative Lyapunov index. This assertion is a consequence of the next result, which shows the strong connection between the upper Lyapunov index of a positively ϕ-invariant compact subset and the ϕ-invariant graphs contained in it.

Proposition 3.6. *Let $M\subset\mathbb{T}^d\times\mathbb{R}$ be a positively ϕ-invariant compact subset and define $M_0=\cap_{k=1}^\infty\phi_1^k M$, a non-empty compact ϕ-invariant set. Then,*

(i) *for each $(\omega_0,x_0)\in M$ there exists a ϕ-invariant measure ν_{ω_0,x_0} concentrated on M_0 such that*

$$\beta_s(\omega_0,x_0)=\int_{\mathbb{T}^d\times\mathbb{R}}\ln\varphi_x(1,\omega,x)\,d\nu_{\omega_0,x_0}\,.$$

(ii) *There exists a ϕ-invariant graph $C_0\subset M_0$ such that*

$$\beta_s(M)=\int_{\mathbb{T}^d\times\mathbb{R}}\ln\varphi_x(1,\omega,x)\,d\mu_{C_0}\,.$$

Proof. (i) Let us fix $(\omega_0,x_0)\in M$. We choose an increasing sequence $(n_k)\uparrow\infty$ with $\beta_s(\omega_0,x_0)=\lim_{k\to\infty}(1/n_k)\sum_{j=0}^{n_k-1}\ln\varphi_x(1,\phi(j,\omega_0,x_0))$ and apply Riesz representation theorem in order to associate to the bounded linear operator $C(M,\mathbb{R})\to\mathbb{R}\,,\ g\mapsto(1/n_k)\sum_{j=0}^{n_k-1}g(\phi(j,\omega,x))$ a Borel normalized measure ν_k concentrated on M. Alaoglu theorem ensures that the sequence (ν_k) admits a subsequence (ν_l) which converges weakly to a measure ν_{ω_0,x_0}: $\lim_{l\to\infty}(1/n_l)\sum_{j=0}^{n_l-1}g(\phi(j,\omega,x))=\int_{\mathbb{T}^d\times\mathbb{R}}g(\omega,x)\,d\nu_{\omega_0,x_0}$ for any continuous g. It is easy to check that ν_{ω_0,x_0} is a ϕ-invariant measure concentrated on M. Since $\nu_{\omega_0,x_0}(\phi_1M)=\nu_{\omega_0,x_0}(\phi_1^{-1}\phi_1M)\geq\nu_{\omega_0,x_0}(M)=1$, it turns out that $\nu_{\omega_0,x_0}(\phi_1^kM)=1$ for every $k\in\mathbb{Z}^+$, and this gives $\nu_{\omega_0,x_0}(M_0)=1$.

(ii) We repeat the argument of Theorem 3.5(i) to show that the semiflow on M_0 admits a flow extension. Clearly, each ϕ-invariant measure ν concentrated on M (and hence on M_0) is invariant for this flow. Statement (i) and (3.6) imply the existence of a sequence (ν_k) of ϕ-invariant measures concentrated on M_0 such that $\beta_s(M)=\lim_{k\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}}\ln\varphi_x(1,\omega,x)\,d\nu_k$. Alaoglu theorem ensures the existence of a subsequence (ν_j) converging weakly to a measure ν_0 concentrated on M_0. In particular, $\beta_s(M)=\int_{\mathbb{T}^d\times\mathbb{R}}\ln\varphi_x(1,\omega,x)\,d\nu_0$. It follows easily that ν_0 is a ϕ-invariant measure. The decomposition into ergodic components theorem for the semiflow $(M,\phi,\mathbb{Z}^+)$ ensures then the existence of a ϕ-ergodic measure μ_0 such that $\beta_s(M)=\int_{\mathbb{T}^d\times\mathbb{R}}\ln\varphi_x(1,\omega,x)\,d\nu_0$ (see Mañé [38]). Statement (ii) follows from the fact that every ϕ-ergodic measure μ concentrated on M_0 agrees with μ_C for a ϕ-invariant graph $C\subset M_0$ (see Arnold [1]). □

As said before, we will establish conditions ensuring the existence of SNAs for certain monotone discrete skew-product flows whose closure is a minimal set M. Therefore, summarizing all the previous assertions, we can conclude the following:

- M contains the graph of a non-continuous ϕ-invariant curve with negative Lyapunov exponent, which is dense in it: the SNA whose closure gives rise to the minimal;
- the section M_ω reduces to a point for a residual subset of the base $\mathbb{T}^d$, since M is an almost automorphic extension of the base;
- M contains the graph of another ϕ-invariant curve with non-negative Lyapunov exponent, since otherwise Theorem 3.5(iii) ensures that M agrees with the graph of a continuous ϕ-invariant curve;
- the section M_ω does not reduce to a point for a full-measure subset of the base, as a consequence of the previous properties.

In fact, for the type of SNAs that we will find, the closure will contain the graph of another ϕ-invariant curve with *positive* Lyapunov exponent. So that, to some extent, the dynamics on this minimal set contents ingredients usually associated to the chaotic dynamics.

We end this section by pointing out again that, of course, this type of strange nonchaotic attractors is not the unique one can find. As mentioned in Subsection 1.2, the example given by Keller [31] corresponds to an SNA which is not contained in a minimal subset. The strong connections between this example and the SNAs we will find has already been commented in the particular case of the almost Mathieu equation (1.5): all of them appear as a consequence of the *collision* (as a parameter varies) of the graphs of two separate invariant continuous curves, and their closures are *pinched* sets (i.e. their sections reduce to a point for a residual subset of base points). These assertions will be clarified in Section 5, in Remark 5.7.

4. The projective flows

The first objective of this section is to explain with some detail the three different frameworks we will simultaneously consider throughout the rest of the paper. This constitutes the starting point in the description of our conditions ensuring the occurrence of SNAs, as well as in the description of the dynamics on these sets. The first step in this description is given in the second part of the section, dedicated mainly (Theorem 4.10) to the analysis of the projective dynamics associated to a non-uniformly hyperbolic system.

Recall that $(\mathbb{T}^d, \sigma_\alpha, \mathbb{Y})$ represents a real or discrete Kronecker flow with rationally independent frequency vector α.

Dirac systems. The first setting we consider is given by a family real two-dimensional linear systems of ordinary differential equations with zero-trace, each one given by the evaluation of a continuous function on $\mathbb{T}^d$ along one of the σ_α-trajectories. That is,

$$\mathbf{z}' = S(\omega{\cdot}t)\,\mathbf{z} = \begin{bmatrix} a(\omega{\cdot}t) & b(\omega{\cdot}t) \\ c(\omega{\cdot}t) & -a(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \quad \omega \in \mathbb{T}^d\,, \tag{4.1}$$

with a, b, $c : \mathbb{T}^d \to \mathbb{R}$ continuous.

We point out that this setting comes frequently from a single two-dimensional system with quasi periodic coefficients with irrationally independent frequency vector: the well known *hull* construction includes one of these systems in a family of type (4.1).

Let $Z(t,\omega)$ represent the fundamental matrix-solution of (4.1) with $Z(0,\omega) = I_2$ Then $\mathbf{z}(t,\omega,\mathbf{z}_0) = Z(t,\omega)\,\mathbf{z}_0$ is the solution of (4.1) with initial data $\mathbf{z}(t,\omega,\mathbf{z}_0) = \mathbf{z}_0$. The family of equations induces a continuous real skew-product flow on the linear two-dimensional bundle $\mathbb{T}^d \times \mathbb{R}^2$,

$$\tau : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{T}^d \times \mathbb{R}^2\,, \quad (t,\omega,\mathbf{z}_0) \mapsto (\omega{\cdot}t, \mathbf{z}(t,\omega,\mathbf{z}_0))\,,$$

which is linear on each fiber $\mathbb{R}^2$. On its turn, this linearity ensures that τ induces a new flow on the real projective bundle $K_\mathbb{R} = \mathbb{T}^d \times \mathbb{P}^1$, where we identify $\mathbb{P}^1$ with $\mathbb{R}/\pi\mathbb{Z}$. In order to describe it, let us take polar-symplectic coordinates $\varphi = \cot^{-1}(z_2/z_1)$ and $\rho = (z_1^2 + z_2^2)/2$ on $\mathbb{R}^2$. We obtain from (4.1) the family of equations

$$\begin{aligned}\varphi' &= f(\omega{\cdot}t,\varphi)\\ &= \frac{1}{2}(b(\omega{\cdot}t)-c(\omega{\cdot}t))+\frac{1}{2}(b(\omega{\cdot}t)+c(\omega{\cdot}t))\cos 2\varphi + a(\omega{\cdot}t)\sin 2\varphi\,, \qquad (4.2)\\ \rho' &= -f_\varphi(\omega{\cdot}t,\varphi)\,\rho = ((b(\omega{\cdot}t)+c(\omega{\cdot}t))\sin 2\varphi - 2\,a(\omega{\cdot}t)\cos 2\varphi)\,\rho\,. \qquad (4.3)\end{aligned}$$

The real projective flow is the skew-product

$$\tau_p : \mathbb{R}\times K_{\mathbb{R}} \to K_{\mathbb{R}}\,,\quad (t,\omega,\bar\varphi_0)\mapsto(\omega{\cdot}t,\bar\varphi(t,\omega,\varphi_0))\,, \qquad (4.4)$$

where $\bar\varphi \in \mathbb{P}^1$ represents the equivalence class of an element $\varphi\in\mathbb{R}$ and $\varphi(t,\omega,\varphi_0)$ is the solution of the equation (4.2) with initial data $\varphi(0,\omega,\varphi_0)=\varphi_0$. Note that simple integration solves equations (4.3) once the orbits of the flow τ_p are known. For further purposes, we also point out that

$$\frac{\partial\varphi}{\partial\varphi_0}\,(t,\omega_0,\varphi_0) = \exp\int_0^t f_\varphi(\omega{\cdot}s,\varphi(s,\omega_0,\varphi_0))\,ds \qquad (4.5)$$

for every $(\omega_0,\bar\varphi_0)\in K_{\mathbb{R}}$.

Schrödinger equations. The second framework considered in this paper corresponds to a family of scalar linear second-order Schrödinger equations

$$-z''+s(\omega{\cdot}t)\,z=0\,,\quad \omega\in\mathbb{T}^d\,, \qquad (4.6)$$

with $s:\mathbb{T}^d\to\mathbb{R}$ continuous, which can be easily included in the previous Dirac setting by taking $\mathbf{z}=\left[\begin{smallmatrix} z\\ z'\end{smallmatrix}\right]$ and $S_0(t)=\left[\begin{smallmatrix}0 & 1\\ s_0(t) & 0\end{smallmatrix}\right]$. (However, we will consider in Section 5 the spectral problems associated to (4.1) and (4.6), and the Schrödinger one is not a particular case of the Dirac one.)

Jacobi equations. A third type of equations admitting a parallel analysis corresponds to the family of one-dimensional Jacobi equations

$$-z(n+1)-z(n-1)+v(\omega{\cdot}n)\,z(n)=0\,,\quad \omega\in\mathbb{T}^d\,, \qquad (4.7)$$

with $v:\mathbb{T}^d\to\mathbb{R}$ continuous. These equations can be written in matrix form as

$$\begin{bmatrix} z(n)\\ z(n+1)\end{bmatrix} = D(\omega{\cdot}n)\begin{bmatrix} z(n-1)\\ z(n)\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -1 & v(\omega{\cdot}n)\end{bmatrix}\begin{bmatrix} z(n-1)\\ z(n)\end{bmatrix}. \qquad (4.8)$$

Analogously to the Dirac case, this family of discrete equations induces a discrete skew-product flow on the real bundle $\mathbb{T}^d\times\mathbb{R}^2$, given by the iterations of the map $T(\omega,\mathbf{z})=(\omega{\cdot}1,D(\omega)\,\mathbf{z})$. Let us represent $Z(n,\omega)=D(\omega{\cdot}(n-1))\cdots D(\omega)$: i.e. $Z(n,\omega)$ is the *propagation matrix* of the equation corresponding to ω. Then, if $z(n,\omega,\mathbf{z}_0)$ represents the solution of (4.7) with initial data $\left[\begin{smallmatrix} z(-1,\omega,\mathbf{z}_0)\\ z(0,\omega,\mathbf{z}_0)\end{smallmatrix}\right]=\mathbf{z}_0$, one has that $T^n(\omega,\mathbf{z}_0)=(\omega{\cdot}n,Z(n,\omega)\,\mathbf{z}_0)=(\omega{\cdot}n,\mathbf{z}(n,\omega,\mathbf{z}_0))$. In order to unify notations, we represent this discrete flow by $\tau:\mathbb{Z}\times\mathbb{T}^d\times\mathbb{R}^2\to\mathbb{T}^d\times\mathbb{R}^2$; that is, $\tau(n,\omega,\mathbf{z})=T^n(\omega,\mathbf{z})$.

As before, one can consider the induced flow on the projective bundle, $(K_{\mathbb{R}},\tau_p,\mathbb{Z})$, defined by the solutions of the equations

$$\varphi(n+1)=\cot^{-1}(-\tan\varphi(n)+v(\omega{\cdot}n))\,,\quad \omega\in\mathbb{T}^d\,, \qquad (4.9)$$

obtained from (4.7) by taking $\varphi=\cot^{-1}(z(n+1)/z(n))$. Appendix B contains the most basic facts concerning the construction of a *suspension* of the discrete flow τ, which provides a Dirac system defining a real continuous skew-product flow over a minimal and almost periodic base whose restriction to discrete instants of time reproduces the initial discrete one.

In what follows we consider simultaneously the Dirac and Jacobi families (4.1) and (4.8), so that t represents an element of $\mathbb{R}$ or $\mathbb{Z}$. It is possible to distinguish between three very different dynamical situations: the elliptic, uniformly hyperbolic and non-uniformly hyperbolic cases. In order to describe them, we recall the fundamental concepts of Lyapunov exponents and exponential dichotomy. As usual, $|\mathbf{z}|$ will represent the Euclidean norm of the vector $\mathbf{z}$ and $\|Z\|$ will denote the induced matrix norm.

Definition 4.1. The four *characteristic exponents* of a pair $(\omega_0, \mathbf{z}_0) \in \mathbb{T}^d \times \mathbb{R}^2$ are the values of the limits

$$\limsup_{t\to\pm\infty} \frac{1}{t}\, \ln |\mathbf{z}(t,\omega_0,\mathbf{z}_0)| \qquad \text{and} \qquad \liminf_{t\to\pm\infty} \frac{1}{t}\, \ln |\mathbf{z}(t,\omega_0,\mathbf{z}_0)|\,, \tag{4.10}$$

which are invariant along the trajectories. In the case that the four limits (4.10) agree, their value is one of the *Lyapunov exponents* of the system.

Remark 4.2. It is known that there exist two Lyapunov exponents, $-\beta$ and β, with $\beta \geq 0$. Moreover,

$$\beta = \lim_{|t|\to\infty} \frac{1}{t}\, \ln \|Z(t,\omega)\|$$

for every element ω of a σ_α-invariant subset $\Omega_0 \in \mathbb{T}^d$ with full measure. These facts are consequences of Oseledets multiplicative ergodic theorem [45] (see also [**?**]), which in addition, in the case of $\beta > 0$, provides a measurable decomposition $\mathbb{T}^d \times \mathbb{R}^2 = W^+ \oplus W^-$ in two one-dimensional subbundles with $\lim_{|t|\to\infty}(1/t)\ln|\mathbf{z}(t,\omega,\mathbf{z}_0)| = \mp\beta$ for $\omega \in \Omega_0$ and $(\omega,\mathbf{z}_0) \in W^\pm$ with $\mathbf{z}_0 \neq \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. The sets W^+ and W^- are usually called *Oseledets subbundles.*

Definition 4.3. A family of m-dimensional linear equations $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ (resp. $\mathbf{z}(t+1) = B(\omega{\cdot}t)\,\mathbf{z}(t)$) has an *exponential dichotomy over* $\mathbb{T}^d$ if there exist two positive constants η, γ and a family of projections $P(\omega) : \mathbb{R}^m \to \mathbb{R}^m$ varying continuously on $\mathbb{T}^d$ such that

(i) $\|Z(t,\omega)\,P(\omega)\,Z^{-1}(s,\omega)\| \leq \eta e^{-\gamma(t-s)}$ for every $t,s \in \mathbb{R}$ (resp. $t,s \in \mathbb{Z}$) with $t \geq s$,

(ii) $\|Z(t,\omega)\,(I_d - P(\omega))\,Z^{-1}(s,\omega)\| \leq \eta e^{\gamma(t-s)}$ for every $t,s \in \mathbb{R}$ (resp. $t,s \in \mathbb{Z}$) with $t \leq s$,

where $Z(t,\omega)$ is the fundamental matrix solution with $Z(0,\omega) = I_d$ (resp. the propagation matrix).

Remark 4.4. In the case of the two-dimensional family (4.1) (resp. (4.8)), the determinant of the fundamental matrix $Z(t,\omega)$ (resp. of the propagation matrix $Z(n,\omega)$) is equal to 1. As a consequence of this fact, if an exponential dichotomy occurs, the linear spaces $l^+(\omega)$ and $l^-(\omega)$ respectively given by the range and kernel of $P(\omega)$ are one-dimensional. Hence they determine closed subbundles $L^\pm = \{(\omega,\mathbf{z}_0) \mid \omega \in \mathbb{T}^d,\ \mathbf{z}_0 \in l^\pm(\omega)\}$ of $\mathbb{T}^d \times \mathbb{R}^2$. According to Sacker and Sell results [48, 49], these subbundles are inlvariant for the corresponding flow (in other words, $P(\omega{\cdot}t)Z(t,\omega) = Z(t,\omega)P(\omega)$) and $\mathbb{T}^d \times \mathbb{R}^2 = L^+ \oplus L^-$ as topological (Whitney) sum. It is habitual to refer to L^+ and L^- as the *stable subbundles at* $-\infty$ *and* $+\infty$, respectively, and also as the *Sacker-Sell subbundles.*

Remark 4.5. The dynamical classification mentioned before for a family of systems of the type (4.1) or (4.8) follow from these results. The three possibilities are

- the *elliptic case*, when $\beta = 0$; in this case 0 is the unique characteristic exponent of any solution of any equation of the family,
- the *uniformly hyperbolic case*, in the case that an exponential dichotomy occurs; in particular, $\beta > 0$,
- and the *non-uniformly hyperbolic case*, when $\beta > 0$ but there is not an exponential dichotomy.

In addition, according to Sacker and Sell results [50], in the case of an exponential dichotomy, $L^{\pm} = W^{\pm}$ and the characteristic exponents of any trajectory lie in $\{-\beta, \beta\}$; while in the non-uniformly hyperbolic case, the Oseledets subbundles are not closed, and the four characteristic exponents of any pair $(\omega_0, \mathbf{z}_0)$ lie in the interval $[-\beta, \beta]$, which is the *dynamical spectrum* of the family.

The following characterization of the presence of exponential dichotomy plays a fundamental role in order to understand the dynamical meaning of this property.

Theorem 4.6. [Sacker and Sell [48, 49], Selgrade [51]] *The family* (4.1) $\mathbf{z}' = A(\omega{\cdot}t)\,\mathbf{z}$ *(resp.* $\mathbf{z}(t+1) = B(\omega{\cdot}t)\,\mathbf{z}(t)$*) has an exponential dichotomy over* $\mathbb{T}^d$ *if and only if no one of its equations admits a non-trivial bounded solution.*

Our next objective is to analyze some facts concerning the dynamical structures of the flows $(\mathbb{T}^d \times \mathbb{R}^2, \tau, \mathbb{Y})$ and $(K_{\mathbb{R}}, \tau_p, \mathbb{Y})$ (which of course are strongly connected) in the hyperbolic cases. In particular we will pay attention to the number of τ_p-minimal subsets of the projective bundle and to the limiting behavior of those τ-trajectories starting at points of these minimal subsets. In this description the following maps and sets play a fundamental role.

Definition 4.7. A τ_p*-invariant curve* is a (Lebesgue) measurable map $\bar{\psi} : \mathbb{T}^d \to \mathbb{P}^1$ defined everywhere such that $\bar{\varphi}(t, \omega, \bar{\psi}(\omega)) = \bar{\psi}(\omega{\cdot}t)$ for every $\omega \in \mathbb{T}^d$ and every $t \in \mathbb{Y}$. The corresponding τ_p*-invariant graph* is the set $K = \{(\omega, \bar{\psi}(\omega)) \mid \omega \in \mathbb{T}^d\} \subset K_{\mathbb{R}}$.

It is easy to check that the measure μ_K defined for the functions $g \in C(K_{\mathbb{R}}, \mathbb{R})$ by

$$\int_{\mathbb{T}^d \times \mathbb{R}} g(\omega, x)\, d\mu_K = \int_{\mathbb{T}^d} g(\omega, \bar{\psi}(\omega))\, d\omega \tag{4.11}$$

is a τ_p-ergodic measure concentrated on K. Note also that $\bar{\psi}$ is continuous if and only if K is closed, in which case this set is a copy of the base.

Remark 4.8. Clearly, there is a strong similitude between the concepts of ϕ-invariant curve and graph given in Definition 3.2 and τ_p-invariant curve and graph given just above. However, note that the first one is associated to a (discrete) flow on $\mathbb{T}^d \times \mathbb{R}$, while the second one corresponds to a (discrete or continuous) flow on $\mathbb{T}^d \times \mathbb{P}^1$. This difference is important enough to justify these new definitions.

The next two results explore the two different possibilities arising in the uniformly and non-uniformly hyperbolic cases. We present a unified approach for the Dirac (4.1), Schrödinger (4.6) and Jacobi (4.7) cases: as explained above, the family (4.6) can be immediately written as a family of systems of the type (4.1); while Appendix B explains how to obtain a real skew-product flow whose restriction to discrete times agrees with the flow associated to (4.7), and this flow is induced by a new system of type (4.1) (for which the base flow is not a Kronecker flow but still is an almost periodic and minimal). So that the statements of these results are valid for the three cases, although they will be enounced for systems (4.1). Also the proofs will refer to the continuous cases: they can be easily adapted to the framework detailed in Appendix B (see Remark 2.1).

Before stating these results, we need some more notation: any $\bar\varphi_0 \in \mathbb{P}^1$ admits a unique representative $\varphi_0 \in [0,\pi)$. For $\varphi_0 \in [0,\pi)$, we define $\mathbf{z}_{\varphi_0} = \left[\begin{smallmatrix}\sin\varphi_0\\ \cos\varphi_0\end{smallmatrix}\right]$ and $\mathbf{z}(t,\omega,\varphi_0) = \mathbf{z}(t,\omega,\mathbf{z}_{\varphi_0})$ (the solution of the system (4.1) with initial data $\mathbf{z}_{\varphi_0}$).

The structure is quite simple in the uniformly hyperbolic case (i.e. in the case of occurrence of exponential dichotomy): the following proposition is a direct consequence of Definition 4.3.

Proposition 4.9. *Assume that the family* (4.1) *is in the uniformly hyperbolic case, and let $\beta > 0$ be its positive Lyapunov exponent. Then $K_\mathbb{R}$ contains exactly two τ_p-minimal subsets, M^+ and M^-, which are copies of the base flow $(\mathbb{T}^d,\sigma_\alpha,\mathbb{R})$: $M^\pm = \{(\omega,\bar\varphi^\pm(\omega)) \mid \omega\in\mathbb{T}^d\}$, with $\bar\varphi^\pm:\mathbb{T}^d\to\mathbb{P}^1$ continuous. In addition,*

$$\mp\beta = \lim_{|t|\to\infty} \frac{1}{t}\,\ln|\mathbf{z}(t,\omega,\varphi^\pm(\omega))|$$

for every $\omega\in\mathbb{T}^d$, the convergence being uniform in $\mathbb{T}^d$.

Proof. The only minimal subsets of $K_\mathbb{R}$ are given by the projections of the Sacker-Sell subbundles $L^\pm$ given by the exponential dichotomy (see Remark 4.4). □

However, the dynamical behavior is quite more complicated in the non-uniformly hyperbolic case, as described in the last result of this section. It is based on some ideas of [22] and [26], to which we add some more precise analysis of the oscillation properties. A complete and detailed proof is included, since the result is fundamental for the rest of the paper.

Theorem 4.10. *Assume that the family* (4.1) *is in the non-uniformly hyperbolic case, and let $\beta > 0$ be its positive Lyapunov exponent. Then,*

(i) *$K_\mathbb{R}$ contains a unique τ_p-minimal subset M. This set M is not uniquely ergodic: it supports two different τ_p-ergodic measures, which in addition are the unique τ_p-ergodic measures in $K_\mathbb{R}$. These measures are associated by relation* (4.11) *to two (non-closed) τ_p-invariant graphs $\{(\omega,\bar\varphi^\pm(\omega)) \mid \omega\in\mathbb{T}^d\}$. In addition, there exists a σ_α-invariant subset Ω_0 of $\mathbb{T}^d$ with full measure such that*

$$\mp\beta = \lim_{|t|\to\infty} \frac{1}{t}\,\ln|\mathbf{z}(t,\omega,\varphi^\pm(\omega))| \tag{4.12}$$

for every $\omega\in\Omega_0$.

(ii) *For every $(\omega,\bar\varphi)$ in a residual subset R of M, there exist four sequences (t_k^i) with $\lim_{k\to\infty} t_k^i = \infty$ for $i=1,3$ and $\lim_{k\to\infty} t_k^i = -\infty$ for $i=2,4$, and such that*

$$\lim_{k\to\infty} \ln|\mathbf{z}(t_k^i,\omega,\varphi)| + \beta\,t_k^i = (-1)^i\infty \quad \textit{for}\ \ i=1,2 \tag{4.13}$$

and

$$\lim_{k\to\infty} \ln|\mathbf{z}(t_k^i,\omega,\varphi)| - \beta\,t_k^i = (-1)^{i+1}\infty \quad \textit{for}\ \ i=3,4\,. \tag{4.14}$$

In particular, for any $(\omega,\bar\varphi)\in R$,

$$\limsup_{t\to\pm\infty}\frac{1}{t}\,\ln|\mathbf{z}(t,\omega,\varphi)|=\beta \quad \textit{and} \quad \liminf_{t\to\pm\infty}\frac{1}{t}\,\ln|\mathbf{z}(t,\omega,\varphi)|=-\beta\,.$$

In other words, the points of the residual subset $R\subset M$ *give rise to* τ*-trajectories which oscillate exponentially both at* $+\infty$ *and* $-\infty$.

Proof. (i) Oseledets multiplicative ergodic theorem [45] ensures the existence of a σ_α-invariant subset $\Omega_0\subset\mathbb{T}^d$ with full measure and two τ_p-invariant graphs in $K_\mathbb{R}$, $\bar W^\pm=\{(\omega,\bar\varphi^\pm(\omega))\mid\omega\in\mathbb{T}^d\}$, such that relations (4.12) hold for $\omega\in\Omega_0$: $\bar W^\pm$ are the projections on $K_\mathbb{R}$ of the Oseledets subbundles (see Remark 4.2 and note that we can define $\bar\varphi^\pm$ to be invariant outside Ω_0). Relations (4.12) and

$$|\mathbf{z}(t,\omega,\varphi^\pm(\omega))|=\exp\left(-\frac{1}{2}\int_0^t f_\varphi(\tau_p(s,\omega,\bar\varphi^\pm(\omega)))\,ds\right)$$

(deduced from equation (4.3)), and Birkhoff ergodic theorem, ensure then that

$$\mp\beta=-\frac{1}{2}\int_{K_\mathbb{R}} f_\varphi(\omega,\bar\varphi)\,d\mu_{\bar W^\pm}\,, \tag{4.15}$$

where $\mu_{\bar W^\pm}$ are the τ_p-ergodic measures defined by (4.11). Assume the existence of any other τ_p-ergodic measure ν on $\mathbb{T}^d\times\mathbb{R}$ and define $\gamma=(-1/2)\int_{K_\mathbb{R}} f_\varphi(\omega,\bar\varphi)\,d\nu$. Assume also that $\gamma\ge 0$ (the proof is symmetric in the other case). Birkhoff ergodic theorem allows us to take $\omega_0\in\Omega_0$ and $\bar\varphi_0\in\mathbb{P}^1-\{\bar\varphi^-(\omega_0)\}$ such that

$$\gamma=\lim_{t\to-\infty}-\frac{1}{2t}\int_0^t f_\varphi(\tau_p(s,\omega_0,\bar\varphi_0))\,ds=\lim_{t\to-\infty}\frac{1}{t}\ln|\mathbf{z}(t,\omega_0,\varphi_0)|\,. \tag{4.16}$$

On the other hand, given two distinct points $\varphi_1,\ \varphi_2\in[0,\pi)$, $\{\mathbf{z}(t,\omega,\varphi_1),\mathbf{z}(t,\omega,\varphi_2)\}$ is a fundamental system of solutions for the equation (4.1) corresponding to ω, and hence, since any fundamental matrix solution has constant determinant,

$$|\mathbf{z}(t,\omega,\varphi_1)||\mathbf{z}(t,\omega,\varphi_2)|\sin(\varphi(t,\omega,\varphi_1)-\varphi(t,\omega,\varphi_2))=\sin(\varphi_1-\varphi_2)\,. \tag{4.17}$$

Therefore, relations (4.12) and (4.16) lead us to the contradiction

$$0<|\sin(\bar\varphi^-(\omega_0)-\bar\varphi_0)|\le\lim_{t\to-\infty}|\mathbf{z}(t,\omega_0,\varphi^-(\omega_0))||\mathbf{z}(t,\omega_0,\varphi_0)|=0\,.$$

Consequently, $\mu_{\bar W^\pm}$ are the unique τ_p-ergodic measures on the projective bundle.

Let M be any τ_p-minimal subset of $K_\mathbb{R}$ (maybe the whole space). According to Krylov-Bogoliubov theorem, the compact metric space M supports a τ_p-ergodic measure, say $\mu_{\bar W^+}$. Assume now that this is the unique τ_p-ergodic measure on M. Using (4.15) and applying Birkhoff ergodic theorem to the uniquely ergodic case,

$$-\beta=\lim_{|t|\to\infty}-\frac{1}{2t}\int_0^t f_\varphi(\tau_p(s,\omega,\bar\varphi))\,ds=\lim_{|t|\to\infty}\frac{1}{t}\ln|\mathbf{z}(t,\omega,\varphi)| \tag{4.18}$$

uniformly on $(\omega,\bar\varphi)\in M$ (see Mañé [38]). On the other hand, the absence of an exponential dichotomy and Theorem 4.6 imply the existence of $(\omega_0,\bar\varphi_0)\in\Omega_0\times\mathbb{P}^1$ with $\sup_{t\in\mathbb{R}}|\mathbf{z}(t,\omega_0,\varphi_0)|<\infty$. In particular, $\bar\varphi_0\ne\bar\varphi^+(\omega_0)$, as deduced from (4.12), while by (4.18) $\lim_{t\to\infty}|\mathbf{z}(t,\omega_0,\varphi_0)|=\lim_{t\to\infty}|\mathbf{z}(t,\omega_0,\varphi^+(\omega_0))|=0$. This contradicts again (4.17).

This shows that M cannot be uniquely ergodic. Consequently, it supports the unique two τ_p-ergodic measures on $K_\mathbb{R}$, which in its turn precludes the existence of any other minimal subset. Obviously, the sets $\{(\omega, \bar\varphi^\pm(\omega)) \mid \omega \in \mathbb{T}^d\}$ cannot be closed: otherwise M would not be minimal. The proof of (i) is complete.

(ii) The proof of this assertion is strongly based on the analysis of the dynamical behavior of the solutions of a family of linear scalar equation, detailed in Appendix A. Let us consider the scalar equations

$$x' = \left(-\frac{1}{2}\, f_\varphi(\tau_p(t,\omega,\bar\varphi)) + \beta\right) x \quad \text{and} \quad x' = -\frac{1}{2}\, f_\varphi(\tau_p(t,\omega,\bar\varphi)) + \beta \tag{4.19}$$

for $(\omega,\bar\varphi) \in M$. According to relation (4.15), the integral of the coefficient function of the first equation with respect to the ergodic measure ν^+ is zero; so that Birkhoff ergodic theorem precludes the existence of an exponential dichotomy for this equation. On its turn, (4.12) guarantees the unboundedness of the function $\ln|\mathbf{z}(t,\omega,\varphi^-(\omega))| + \beta t$, a solution of the second equation in (4.19). Hence, since the restriction to the flow τ_p to M is minimal, the hypotheses of Theorem A.2 are satisfied. Consequently, for any point $(\omega,\bar\varphi)$ of a residual subset R_1 of M there exist two sequences $(t_k^1) \uparrow \infty$ and $(t_k^2) \downarrow -\infty$ satisfying (4.13).

An analogous argument applied to the equations obtained substituting β by $-\beta$ in (4.19) shows the existence of a residual R_2 and sequences $(t_k^3) \uparrow \infty$ and $(t_k^4) \downarrow -\infty$ for which (4.14) hold. The residual set R is given by $R_1 \cap R_2$.

Finally, from (4.13) and (4.14) it follows easily that, for any $(\omega,\bar\varphi) \in R$,

$$\limsup_{t\to\pm\infty} \frac{1}{t}\, \ln|\mathbf{z}(t,\omega,\varphi)| \geq \beta \quad \text{and} \quad \liminf_{t\to\pm\infty} \frac{1}{t}\, \ln|\mathbf{z}(t,\omega,\varphi)| \leq -\beta\,.$$

Hence the last assertion of the theorem follows from the fact that all the characteristics exponents lie in the dynamical spectrum $[-\beta,\beta]$ (see Definition 4.1 and Remark 4.5). This completes the proof. □

Remarks 4.11. 1. Nothing in Theorem 4.10 allows one to distinguish a priori between the different possibilities for the unique τ_p-minimal set M appearing in the non-uniformly hyperbolic case. There are known examples of this kind of dynamics for which the whole space $K_\mathbb{R}$ is minimal, as those found by Bjerklöv [4], as well as situations in which M is an almost automorphic extension of the base flow, being classical the examples due to Millionščikov [39, 40] and Vinograd [61]. And nothing precludes other different situations.

2. In fact the situation in which M is an almost automorphic extension is the interesting one to our purposes: it will constitute the framework in which we will be able to find SNAs on $\mathbb{R}$, as we will see in Section 6. But something more than non-uniform hyperbolicity is required to guarantee it. These additional conditions will be establishes in next section.

3. However, we have already pointed out in the Introduction that some authors do not impose the real character of the SNA in their definition. For the interested reader we also recall that in the case that the projective flow $(K_\mathbb{R}, \tau_p, \mathbb{Y})$ admits a *bounded mean motion* (i.e. if $\sup_{t\in\mathbb{R}} |\varphi(t,\omega_0,\bar\varphi_0) - \bar\varphi_0 - \rho\, t| < \infty|$ for every $(\omega_0,\bar\varphi_0) \in K_\mathbb{R}$, where ρ is the *rotation number* of the flow), the minimal M is an almost automorphic extension of the base (see [63]). So that the other possibilities for M correspond to the case of unbounded mean motion. According to some authors, in these cases, the non-continuous invariant graph of the map $\bar\varphi^- : \mathbb{T}^d \to \mathbb{P}^1$ would be an *unbounded* SNA on $\mathbb{P}^1$.

5. Extreme points of spectral gaps

In this section we consider the three families of spectral problems associated with the Dirac, Schrödinger and Jacobi operators respectively defined on $L^2(\mathbb{R}^2)$, $L^2(\mathbb{R})$ and $L^2(\mathbb{Z})$ by the families (4.1), (4.6) and (4.7); namely

$$\mathcal{L}^1_\omega \mathbf{z} = \lambda\,\mathbf{z} \quad \text{with} \quad \mathcal{L}^1_\omega z(t) = J(\mathbf{z}'(t) - S(\omega{\cdot}t)\,\mathbf{z}(t)) \ \left(\text{where } J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right), \tag{5.1}$$

$$\mathcal{L}^2_\omega z = \lambda\,z \quad \text{with} \quad \mathcal{L}^2_\omega z(t) = -z''(t) + s(\omega{\cdot}t)\,z(t)\,, \tag{5.2}$$

$$\mathcal{L}^3_\omega z = \lambda\,z \quad \text{with} \quad \mathcal{L}^3_\omega z(n) = -z(n+1) - z(n-1) + v(\omega{\cdot}n)\,z(n)\,. \tag{5.3}$$

Here ω is a fixed point of $\mathbb{T}^d$ and λ represents a complex parameter. We recall that the (classical) *spectrum* of $\mathcal{L}^j_\omega$ is defined as the set of $\lambda \in \mathbb{C}$ such that the operator does not admit a bounded inverse operator. The *resolvent* is the complementary of the spectrum in the complex plane.

As stated below, it turns out that the spectrum is a closed real subset independent of the choice of ω. This section is mainly dedicated to analyze the dynamical behavior of the projective flow given by one of the previous equations when λ in the spectrum is the extreme point of a real interval of the resolvent. This will be the second condition allowing us to define an SNA on $\mathbb{R}$: we will prove (Theorem 5.3) that this hypothesis guarantees that the corresponding projective flow admits at least one almost automorphic extension of the base flow; consequently (Remark 5.7) exactly one –appearing as consequence of the collision of the graphs of two continuous invariant curves and with a very complicate dynamical behavior– when this spectral condition is added to the non-uniform hyperbolicity of the system studied, described in the previous section.

In order to formulate these assertions with more precision, we need to recall previously the strong connection between resolvent and occurrence of an exponential dichotomy.

Remark 5.1. The definitions of characteristic and Lyapunov exponents (Definition 4.1) and the dynamical description provided by Oseledets theorem (Remark 4.2), as well as the concept of exponential dichotomy over Ω (Definition 4.3) and the characteristics of the associated splitting (Remark 4.4), are valid for systems with complex coefficients, like the ones we are now considering: one must just replace $\mathbb{R}^d$ by $\mathbb{C}^d$. Both definitions coincide in the case of real coefficients: the invariant subbundles are real in this case.

Theorem 5.2. [Johnson [25]]

(i) *The family of systems* (5.1) *(resp.* (5.2)*,* (5.3)*) corresponding to a value* $\lambda \in \mathbb{C}$ *of the parameter has an exponential dichotomy over* $\mathbb{T}^d$ *if and only if* λ *belongs to the resolvent of the operator* $\mathcal{L}^1_\omega$ *(resp.* $\mathcal{L}^2_\omega$*,* $\mathcal{L}^3_\omega$*), which is common for every* $\omega \in \mathbb{T}^d$*.*

(ii) *The family* (5.1) *(resp.* (5.2)*,* (5.3)*) corresponding to* $\lambda \in \mathbb{C} - \mathbb{R}$ *has an exponential dichotomy over* $\mathbb{T}^d$*.*

In particular, the spectrum is a closed subset of the real line. So that its complementary on $\mathbb{R}$ is an open set, and hence it is composed by an at most countable union of disjoint open intervals. These intervals are called *spectral gaps*. We point again that the system corresponding to any value of the parameter λ_0 in one of these spectral gaps has an exponential dichotomy over $\mathbb{T}^d$, and we represent by

$\bar\varphi^{\pm}_{\lambda_0}(\omega)$ the continuous maps given by the projective coordinates of the Oseledets subbundles $L^{\pm}_{\lambda_0}$ (see Remarks 4.2 and 4.5 and Proposition 4.9).

The main objective of this section is to prove in detail the following result, in which the behavior of the functions $\bar\varphi^{\pm}_{\lambda_0}$ as λ_0 tends to an extreme point of a spectral gap is analyzed.

Theorem 5.3. *Let $J=(\lambda_1,\lambda_2)$ be a spectral gap. Then $\mp\bar\varphi^{\pm}_{\lambda_0}(\omega)$ moves counterclockwisely along $\mathbb{P}^1$ as λ_0 increases in J for any $\omega\in\mathbb{T}^d$. In the case that $\lambda_2<\infty$ (resp. $\lambda_1>\infty$), there exist the limits $\bar\varphi^{\pm}_{\lambda_2}(\omega)=\lim_{\lambda_0\to\lambda_2^-}\bar\varphi^{\pm}_{\lambda_0}(\omega)$ (resp. $\bar\varphi^{\pm}_{\lambda_1}(\omega)=\lim_{\lambda_0\to\lambda_1^+}\bar\varphi^{\pm}_{\lambda_0}(\omega)$) for any $\omega\in\mathbb{T}^d$. In addition, the functions $\bar\varphi^{\pm}_{\lambda_2}:\mathbb{T}^d\to\mathbb{P}^1$ (resp. $\bar\varphi^{\pm}_{\lambda_1}:\mathbb{T}^d\to\mathbb{P}^1$) are continuous at the points of a σ_α-invariant residual set $R\subset\mathbb{T}^d$. Furthermore, if $\omega_1\in\mathbb{R}$, the sets*

$$M^{\pm}_{\lambda_j}=\text{closure}_{K_\mathbb{R}}\{(\omega_1{\cdot}t,\bar\varphi^{\pm}_{\lambda_j}(\omega_1{\cdot}t))\mid t\in\mathbb{R}\}\,,\quad j=2\quad(\textit{resp. } j=1)\qquad(5.4)$$

are $\tau_{\lambda_j,p}$-invariant and determine almost automorphic extensions of the base flow.

The proof reproduces arguments used in [23]. The version here included is a bit more detailed, since its ideas are fundamental for us in the next section, in which we will obtain conclusions about the existence of SNAs. This proof is strongly based on a careful analysis of the limiting behavior of the Weyl functions, associated a priori to the stable and unstable directions in the case of uniform hyperbolicity. Before recalling their definitions, let us define a new skew-product flow associated to the family of systems that we are working with.

As seen in Section 4, equations (5.2) and (5.3) can be written in a equivalent way as families of two-dimensional systems for each $\lambda\in\mathbb{C}$ fixed. In this way, these families induce linear skew-product flows τ_λ on $\mathbb{T}^d\times\mathbb{C}^2$. For our next purposes, it is convenient to consider the skew-product flows $\widetilde\tau_{\lambda,p}$ determined from the previous ones by the projection onto the complex projective bundle

$$\Pi:\mathbb{T}^d\times\mathbb{C}^2\to K_\mathbb{C}\,,\quad (\omega,\left[\begin{smallmatrix}z_1\\z_2\end{smallmatrix}\right])\mapsto(\omega,z_2/z_1)\,.\qquad(5.5)$$

Here, $K_\mathbb{C}=\mathbb{T}^d\times\mathbb{P}^1_\mathbb{C}$, where $\mathbb{P}^1_\mathbb{C}$ represents the complex projective line: each line in $\mathbb{C}^2$ passing trough the origin, determined by a non-null vector $\mathbf{z}=\left[\begin{smallmatrix}z_1\\z_2\end{smallmatrix}\right]$, is identified with the value $m=z_2/z_1$ in the compactification of $\mathbb{C}$. The complex projective flow is then given by

$$\widetilde\tau_{\lambda,p}:\mathbb{R}(\mathbb{Z})\times K_\mathbb{C}\to K_\mathbb{C}\,,\quad (t,\omega,m_0)\mapsto(\omega{\cdot}t,m_\lambda(t,\omega,m_0))\,,$$

where $m_\lambda(t,\omega,m_0)$ is the solution with $m_\lambda(0,\omega,m_0)=m_0$ of the Ricatti equation

$$m'=(c(\omega{\cdot}t)-\lambda)-2a(\omega{\cdot}t)\,m-(b(\omega{\cdot}t)+\lambda)\,m^2\qquad(5.6)$$

in the case of (5.1); of

$$m'=s(\omega{\cdot}t)-\lambda-m^2\qquad(5.7)$$

in the case of (5.2); and of

$$-m(n+1)-\frac{1}{m(n)}+v(\omega{\cdot}n)=\lambda\qquad(5.8)$$

in the case of (5.3). These equations are respectively obtained from (5.1), (5.2) and (5.3) by defining $m(t)$ as $z_2(t)/z_1(t)$, $z'(t)/z(t)$ and $z(n)/z(n-1)$ (for $t=n\in\mathbb{Z}$ in this last case). Note also that $K_\mathbb{R}\subset K_\mathbb{C}$: the change of variable $m=\cot\bar\varphi$ gives the relation between the new coordinate and the previously considered one and gives the flow isomorphism between the restriction of $\widetilde{\tau}_{\lambda,p}$ to the real projective bundle and the flow $\tau_{\lambda,p}$ defined by the corresponding expression (4.4).

The following result is also proved in [25].

Theorem 5.4. *Let $\mathbb{T}^d \times \mathbb{C}^2 = L^+_\lambda \oplus L^-_\lambda$ be the splitting in two one-dimensional closed τ_λ-invariant subbundles for $\lambda \in \mathbb{C} - \mathbb{R}$. Define $M^\pm_\lambda = \Pi(L^\pm_\lambda)$ (Π defined by (5.5)). Then*

$$M^\pm_\lambda = \big\{(\omega, m^\pm(\omega,\lambda)) \mid \omega \in \mathbb{T}^d\big\} \tag{5.9}$$

with $\pm \operatorname{Im}\lambda \operatorname{Im} m^\pm(\omega,\lambda) > 0$.

Definition 5.5. The functions $m^\pm : \mathbb{T}^d \times (\mathbb{C} - \mathbb{R}) \to K_{\mathbb{C}}$, $(\omega,\lambda) \mapsto m^\pm(\omega,\lambda)$ given by (5.9) are the *Weyl functions* or *m-functions* associated to the family of equations (5.1) (resp. (5.2), (5.3)).

Note that the splitting and hence the m-functions depend on the index j, although this dependence will be omitted in the notation. A different approach to these functions in terms of the unique $L^2(\mathbb{R})$-solution of the equation corresponding to $\lambda \in \mathbb{C} - \mathbb{R}$, based on classical Weyl's classification of two-dimensional systems into point-limit and circle-limit cases [62], can be found e.g. in Kotani [35], Deift and Simon [10] and Sun [58].

Definition 5.5 shows that the functions $m^\pm(\omega,\lambda)$ represent the complex projective coordinates of the stable directions at $\mp\infty$, respectively. Sacker-Sell spectral theory [50] ensures that they are continuous in both variables on the sets $\mathbb{T}^d \times \mathbb{C}^\pm$, where $\mathbb{C}^\pm = \{\lambda \in \mathbb{C} \mid \pm \operatorname{Im}\lambda > 0\}$. It is well known that, for each fixed $\omega \in \mathbb{T}^d$, the maps $\lambda \mapsto m^\pm(\omega,\lambda)$ are analytic on $\mathbb{C}^+$ and $\mathbb{C}^-$; and this and the condition $\pm \operatorname{Im}\lambda \operatorname{Im} m^\pm(\omega,\lambda)$ mean that $\pm m^\pm(\omega,\cdot)$ are Herglotz functions. From their definition it follows easily that $\overline{m^\pm(\omega,\lambda)} = m^\pm(\omega,\overline{\lambda})$. Finally, the τ_λ-invariance of the subbundles $L^\pm_\lambda$ implies that, fixed $\omega \in \mathbb{T}^d$ and $\lambda \in \mathbb{C} - \mathbb{R}$, the functions $t \mapsto m^\pm(\omega{\cdot}t,\lambda)$ are solutions of the associated equation (5.6), (5.7) or (5.8).

The Herglotz character of the Weyl functions guarantees the existence of the radial limits

$$l^\pm_{\lambda_0}(\omega) = \lim_{\varepsilon\to 0^+} m^\pm(\omega, \lambda_0 + i\varepsilon)$$

at (Lebesgue) a.e. $\lambda_0 \in \mathbb{R}$ for each $\omega \in \mathbb{T}^d$ fixed (see Koosis [34]). Fubini theorem shows then that for a.e. $\lambda_0 \in \mathbb{R}$ the limits exist a.e. in $\mathbb{T}^d$. In addition, if $\beta(\lambda_0)$ represents the Lyapunov exponent of the system corresponding to λ_0, then:

Proposition 5.6. *If $\beta(\lambda_0) > 0$, then the limits $l^\pm_{\lambda_0}(\omega)$ exist and belong to $\mathbb{P}^1$ (i.e. $\operatorname{Im} l^\pm\lambda_0(\omega) = 0$) for a.e. $\omega \in \mathbb{T}^d$.*

The proof of this result uses known arguments in spectral theory; it is due to Titchmarsh [60] in the Schrödinger case and extended by Giachetti and Johnson [16] to two-dimensional systems. An alternative proof, based on the properties of the Floquet coefficient, is made in [10] by Deift and Simon, who also prove the result in the Jacobi case.

We recall that $\beta(\lambda_0) > 0$ corresponds to two different situations: uniform and non-uniform hyperbolicity. In the uniformly hyperbolic case, more can be said about this limiting behavior: Sacker-Sell spectral theory implies that the limit exists for every $\omega \in \mathbb{T}^d$, and if $\mathbb{T}^d \times \mathbb{R}^2 = L^+_{\lambda_0} \oplus L^-_{\lambda_0}$, then $\Pi(L^\pm_{\lambda_0}) = \{(\omega, l^\pm_{\lambda_0}(\omega)) \mid \omega \in \mathbb{T}^d\} \subset \mathbb{T}^d \times \mathbb{P}^1$. In particular, the functions $l^\pm_{\lambda_0}$ are continuous functions on $\mathbb{T}^d$. Furthermore, the analytic variation of the spectral subbundles ensures that the functions $m^{\pm}(\omega,\cdot)$ admit a meromorphic extension to $\mathbb{C}-\Sigma^j$, where Σ^j represents the (common) spectrum of $\mathcal{L}^j_\omega$: one must just define $m^{\pm}(\omega,\lambda_0)=l^{\pm}_{\lambda_0}(\omega)$ for each $\lambda_0\in\mathbb{R}-\Sigma^j$.

Note that in the previous paragraphs we are identifying $\mathbb{P}^1$ with the compactification of $\mathbb{R}$, contained in $\mathbb{P}^1_{\mathbb{C}}$: the limits $l^{\pm}_{\lambda_0}(\omega)$ can take the value ∞. In order to work with our usual angular coordinate we should consider $\widetilde{\varphi}^{\pm}_{\lambda_0}(\omega)=\cot^{-1} l^{\pm}_{\lambda_0}(\omega)\in\mathbb{R}/\pi\mathbb{Z}$. Note that $(\omega{\cdot}t,\widetilde{\varphi}^{\pm}_{\lambda_0}(\omega{\cdot}t))=\tau_{\lambda_0,p}(t,\omega,\widetilde{\varphi}^{\pm}_{\lambda_0}(\omega))$: this is easily deduced from the properties of the Weyl functions before summarized. In addition, in the case that λ_0 belongs to a spectral gap, we have $\bar{\varphi}^{\pm}_{\lambda_0}(\omega)=\widetilde{\varphi}^{\pm}_{\lambda_0}(\omega)$, since these functions represent the stable and unstable directions given by the exponential dichotomy.

Proof of Theorem 5.3. Let us fix $\omega_0\in\mathbb{T}^d$. We represent $l^{\pm}_{\omega_0}(\lambda_0)=l^{\pm}_{\lambda_0}(\omega_0)$ for $\lambda_0\in J$ and $m^{\pm}_{\omega_0}(\lambda)=m^{\pm}(\omega_0,\lambda)$ for $\lambda\in\mathbb{C}-\mathbb{R}$. Then,

$$(l^{\pm}_{\omega_0})'(\lambda_0)=\lim_{\varepsilon\to0^+}\frac{\partial\, m^{\pm}_{\omega_0}}{\partial\,\mathrm{Re}\,\lambda}\,(\lambda_0+i\varepsilon)=-\lim_{\mu\to0^+}\frac{\mathrm{Im}\, m^{\pm}_{\omega_0}(\lambda_0+i\mu)}{\mu}\,.$$

This means that $\mp(l^{\pm}_{\omega_0})'(\lambda_0)\ge0$. It follows easily that $\mp\bar{\varphi}^{\pm}_{\lambda_0}(\omega_0)$ moves counterclockwisely along $\mathbb{P}^1$ as λ_0 increases in J, this behavior being independent of the choice of ω_0.

We will assume that λ_2 is a real point and analyze the variation of $\bar{\varphi}^{\pm}_{\lambda_0}$ as $\lambda_0\to\lambda_2^-$, the other case being similar. Let us fix $\lambda_*\in J$. Given ω_0, we can choose representatives $\varphi^{\pm}_{\lambda_*}(\omega_0)$ of $\bar{\varphi}^{\pm}_{\lambda_*}(\omega_0)$ and $\delta_0>0$ with

$$\varphi^{-}_{\lambda_*}(\omega_0)<\varphi^{+}_{\lambda_*}(\omega_0)\le\varphi^{-}_{\lambda_*}(\omega_0)+\pi-\delta_0\,. \tag{5.10}$$

Then, for each $\lambda_0\in(\lambda_*,\lambda_2)$, we can find unique representatives $\varphi^{\pm}_{\lambda_0}(\omega_0)$ of $\bar{\varphi}^{\pm}_{\lambda_0}(\omega_0)$ with

$$\varphi^{-}_{\lambda_*}(\omega_0)\le\varphi^{-}_{\lambda_0}(\omega_0)<\varphi^{+}_{\lambda_0}(\omega_0)\le\varphi^{+}_{\lambda_*}(\omega_0)\,. \tag{5.11}$$

We are using the fact that $\bar{\varphi}^{+}_{\lambda_0}(\omega_0)\neq\bar{\varphi}^{-}_{\lambda_0}(\omega_0)$ for every $\lambda_0\in J$, as well as the continuity and monotonicity of these functions with respect to λ_0. This ensures the existence of $\lim_{\lambda_0\to\lambda_2^-}\varphi^{\pm}_{\lambda_0}(\omega_0)\in\mathbb{R}$. Hence, there also exists $\bar{\varphi}^{\pm}_{\lambda_2}(\omega_0)=\lim_{\lambda\to\lambda_2^-}\bar{\varphi}^{\pm}_{\lambda_0}(\omega_0)\in\mathbb{P}^1$.

This can be done for each $\omega\in\mathbb{T}^d$. In addition, by construction, $\varphi^{\pm}_{\lambda_0}(\omega)-\varphi^{-}_{\lambda_*}(\omega)$ belong to $[0,\pi-\delta_0]$ for each $\omega\in\mathbb{T}^d$ and $\lambda_0\in(\lambda_*,\lambda_2)$. This uniqueness in the choice of representatives ensures the continuity of the functions $\varphi^{\pm}_{\lambda_0}-\varphi^{-}_{\lambda_*}:\mathbb{T}^d\to[0,\pi-\delta_0]$. The limit of a monotone sequence of continuous functions is semicontinuous. This ensures that the functions $\lim_{\lambda_0\to\lambda_2^-}\varphi^{\pm}_{\lambda_0}-\varphi^{-}_{\lambda_*}:\mathbb{T}^d\to[0,\pi-\delta_0]$ (and hence also $\bar{\varphi}^{\pm}_{\lambda_2}-\bar{\varphi}^{-}_{\lambda_*}:\mathbb{T}^d\to\mathbb{P}^1$) are continuous at the points of a residual set $R\in\mathbb{T}^d$. The continuity of $\bar{\varphi}^{-}_{\lambda_*}$ ensures then that the limits $\bar{\varphi}^{\pm}_{\lambda_2}$ are continuous at the points of R. In addition, this set is σ_α-invariant, since for any $t\in\mathbb{Y}$,

$$(\omega{\cdot}t,\bar{\varphi}^{\pm}_{\lambda_2}(\omega{\cdot}t))=(\omega{\cdot}t,\lim\nolimits_{\lambda_0\to\lambda_2^-}\bar{\varphi}^{\pm}_{\lambda_0}(\omega{\cdot}t))=\tau_{\lambda_2,p}(t,\omega,\bar{\varphi}^{\pm}_{\lambda_2}(\omega))\,. \tag{5.12}$$

Fix now $\omega_1\in R$ and note that (5.12) implies the $\tau_{\lambda,p}$-invariance of the compact sets $M^{\pm}_{\lambda_2}$ defined by (5.4). The minimality of the base flow ensures that the projections $M^{\pm}_{\lambda_2}\to\mathbb{T}^d$, $(\omega,\bar{\varphi})\mapsto\omega$ determine surjective flow homomorphisms. Let $\omega_*\in R$ be other continuity point of $\bar{\varphi}^{\pm}_{\lambda_2}$ and assume that $(\omega_*,\bar{\varphi}_*)\in M^{\pm}_{\lambda_2}$. Then there exists a sequence $(t_n)\uparrow\infty$ such that $(\omega_*,\bar{\varphi}_*)=\lim_{n\to\infty}(\omega_1{\cdot}t_n,\bar{\varphi}_{\lambda_2}(\omega_1{\cdot}t_n))$, which gives $\bar{\varphi}_*=\lim_{n\to\infty}\bar{\varphi}_{\lambda_2}(\omega_1{\cdot}t_n)=\bar{\varphi}_{\lambda_2}(\omega_0)$; in other words, the sets $M^{\pm}_{\lambda_2}$ contain a unique element of the fiber over each continuity point ω_*. In particular, $M^{\pm}_{\lambda_2}$ are $\tau_{\lambda_2,p}$-minimal: any non-empty $\tau_{\lambda_2,p}$-minimal subset $K \subset M^{\pm}_{\lambda_2}$ projects onto the whole $\mathbb{T}^d$ and, consequently, it contains $(\omega_1, \bar{\varphi}^{\pm}_{\lambda_2}(\omega_1))$, which implies $K = M^{\pm}_{\lambda_2}$. These properties mean that $(M^{\pm}_{\lambda_2}, \tau_{\lambda_2,p})$ are almost automorphic extensions of $(\mathbb{T}^d, \sigma_\alpha)$, as asserted. Note finally that the sets $M^{\pm}_{\lambda_2}$ are independent of the continuity point ω_1 chosen to define them. □

Remark 5.7. An extreme point λ^* of a spectral gap belongs to the spectrum. Hence there are two possibilities for the family of systems (5.1), (5.2) or (5.3) corresponding to λ^*: either it is elliptic or it is non-uniformly hyperbolic. Assume that it is non-uniformly hyperbolic. Then,

- Theorem 4.10 ensures the existence of a unique $\tau_{\lambda^*,p}$-minimal subset M_{λ^*} (and describes the complicate dynamics on it). Therefore, necessarily $M^{+}_{\lambda^*} = M^{-}_{\lambda^*} = M_{\lambda^*}$.
- Theorem 4.10 also establishes the existence of two different $\tau_{\lambda^*,p}$-invariant graphs contained in M_{λ^*} and associated to $\mp\beta(\lambda^*)$, which clearly agree with $\{(\omega, \bar{\varphi}^{\pm}_{\lambda^*}(\omega)) \mid \omega \in \mathbb{T}^d\}$.
- Proposition 5.6 asserts the existence of the limits $l^{\pm}_{\lambda^*}(\omega) \in \mathbb{P}^1$. It is easy to see that in fact both limits agree: $l^{\pm}_{\lambda^*}(\omega) = \cot \bar{\varphi}^{\pm}_{\lambda^*}(\omega, \lambda^*)$; or, in other words, $\widetilde{\varphi}^{\pm}_{\lambda^*}(\omega) = \bar{\varphi}^{\pm}_{\lambda^*}(\omega)$, as it happened for the values of the parameter in the spectral gap.
- The functions $\bar{\varphi}^{+}_{\lambda^*}(\omega)$ and $\bar{\varphi}^{-}_{\lambda^*}(\omega)$ are different in a σ_α-invariant subset $\Omega_0 \subset \mathbb{T}^d$ of full measure, at which the corresponding points $(\omega, \bar{\varphi}^{\pm}(\omega, \lambda^*))$ satisfy (4.12) for $\beta = \beta(\lambda^*)$; whereas there exists a σ_α-invariant residual set for which the corresponding orbits oscillate in the way described by relations (4.13) and (4.14).

Note the connections between this description and the one of a minimal set containing an SNA made at the end of Section 3. In fact the existence of a set M_{λ^*} of this type will be the starting point to show the existence of SNAs on $\mathbb{R}$ for suitable monotone discrete flows. And note finally that M_{λ^*} appears as a consequence of the collision, as λ_0 (in the spectral gap) tends to λ^* (in the spectrum), of the two copies of the base associated to the exponential dichotomy occurring for λ_0.

6. Examples of SNAs

The information summarized in the previous pages will be finally used in what follows in order to describe situations giving rise to the occurrence of SNAs on the real line. Recall that these sets are associated to discrete flows or semiflows on $\mathbb{T}^d \times \mathbb{R}$. . These flows will be closely related to the flow τ_p induced on $K_{\mathbb{R}}$ by a family of equations of type (4.1), (4.6) or (4.7). So that we will make use of all the notation introduced in Section 4. In our main results (Theorems 6.3 and 6.4), the two hypotheses described in the previous sections will be imposed (non uniformly hyperbolic dynamics at the extreme point of a spectral gap): roughly speaking, these conditions will provide the SNA on $\mathbb{R}$ after an *unwinding procedure* (in some cases) and a discretization (in the case of $\mathbb{Y} = \mathbb{R}$).

6.1. **Vinograd's case.** In order to clarify the analysis, we begin by recalling a well-known example. Vinograd [61] constructs a two-dimensional linear ordinary differential equation of the form

$$\mathbf{z}' = \begin{bmatrix} 0 & 1 + a((y_1, y_2)\cdot t) \\ 1 - a((y_1, y_2)\cdot t) & 0 \end{bmatrix} \mathbf{z}\,, \quad (y_1, y_2) \in \mathbb{T}^2\,, \tag{6.1}$$

where the Kronecker flow on the base $\mathbb{T}^2$ is given by a frequency vector $\alpha = (1, \alpha_2)$ for an irrational number α_2: $(y_1, y_2)\cdot t = (y_1 + t, y_2 + \alpha_2 t)$ for $(y_1, y_2) \in \mathbb{T}^2$. By taking angular coordinate $\varphi = \cot^{-1} z_2/z_1$ we obtain the family of equations

$$\varphi' = a((y_1, y_2)\cdot t) + \cos 2\varphi\,, \quad (y_1, y_2) \in \mathbb{T}^2\,. \tag{6.2}$$

The function a is obtained as $a = \lim_{n\to\infty} a_n$ for a non-decreasing sequence of non-negative functions (a_n), which in its turn is defined by means of an iterative procedure designed to get the following properties: the angular equation $\varphi' = a_n((0,0)\cdot t) + \cos 2\varphi$ has two solutions $\psi_n^{\pm}(t)$ such that

$$\frac{\pi}{4} < \psi_n^-(t) < \psi_{n+1}^-(t) < \psi_{n+1}^+(t) < \psi_n^+(t) < \frac{3\pi}{4}\,,$$

with $\inf_{t\in\mathbb{R}} |\psi_n^+(t) - \psi_n^-(t)| = \delta_n > 0$ and $\lim_{n\to\infty} \delta_n = 0$, and the system obtained by replacing a by a_n in (6.1) has two Lyapunov exponents $\pm\beta_n$ with $\beta_n > 1/2$.

This behavior for one of the systems (6.1) implies that the family is non-uniformly hyperbolic: the positive Lyapunov exponent of (6.1) is $\beta \geq 1/2$ and the family of systems does not have an exponential dichotomy. Theorem 4.10 ensures the occurrence of a unique τ_p-minimal subset $M \subset K_{\mathbb{R}}$ containing two τ_p-invariant graphs $\{((y_1, y_2), \bar\varphi^{\pm}(y_1, y_2)) \mid (y_1, y_2) \in \mathbb{T}^2\}$, whose elements determine solutions of (6.1) which behave as described by (4.12) for every (y_1, y_2) in a σ-invariant subset Ω_0 with full measure. In addition, $M \subset \mathbb{T}^2 \times [\pi/4, 3\pi/4]$, and hence no one of the maps $\varphi(t, (y_1, y_2), \varphi_0)$ for $((y_1, y_2), \varphi_0) \in M$ crosses the infinite point of $\mathbb{P}^1$ (we recall the identification $\mathbb{P}^1 \equiv \mathbb{R}/\pi\mathbb{Z}$). By repeating the arguments of Theorem 3.5, one shows that M is in fact an almost automorphic extension of the base flow. All these properties were already described in the exhaustive analysis of this example made by Johnson in [24].

Now we will explain how M determines an SNA on the real line for a discrete flow. Let us identify $y \in \mathbb{T}^1$ with the point $(0, y) \in \mathbb{T}^2$, so that $\mathbb{T}^1$ is a fixed section of $\mathbb{T}^2$. The restriction of σ_α to $\mathbb{Z} \times \mathbb{T}^1$ is then $\sigma_{\alpha_2}(n, y) = y + n\alpha_2 \equiv y\cdot n$, a one-dimensional Kronecker flow with irrational frequency on the circle. And the skew-product flow

$$\phi : \mathbb{Z} \times \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}\,, \quad (n, y, \varphi_0) \mapsto (y\cdot n, \varphi(n, (0, y), \varphi_0))\,,$$

where $\varphi(t, (y_1, y_2), \varphi_0)$ represents the solution of (6.2) with initial data φ_0, satisfies all the conditions imposed at the beginning of Section 3.

We define $M_0 = \{(y, \varphi) \in \mathbb{T}^1 \times \mathbb{R} \mid ((0, y), \varphi) \in M\} \equiv M \cap (\mathbb{T}^1 \times \mathbb{R})$. Since $M \subset \mathbb{T}^2 \times (0, \pi)$, M_0 is a ϕ-invariant subset. In addition, M_0 is minimal for $(\mathbb{T}^1 \times \mathbb{R}, \phi, \mathbb{Z})$: one shows easily that any one of its trajectories is dense on it. Hence M_0 is an almost automorphic minimal extension of the base flow which contains two different ϕ-invariant graphs, $\{(y, \bar\varphi^{\pm}(0, y)) \mid y \in \mathbb{T}^1\}$. Since the index $\beta_s((0, y), \varphi_0)$ agrees with $\limsup_{t\to\infty} (-1/2t) \ln |\mathbf{z}(t, (0, y), \varphi_0)|$, as deduced from (4.5), we conclude from (4.12) that the ϕ-invariant curves $\mathbb{T}^1 \to (0, \pi)$, $y \mapsto \bar\varphi^{\pm}(0, y)$ have non-null and opposite Lyapunov exponents (note that any σ_{α_2}-invariant subset Ω_0 of $\mathbb{T}^2$ must have points with null first component). In addition, no one of these curves is continuous: otherwise M_0 would not be minimal. Consequently, the ϕ-invariant graph of the map $\bar\varphi^-(0, \cdot)$ is an SNA contained in M_0.

Note that $\bar{\varphi}^+(0,y) \neq \bar{\varphi}^-(0,y)$ in the σ_{α_2}-invariant set $\{y \mid (0,y) \in \Omega_0\}$, which has full measure. Finally, Theorem 4.10 ensures that M_0 does not contain any other ϕ-invariant graph, and hence it is proved in Theorem 3.5(i) that $\bar{\varphi}^+(0,y)$ and $\bar{\varphi}^-(0,y)$ are continuous and coincident at the points y of a residual $\widetilde{\sigma}$-invariant subset of $\mathbb{T}^1$, and that they determine at these points trajectories with the complicated oscillating behavior described by (4.13) and (4.14).

We point out that the point $\lambda^* = 0$, which belongs to the spectrum of the corresponding problem (5.1), is in fact the extreme point of a spectral gap. This assertion follows from a simple analysis of the rotation number of the family of systems (6.1): it is a non-decreasing function of λ, it is positive in the interval (-1,0) (since $1 + a(\cdot) + \lambda \geq 0$), and it vanishes at $\lambda_0 = 0$. Consequently, the rotation number vanishes in the whole interval, and hence Johnson [25] results assure that the family (6.1) has an exponential dichotomy over $\mathbb{T}^2$. Theorem 5.2 implies that $(-1,0)$ is contained in the resolvent.

We also mention that Vinograd example [61] is based on the previous works by Millionščikov [39, 40], who first describes cases of non-uniformly hyperbolic two-dimensional systems with smooth almost periodic coefficients and later refines the analysis in order to obtain quasi periodic examples. Vinograd goes further: in his example the coefficient matrix is analytic and the irrational value of α_2 is arbitrarily chosen.

6.2. **The real unwound cases.** Let us see how to extend the previous ideas to the general almost periodic case. We consider first the Dirac and Schrödinger cases. Assume that our system (4.1) (resp. (4.6)), like Vinograd's one, fits in the situation described in Theorem 4.10 and Theorem 5.3 (see Remark 5.7); i.e. that the dynamics corresponds to the non-uniformly hyperbolic case and $\lambda^* = 0$ is the extreme point of a spectral gap of the spectral problem (5.1) (resp. (5.2)). We represent by $\bar{\varphi}^\pm : \mathbb{T}^d \to \mathbb{P}^1$ the τ_p invariant curves associated to $\mp\beta$ by relation (4.12) and by M the unique τ_r-invariant minimal subset of $K_\mathbb{R}$, which according to Theorem 5.3 is an almost automorphic extension of the base flow.

Note that the solutions of the angular equations (4.2) ($\varphi' = f(\omega{\cdot}t, \varphi)$) define a global skew-product flow on $\mathbb{T}^d \times \mathbb{R}$,

$$\tau_r : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}\,, \quad (t, \omega, \varphi_0) \mapsto (\omega{\cdot}t, \varphi(t, \omega, \varphi_0))\,.$$

In fact $(K_\mathbb{R}, \tau_p, \mathbb{R})$ is obtained from this one by projecting the solutions to $\mathbb{P}^1$. Note also that

$$\ln \frac{\partial \varphi}{\partial \varphi_0}\,(t, \omega_0, \varphi_0 + n\pi) = -2\,\ln |\mathbf{z}(t, \omega_0, \varphi_0)|$$

for every $(\omega_0, \varphi_0) \in \mathbb{T}^d \times [0, \pi)$ and $n \in \mathbb{Z}$, as deduced from (4.5) and (4.3). Consequently, relation (4.12) ensures that

$$\lim_{|t|\to\infty} \frac{1}{t} \ln \frac{\partial \varphi}{\partial \varphi_0}\,(t, \omega, \bar{\varphi}^\pm(\omega)) = \lim_{|t|\to\infty} -\frac{2}{t}\,\ln |\mathbf{z}(t, \omega, \varphi^\pm(\omega)| = \pm 2\beta \tag{6.3}$$

for every ω in a full measure σ_α-invariant subset Ω_0 of $\mathbb{T}^d$.

Assume that the graphs of the curves $\bar{\varphi}^\pm$ are contained in $\mathbb{T}^d \times [\delta, \pi - \delta]$ for a positive value of δ. Or equivalently, that M is contained in $\mathbb{T}^d \times [\delta, \pi - \delta]$. In this case we say that M is *unwound*. As pointed out before, this hypothesis also holds in Vinograd's example. Clearly, in this case, the representatives $\varphi^\pm : \mathbb{T}^d \to (0, \pi)$ of the τ_p-invariant curves $\bar{\varphi}^\pm : \mathbb{T}^d \to \mathbb{P}^1$ are τ_r-invariant curves, and M is also an almost automorphic extension of the base flow for $(\mathbb{T}^d, \tau_r, \mathbb{R})$. Relation (6.3) shows the connection between the Lyapunov exponents of the family (4.1) or (4.7) and the Lyapunov exponents of the τ_r-invariant maps $\varphi^{\pm} : \Omega \to \mathbb{R}$ (which are defined in the same way as in the discrete case, see Definition 3.3)

Let us identify now $\mathbb{T}^{d-1}$ with the points of $\mathbb{T}^d$ with null first component, and note that there are points of Ω_0 in this section of the torus. The vector $\alpha^* = (\alpha_2/\alpha_1, \dots, \alpha_d/\alpha_1)$ (where $\alpha = (\alpha_1, \dots, \alpha_d)$) defines a discrete minimal Kronecker flow σ_{α^*} on $\mathbb{T}^{d-1}$. As before, we define $M_0 = \{(\omega, \varphi) \in \mathbb{T}^{d-1} \times \mathbb{R} \,|\, ((0,\omega),\varphi) \in M\} \equiv M \cap (\mathbb{T}^{d-1} \times \mathbb{R})$ and conclude that M_0 is an almost automorphic extension of $(\mathbb{T}^{d-1}, \sigma_{\alpha^*}, \mathbb{Z})$ for the skew-product flow

$$\phi : \mathbb{Z} \times \mathbb{T}^{d-1} \times \mathbb{R} \to \mathbb{T}^{d-1} \times \mathbb{R}\,, \quad (n, \omega, \varphi_0) \mapsto (\omega + n\alpha^*, \varphi(n/\alpha_1, (0,\omega), \varphi_0))\,.$$

Note that this discrete flow satisfies the conditions imposed at the beginning of Section 3. By repeating the arguments of the previous example, we show that M_0 contains a non-closed ϕ-invariant graph with Lyapunov exponent $-2\beta/\alpha_1$; i.e. an SNA for the flow ϕ_ρ.

Remark 6.1. As seen in Section 5, there is another way to define skew-product flows on $\mathbb{T}^d \times \mathbb{R}$ from the families of equations (4.1) and (4.6): the ones induced by the solutions of the equations (5.6) and (5.7) (for $\lambda = 0$), which are obtained from the initial ones by taking coordinate $m = z_2/z_1$. But note that these flows are not global but just *local*. This is the reason for which we take angular coordinate $\varphi = \cot^{-1}(z_2/z_1)$ in order to find the suitable framework for the existence of SNAs.

Remark 6.2. We point out that the previous arguments provide an infinite number of identic SNAs for each one of the constructed flows on $\mathbb{T}^d \times \mathbb{R}$: each string $\mathbb{T}^d \times [n\pi, (n+1)\pi)$, $n \in \mathbb{Z}$, contains one of these sets.

6.3. **The real general cases.** Not only the dynamics on the τ_p-minimal set M is important in the previous arguments: its unwound character is crucial in order to obtain an SNA *on the real line*. Nevertheless, the next result shows that it is not necessary to impose this condition a priori: a suitable flow transformation provides a new flow admitting an unwound almost automorphic minimal set M^* with the same dynamical behavior as M. Consequently, the previous arguments can be applied to this new flow in order to describe situations of occurrence of SNAs.

Theorem 6.3. *Assume that the family* (4.1) *(resp.* (4.6)*) is in the non-uniformly hyperbolic case and that* $\lambda^* = 0$ *is the extreme point of any spectral gap of the problem* (5.1) *(resp.* (5.2)*). Then,*

(i) *there exists a flow homeomorphism taking* $(K_\mathbb{R}, \tau_p, \mathbb{R})$ *to a new skew-product flow* $\tau_p^* : \mathbb{R} \times K_\mathbb{R} \to K_\mathbb{R}\,,\ (t, \omega, \bar\psi_0) \mapsto (\omega{\cdot}t, \bar\psi(t, \omega, \bar\psi_0))$ *for which there exists a unique minimal set* M^* *given by an unwound almost automorphic extension of the base flow.*

(ii) *The flow* τ_p^* *agrees with the projection onto* $K_\mathbb{R}$ *of the (global) flow* $\tau_r^* : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}\,,\ (t, \omega, \psi_0) \mapsto (\omega{\cdot}t, \psi(t, \omega, \psi_0))$ *induced by a family of differential equations*

$$\psi' = g(\omega{\cdot}t, \psi)\,, \tag{6.4}$$

given by a function $g : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ *which is* π*-periodic in* ψ *and continuous. The set* M^* *is also a* τ_r^**-minimal almost automorphic extension of* $(\mathbb{T}^d, \sigma)$*.*

(iii) *M^* contains exactly two τ_r^*-invariant graphs $\{(\omega,\psi^\pm(\omega))\,|\,\omega\in\mathbb{T}^d\}$ which are non-closed and such that*

$$\lim_{|t|\to\infty}\frac{1}{t}\ln\frac{\partial\psi}{\partial\psi_0}\,(t,\omega,\psi^\pm(\omega))=\pm 2\beta$$

for a.e. $\omega\in\mathbb{T}^d$, where $\beta>0$ is the Lyapunov exponent of the initial family (4.1) *(resp.* (4.6)*).*

Proof. (i) To fix ideas, we assume that $(\lambda_1,0)$ is one of the spectral gaps. We maintain the notation of Theorem 5.3 (in particular, $\tau_p=\tau_{0,p}$) and use the construction made in its proof, for which we fix $\lambda_0\in(\lambda_1,0)$.

Let us define

$$h:K_\mathbb{R}\to K_\mathbb{R},\quad (\omega,\bar\varphi)\mapsto(\omega,\bar\varphi-\bar\varphi^-_{\lambda_0}(\omega))\,,$$

and consider the flow $(K_\mathbb{R},\tau_p^*,\mathbb{R})$ obtained from τ_p by the relation $\tau_p^*(t,\omega,\bar\psi_0)=h(\tau_p(t,h^{-1}(\omega,\bar\psi_0)))$:

$$\tau_p^*(t,\omega,\bar\psi_0)=(\omega{\cdot}t,\bar\psi(t,\omega,\bar\psi_0))=\big(\omega{\cdot}t,\bar\varphi(t,\omega,\bar\psi_0+\bar\varphi^-_{\lambda_0}(\omega))-\bar\varphi^-_{\lambda_0}(\omega{\cdot}t)\big)\;.$$

Let M be the unique minimal set of the initial flow. Then $M^*=h(M)$ is the unique minimal set for τ_p^* and it is an almost automorphic extension of the base flow. Moreover, it is contained in $\mathbb{T}^d\times[0,\pi-\delta_0]$, as deduced from (5.10), (5.11) and (5.4). Hence, by compactness, it is also contained in $\mathbb{T}^d\times[\delta,\pi-\delta]$ for $\delta>0$ small enough.

(ii) It follows from its definition that τ_p^* comes from the family of differential equations (6.4) for

$$g(\omega,\psi)=f_0(\omega,\psi+\bar\varphi^-_{\lambda_0}(\omega))-f_{\lambda_0}(\omega,\bar\varphi^-_{\lambda_0}(\omega))\,,$$

where $f_\lambda:\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ is given by

$$f_\lambda(\omega,\varphi)=\lambda+\frac{1}{2}\,(b(\omega)-c(\omega))+\frac{1}{2}\,(b(\omega)+c(\omega))\cos 2\varphi+a(\omega)\sin 2\varphi$$

in the Dirac case and by

$$f_\lambda(\omega,\varphi)=\cos^2\varphi-(s(\omega)-\lambda)\,\sin^2\varphi$$

in the Schrödinger case. Note that, since f_λ is π-periodic in φ, also g is π-periodic in ψ, and hence $\bar\psi(t,\omega,\bar\psi_0)$ is the class in $\mathbb{P}^1$ of the solution of (6.4) with initial data $\bar\psi_0$.

(iii) Obviously, the sets $\{(\omega,\psi^\pm(\omega))\,|\,\omega\in\mathbb{T}^d\}=h(\{(\omega,\bar\varphi^\pm(\omega))\,|\,\omega\in\mathbb{T}^d\})$, where as before $\bar\varphi^\pm:\mathbb{T}^d\to\mathbb{P}^1$ the τ_p invariant curves associated to $\mp\beta$ by relation (4.12), are the unique τ_r^*-invariant graphs contained in M^*, and they are non-closed. Fix $(\omega,\psi_0)\in M^*$. Then

$$\psi(t,\omega,\psi_0)=\varphi(t,\omega,\psi_0+\bar\varphi^-_{\lambda_0}(\omega))-\varphi^-_{\lambda_0}(\omega{\cdot}t)$$

for a suitable choice of the representative $\varphi^-_{\lambda_0}(\omega{\cdot}t)$, and

$$(\partial\psi/\partial\psi_0)(t,\omega,\psi_0)=(\partial\varphi/\partial\varphi_0)(t,\omega,\psi_0+\bar\varphi^-_{\lambda_0}(\omega))\,.$$

In particular,

$$\lim_{|t|\to\infty}\frac{1}{t}\ln\frac{\partial\psi}{\partial\psi_0}\,(t,\omega,\psi^\pm(\omega))=\lim_{t\to\infty}\frac{1}{|t|}\,\ln\frac{\partial\varphi}{\partial\varphi_0}\,(t,\omega,\bar\varphi^\pm(\omega))=\pm 2\beta$$

for a.e. $\omega\in\mathbb{T}^d$, as shown in (6.3). □

6.4. **The Jacobi case.** Let us now consider the Jacobi case (4.7). The same hypotheses as in the continuous cases ensure the existence of a unique τ_p-minimal subset $M \subset K_{\mathbb{R}}$, given by an almost automorphic extension of the base flow, with the complicate dynamical behavior described by Theorem 5.3 and Remark 5.7. Let us see the way in which this set determines an SNA on $\mathbb{R}$ for a flow satisfying the conditions imposed at the beginning of this section.

There is an important difference with the continuous case: since the function $\cot^{-1}$ is multivalued, there is no a direct way to find a continuous (monotone) global flow on $\Omega \times \mathbb{R}$, analogous to the flow τ_r, from the angular equation (4.9). The way to solve this problem is again to suspend the discrete flow.

Theorem 6.4. *Assume that the family* (4.8) *is in the non-uniformly dynamics case and that* $\lambda^* = 0$ *is the extreme point of any spectral gap of the problem* (5.3)*. Then,*

(i) *there exists a flow homeomorphism taking* $(K_{\mathbb{R}}, \tau_p, \mathbb{Z})$ *to a new skew-product flow* $(K_{\mathbb{R}}, \tau_p^*, \mathbb{Z})$ *for which there exists a unique minimal set* M^* *given by an unwound almost automorphic extension of the base flow.*

(ii) *The flow* τ_p^* *is given by the projection onto* $K_{\mathbb{R}}$ *of a (global) monotone flow* $\phi^* : \mathbb{Z} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\,,\ (n, \omega, \psi_0) \mapsto (\omega{\cdot}n, \psi(n, \omega, \psi_0))$*. The set* M^* *is also a* ϕ^**-minimal almost automorphic extension of* (Ω, σ)*.*

(iii) M^* *contains exactly two* ϕ^**-invariant graphs* $\{(\omega, \psi^{\pm}(\omega)) \mid \omega \in \mathbb{T}^d\}$ *which are non-closed and such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\partial \psi}{\partial \psi_0} (n, \omega, \psi^{\pm}(\omega)) = \pm 2\beta$$

for a.e. $\omega \in \mathbb{T}^d$*, where* β *is the Lyapunov exponent of the initial family* (4.8)*. In particular,* $\beta_s(\psi^{\pm}) = \pm 2\beta$*, which means that the* ϕ^**-invariant curve* ψ^- *is an SNA.*

Proof. The construction detailed in Section B allows us to obtain the conclusions of Theorem 6.3 taking as starting point the suspended flow $(\widehat{K_{\mathbb{R}}}, \widehat{\tau}_p, \mathbb{R})$ and the corresponding unique minimal set $\widehat{M}$. This requires to adapt to this setting the construction made in Theorem 5.3, which is possible since the sense of rotation of the Weyl functions with respect to the parameter λ is the same for the suspended and the initial flows. Once this is done, and using the arguments previous to Theorem 6.3, we define, from the transformed global flow $(\widehat{\Omega} \times \mathbb{R}, \widehat{\tau}_r^*, \mathbb{R})$ and its minimal set $\widehat{M}^*$, a monotone discrete flow ϕ^* admitting an SNA on $\mathbb{R}$ contained in a ϕ^*-minimal subset of $\widehat{M}^*$. It is easy to realize that this subset can be taken as the Poincaré section corresponding to $\Omega \subset \widehat{\Omega}$, and hence it is identified with the minimal set M^* obtained directly from M by means of the transformation h applied to the initial flow. □

Remark 6.5. It is well-known that the spectrum of the problem (5.3) is a compact subset of $\mathbb{R}$, and that of (5.2) is contained in a positive half-line. In particular, in both cases, there is a first point in the spectrum, the right extreme point of a spectral half-line. It is possible to show that, in the case of non-uniformly hyperbolic dynamics, the minimal set M is unwound. This assertion follows immediately from the fact that the corresponding linear systems is *disconjugate*. The interested reader is referred to [26].

6.5. **Non-uniformly hyperbolic dynamics examples.** We complete this section by recalling some cases for which the dynamics is in the non-uniformly hyperbolic case. In the continuous cases, apart from the already mentioned Dirac systems constructed by Millionščikov [39, 40] and Vinograd [61] (see also Lipnitskii [37] for some technical improvements), there are also known examples of Schrödinger type (4.6), like the limit-periodic one constructed by Johnson [23], for which $\lambda_0 = 0$ is exactly the first point of the spectrum of the corresponding operator (5.2); or the quasi periodic Schrödinger equation described by Koltyzhenkov [33].

Let us now consider the Jacobi case (4.7) with quasi periodic potential, for which the spectrum of the corresponding problem (5.3) is known to be a compact subset of $\mathbb{R}$. Herman [20] studies the case of Kronecker flow with irrational frequency over base $\mathbb{T}^1$ and coefficient function $v(\omega) = \rho w(\omega)$ given by a non-constant trigonometric polynomial. He shows that the Lyapunov exponents $\pm\beta(\lambda)$ of (5.3) never vanish provided that ρ is large enough. The example of almost Mathieu equation (1.5) described in the Introduction fits in this situation, for $v(\omega) = -2b\cos(\omega)$: as recalled in Subsection 1.1, Herman [20] proved that when $|b|$ is larger than 1, all the values of λ in the spectrum give rise to non-null Lyapunov exponents.

Sorets and Spencer [55] extend Herman's result to the case of any non-constant real-analytic w. A different approach is made by Bourgain and Goldstein [7] in order to show that the same property hods for non-constant real-analytic potentials on $\mathbb{T}^d$ provided that the base frequency vector is diophantine. The results in this section show that these examples provide situations giving rise to occurrence of SNAs at the extreme point λ of any of the spectral gaps.

On the other hand, the results of Eliasson [11] show the existence non-null Lyapunov exponents for Lebesgue a.e. point in the spectrum for a Gevrey potential w and irrational and diophantine frequency (again for $d = 1$). Similar results can be found in Sinai [54], Frölich *et al.* [15] and Surace [59], among others. In fact, Bjerklöv [4] shows the existence of examples of C^1-potential $\rho w(\omega)$ over an irrational base flow $(\mathbb{T}^1, \sigma)$ and values of λ in the spectrum for which the Lyapunov exponents do not vanish but the corresponding projective flow is minimal, conditions in which our results do not guarantee the existence of an SNA on the real line.

We conclude by pointing out that, in despite of the situations above described, the non-uniform hyperbolicity is an unfrequent property from a topological point of view in the space of continuous potentials. The interested reader is referred to Fabbri [13], Bochi [5], Novo *et al.* [43], Bochi and Viana [6], and references therein.

Appendix A. One-dimensional dynamics

Let (Ω, σ) represent a (real or discrete) continuous flow on a compact metric space which is minimal but not necessarily uniquely ergodic, and $g : \Omega \to \mathbb{R}$ will be a continuous function. Our objective in what follows is to analyze the behavior of the solutions of the family of scalar differential equations $x' = g(\omega{\cdot}t)$, namely $x(t, \omega, x_0) = x_0 + \int_0^t g(\omega{\cdot}s)\, ds$ and their discrete analogous. The knowledge of this behavior has been necessary to prove Theorem 4.10, which describes the complicate structure of a set giving rise, a posteriori, to an SNA on $\mathbb{R}$.

In particular, we concentrate our attention in the two different frameworks which can occur in the case that the family of linear scalar differential equations $x' = g(\omega{\cdot}t)\, x$ does not have an exponential dichotomy over Ω: either all the solutions are bounded or all the semitrajectories of the equations corresponding to a residual subset of Ω oscillate from $-\infty$ to $+\infty$ as $|t|\uparrow\infty$. These are well known properties (see [17], [22] and references therein). Nevertheless, we include all the proofs by completeness and give a shorter one for Theorem A.2.

Proposition A.1. *The following assertions are equivalent:*

(1) *there exists a point* $\widetilde{\omega}\in\Omega$ *such that* $|\int_0^t g(\widetilde{\omega}{\cdot}s)\,ds|\le c_1<\infty$ *for every* $t\ge 0$ *or for every* $t\le 0$*;*
(2) *there exist* $G:\Omega\to\mathbb{R}$ *continuous with* $G(\omega{\cdot}t)-G(\omega)=\int_0^t g(\omega{\cdot}s)\,ds$ *and, consequently, a constant* c *such that* $|\int_0^t g(\omega{\cdot}s)\,ds|\le c$ *for every* $\omega\in\Omega$ *and* $t\in\mathbb{R}$.

Proof. Assume that (1) holds for $\widetilde{\omega}\in\Omega$ and $t\ge 0$; i.e. that the flow ϕ induced on $\Omega\times\mathbb{R}$ by the family of scalar equations $x'=g(\omega{\cdot}t)$ has a bounded positive semitrajectory $\{(\widetilde{\omega}{\cdot}t,\int_0^t g(\widetilde{\omega}{\cdot}s)\,ds)\mid t\ge 0\}$. The omega-limit set of this semitrajectory contains a minimal subset, say M. As said before, minimality of the base flow ensures that M projects onto the whole of Ω. Let us check that this projection is in fact a bijection; or, in other words, that M is a copy of the base. Note that, for every $s\in\mathbb{R}$, the set $M_s=\{(\omega,x+s)\mid(\omega,x)\in M\}$ is also minimal. Assume by contradiction the existence of (ω_*,y_1), $(\omega_*,y_2)\in M$ with $y_2>y_1$, and set $s=y_2-y_1$. Then $(\omega_*,y_2)\in M\cap M_s$, which means that $M=M_s$. Analogously, $M=M_{ks}$ for every $k\in\mathbb{N}$, and this contradicts the boundedness of M.

Consequently, $M=\{(\omega,G(\omega))\mid\omega\in\Omega\}$, for a continuous $G:\Omega\to\mathbb{R}$ and, since M is ϕ-invariant, $G(\omega{\cdot}t)=G(\omega)+\int_0^t g(\omega{\cdot}s)\,ds$ for every $\omega\in\Omega$ and $t\in\mathbb{R}$. This proves (2). The converse implication is obvious. □

Theorem A.2. *Assume that statement* (1) *of Proposition* A.1 *does not hold and that the family* $x'=g(\omega{\cdot}t)\,x$ *does not have an exponential dichotomy over* Ω*. Then, for every* ω *in a residual subset* R*, there exist four sequences* (t_k^i) *with* $\lim_{k\to\infty}t_k^i=\infty$ *for* $i=1,3$ *and* $\lim_{k\to\infty}t_k^i=-\infty$ *for* $i=2,4$*, and such that*

$$\lim_{k\to\infty}\int_0^{t_k^i} g(\omega{\cdot}s)\,ds=-\infty \text{ for } i=1,2 \quad\text{and}\quad \lim_{k\to\infty}\int_0^{t_k^i} g(\omega{\cdot}s)\,ds=\infty \text{ for } i=3,4\,.$$

Proof. Let us define

$$e_1(\omega)=\inf_{t\ge 0}\exp\int_0^t g(\omega{\cdot}s)\,ds\,.$$

The function e_1 is the limit of a decreasing sequence of continuous functions, and hence it is upper semicontinuous. In particular, it admits a residual set R_1 of continuity points. In addition, for any $r\ge 0$,

$$e_1(\omega{\cdot}r)=\inf_{t\ge 0}\exp\int_0^t g(\omega{\cdot}(r+s))\,ds\ge e_1(\omega)\exp\left(-\int_0^r g(\omega{\cdot}s)\,ds\right). \tag{A.1}$$

Let us assume the existence of $\omega_1\in R_1$ with $e_1(\omega_1)>0$. Then $e_1(\omega)>\rho>0$ for any ω belonging to an open neighborhood V of ω_1. By minimality of the base flow, $\Omega=\cup_{r\in[0,r_1]}\sigma_r(V)$ for an $r_1>0$. Take any $\omega\in\Omega$ and choose $r\in[0,r_1]$ with $\omega{\cdot}(-r)\in V$. Then, by (A.1),

$$e_1(\omega)\ge e_1(\omega{\cdot}(-r))\exp\left(-\int_0^r g((\omega{\cdot}(-r){\cdot}s))\,ds\right)\ge\rho\,e^{-r_1\|g\|_\infty}\,.$$

It follows the existence of a constant $k \in \mathbb{R}$ with $-k \le \int_0^t g(\omega{\cdot}s)\,ds$ for every $t \ge 0$ and $\omega \in \Omega$. On the other hand, the absence of exponential dichotomy and Theorem 4.6 ensure the existence of $\widetilde{\omega} \in \Omega$ with $\{\exp \int_0^t g(\widetilde{\omega}{\cdot}s)\,ds \mid t \in \mathbb{R}\}$ bounded. We conclude that $|\int_0^t g(\widetilde{\omega}{\cdot}s)\,ds|$ is bounded for every $t \ge 0$, which contradicts our hypothesis.

Consequently, $e_1(\omega_1) = 0$ for every $\omega_1 \in R$, which ensures the existence of a sequence (t_k^1) with the required properties. The existence of (t_k^2) for any ω_2 in a residual set R_2 follows from the application of the same arguments to the function

$$e_2(\omega) = \inf_{t\le 0} \exp \int_0^t g(\omega{\cdot}s)\,ds\,,$$

while repeating the same with the function $-g$ provides the sequences (t_k^3) and (t_k^4) for the points of the residual sets R_3 and R_4. The set R is then given by the intersection of this four residual subsets, and the proof is complete. □

The same properties hold for the solutions of the family of scalar difference equations $x(n+1) = x(n) + g(\omega{\cdot}n)$, namely $x(n) = x(0) + \sum_{j=0}^{n-1} g(\omega{\cdot}j)$. The proofs are completely analogous to the previous ones.

Proposition A.3. *Let $g : \Omega \to \mathbb{R}$ be a continuous function. The following assertions are equivalent:*

(1) *there exists a point $\widetilde{\omega} \in \Omega$ such that $\sum_{j=0}^{n-1} g(\widetilde{\omega}{\cdot}j) \le k_1 < \infty$ for every $n \ge 0$ or for every $n \le 0$;*
(2) *there exist $G : \Omega \to \mathbb{R}$ continuous with $G(\omega{\cdot}n) - G(\omega) = \sum_{j=0}^{n-1} g(\omega{\cdot}j)$ and, consequently, a constant k such that $|\sum_{j=0}^{n-1} g(\omega{\cdot}j)| \le k$ for every $\omega \in \Omega$ and $n \in \mathbb{Z}$.*

Theorem A.4. *Assume that statement* (1) *of Proposition* A.3 *does not hold and that the family of linear difference equations $x(n+1) = e^{g(\omega\cdot n)}x(n)$ does not have an exponential dichotomy over Ω. Then, for every ω in a residual subset R, there exist four sequences (n_k^i) with $\lim_{k\to\infty} n_k^i = \infty$ for $i = 1,3$ and $\lim_{k\to\infty} n_k^i = -\infty$ for $i = 2,4$, and such that*

$$\lim_{k\to\infty} \sum_{j=0}^{n_k^i} g(\omega{\cdot}j) = -\infty \ \text{ for } i = 1,2 \quad \text{and} \quad \lim_{k\to\infty} \sum_{j=0}^{n_k^i} g(\omega{\cdot}j) = \infty \ \text{ for } i = 3,4\,.$$

Appendix B. The suspensions of the discrete flows

The interpolation of the discrete flows $(\mathbb{T}^d, \sigma)$, $(\mathbb{T}^d \times \mathbb{R}^2, \tau)$ and $(K_\mathbb{R}, \tau_p)$ associated to the family of Jacobi equations (4.7) by real flows, known as *suspensions* of the initial ones, allows as to unify the analysis of the dynamics on $K_\mathbb{R}$ in the continuous and discrete cases. In this pages we summarize the most basic facts concerning the suspension construction, following the approach made by Johnson in [25].

Let us consider a discrete flow $\varsigma : \mathbb{Z} \times X \to X$, $(n,x) \mapsto \varsigma(n,x) \equiv x{\cdot}n$ on a locally compact Hausdorff space X. We define on the product space $X \times \mathbb{R}$ an equivalence relation by

$$(x_1, s_1) \sim (x_2, s_2) \quad \Leftrightarrow \quad s_1 - s_2 = n \in \mathbb{Z} \quad \text{and} \quad x_2 = x_1{\cdot}n\,.$$

Then the quotient space $\widehat{X}$ is also a locally compact Hausdorff space, and it is compact in the case X is. We will represent the equivalence class of the pair (x,s) by $[x,s]$; so that $[x,s]=[x{\cdot}n,s-n]$ for every $n\in\mathbb{Z}$. Each equivalence class admits a unique representative (x,s) with $s\in[0,1)$, which will chosen if fixing one is needed. The map $\widehat{\varsigma}:\mathbb{R}\times\widehat{X}\to\widehat{X},\ (t,[x,s])\mapsto[x,s+t]$ defines a real continuous flow $(\widehat{X},\widehat{\varsigma})$, which is called the *suspension* of the discrete flow (X,ς). By defining $i_s:X\to\widehat{X},\ x\mapsto[x,s]$ for each $s\in\mathbb{R}$, we can identify X with $i_0(X)\subset\widehat{X}$, so that the restriction of $\widehat{\varsigma}$ to $\mathbb{Z}\times X$ coincides with the discrete initial flow: $\widehat{\varsigma}(n,[x,0])=[x,n]=[x{\cdot}n,0]\equiv x{\cdot}n=\varsigma(n,x)$.

It is immediate to check that if the ς-trajectory $\{x{\cdot}n\mid n\in\mathbb{Z}\}$ of an element $x\in X$ is dense on X, then so is the $\widehat{\varsigma}$-trajectory $\{[x,s+t]\mid t\in\mathbb{R}\}$ on $\widehat{X}$ for every $s\in\mathbb{R}$. In particular, the new flow is minimal in the case that the discrete one is.

Given a measure μ on X, we define

$$\int_{\widehat{X}} g([x,s])\,d\widehat{\mu}=\int_0^1\int_X g\circ i_s(x)\,d\mu\,ds\quad\forall\,g\in C(\widehat{X})\,.$$

It is easy to check that $\widehat{\mu}$ is a measure, and that it is $\widehat{\varsigma}$-invariant if μ is ς-invariant; in fact, if μ is ergodic so $\widehat{\mu}$ is. In addition, note that if $\widehat{X}_0\subset\widehat{X}$ is $\widehat{\varsigma}$-invariant with $\widehat{\mu}(\widehat{X}_0)=1$, then $\mu(\{x\mid[x,s]\in\widehat{X}_0\})=1$ for every $s\in[0,1]$.

This procedure allows us to construct $(\widehat{\mathbb{T}}^d,\widehat{\sigma})$ and $(\widehat{\mathbb{T}^d\times\mathbb{R}^2},\widetilde{\tau})$, suspensions of $(\mathbb{T}^d,\sigma)$ and $(\mathbb{T}^d\times\mathbb{R}^2,\tau)$ respectively. One can check that the flow $(\widehat{\mathbb{T}}^d,\widehat{\sigma})$ inherits the almost periodic character of the discrete base flow. In addition, it turns out that the space $\widehat{\mathbb{T}^d\times\mathbb{R}^2}$ is a trivial bundle over $\mathbb{T}^d$, and this trivialization takes $\widetilde{\tau}$ to a new flow, of linear skew-product type. Let us see how. Let $F:\mathbb{T}^d\times[0,1]\to\mathrm{GL}(2,\mathbb{C})$ be the homotopy between the maps $\omega\mapsto\mathrm{Id}$ and $\omega\mapsto\left[\begin{smallmatrix}0&1\\-1&v(\omega)\end{smallmatrix}\right]$ given by

$$F(\omega,s)=\begin{bmatrix}\cos\theta(s)&\sin\theta(s)\\-\sin\theta(s)&\cos\theta(s)+\vartheta(s)\,v(\omega)\end{bmatrix},$$

where θ and ϑ are C^∞-functions satisfying

$$\begin{array}{lll}\theta:[0,1]\to[0,\pi/2]\,, & \theta|_{[0,\delta]}=0\,, & \theta|_{[1/2-\delta,1]}=\pi/2\,,\\ \vartheta:[0,1]\to[0,1]\,, & \vartheta|_{[0,1/2+\delta]}=0\,, & \vartheta|_{[1-\delta,1]}=1\end{array}$$

for a fixed $\delta\in(0,1/4)$. These conditions provide periodic C^∞ extensions of θ' and ϑ' to the whole real line. The map

$$\zeta:\widehat{\mathbb{T}^d\times\mathbb{R}^2}\longrightarrow\widehat{\mathbb{T}}^d\times\mathbb{R}^2\,,\quad[(\omega,\mathbf{z}){\cdot}s]\mapsto([\omega,s],F(\omega,s)\,\mathbf{z})$$

(with $s\in[0,1)$) is a homeomorphism. Let us denote by $\widehat{\tau}$ the flow obtained by translation of $\widetilde{\tau}$ vía ζ. For $t\in\mathbb{R}$ and $s\in[0,1)$,

$$\widehat{\tau}(t,[\omega,s],\mathbf{z})=\Big([\omega,(s+t)],\widehat{Z}(t,[\omega,s])\,\mathbf{z}\Big)\,,$$

where

$$\widehat{Z}(t,[\omega,s])=F(\omega{\cdot}n,l)\,Z(n,\omega)\,F(\omega,s)^{-1}\tag{B.1}$$

with $n=[s+t]$ and $l=s+t-n$. (Here, as usual, $[\cdot]$ represents the integer part of a real number.) The restriction to $\mathbb{Z}\times\mathbb{T}^d\times\mathbb{R}^2\subset\mathbb{R}\times\widehat{\mathbb{T}}^d\times\mathbb{R}^2$ is then

$$\widehat{\tau}(n,[\omega,0],\mathbf{z})=(\omega{\cdot}n,F(\omega{\cdot}n,0)\,U(n,\omega)\,F(\omega,0)^{-1}\mathbf{z})=(\omega{\cdot}n,U(n,\omega)\,\mathbf{z})=\tau(n,\omega,\mathbf{z})\,;$$

that is, the discrete flow τ agrees with this restriction. The following result is proved in Núñez and Obaya [44].

Proposition B.1. *The continuous flow $\widehat{\tau}$ is given on $\widehat{\mathbb{T}}^d \times \mathbb{R}^2$ by the family of two-dimensional linear systems*

$$\mathbf{z}' = \widehat{D}([\omega, s+t])\,\mathbf{z}\,, \quad [\omega, s] \in \widehat{\mathbb{T}}^d\,, \tag{B.2}$$

where $\widehat{D} : \widehat{\mathbb{T}}^d \to GL(2,\mathbb{R})$ is the continuous map defined by

$$\widehat{D}([\omega, s]) = \begin{bmatrix} 0 & \theta'(s) \\ -\theta'(s) + \vartheta'(s)\,v(\omega{\cdot}[s]) & 0 \end{bmatrix}$$

for $\omega \in \mathbb{T}^d$ and $s \in \mathbb{R}$.

Let $\widehat{\mathbf{z}}(t, [\omega, s], \mathbf{z}_0)$ be the solution of (B.2) with initial data $\widehat{\mathbf{z}}(0, [\omega, s], \mathbf{z}_0) = \mathbf{z}_0$; then $\widehat{\tau}(t, [\omega, s], \mathbf{z}_0) = ([\omega, s+t], \widehat{\mathbf{z}}(t, [\omega, s], \mathbf{z}_0))$. In addition, by taking angular coordinate $\varphi = \cot^{-1}(z_2/z_1)$, we obtain the flow $\widehat{\tau}_p$ induced by (B.2) on the projective bundle $\widehat{K_{\mathbb{R}}} = \widehat{\mathbb{T}}^d \times \mathbb{P}^1$. It is defined as $\widehat{\tau}_p(t, [\omega, s], \bar{\varphi}_0) = ([\omega, s+t], \bar{\widehat{\varphi}}(t, [\omega, s], \varphi_0))$, $\bar{\widehat{\varphi}}(t, [\omega, s], \varphi_0)$ being the projection on $\mathbb{P}^1$ of the solution $\widehat{\varphi}(t, [\omega, s], \varphi_0)$ with initial data φ_0 of the equation

$$\varphi' = \theta'(s+t) - \vartheta'(s+t)\,v(\omega)\sin^2\varphi = f([\omega, s+t], \varphi)\,. \tag{B.3}$$

Relation (B.1) provides the expressions of $\widehat{\mathbf{z}}(t, [\omega, 0], \mathbf{z}_0)$ (resp. $\widehat{\varphi}(t, [\omega, 0], \varphi_0)$) in terms of $z(n, \omega, \mathbf{z}_0)$ (resp. $\varphi(n, \omega, \varphi_0)$). In particular, the dynamical spectrums of (4.8) and (B.2) agree, and the restriction to $\widehat{\tau}_p$ to $\mathbb{Z} \times K_{\mathbb{R}} \subset \mathbb{Z} \times \widehat{K_{\mathbb{R}}}$ coincides with the discrete projective flow τ_p defined from (4.9). The details can be found in [44]. It follows easily that the discrete solutions and flows inherit the almost periodicity properties of the real solutions and flows. In addition, given a $\widehat{\tau}_p$-minimal subset $\widehat{M} \subset \widehat{K_{\mathbb{R}}}$, the set $M = \{(\omega, \bar{\varphi}) \,|\, [(\omega, \bar{\varphi}), 0] \in \widehat{M}\}$ is a τ_p-minimal subset of $K_{\mathbb{R}}$; and conversely, given a τ_p-minimal subset $M \subset K_{\mathbb{R}}$, the set $\widehat{M} = \{[(\omega, \bar{\varphi}), s] \,|\, (\omega, \bar{\varphi}) \in M,\ s \in [0, 1)\} \subset \widehat{K_{\mathbb{R}}}$ is $\widehat{\tau}_p$-minimal. And finally, a $\widehat{\tau}_p$-ergodic sheet of the suspension $\widehat{K_{\mathbb{R}}}$ determines a τ_p-ergodic sheet of $K_{\mathbb{R}}$.

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