Integrability of Hamiltonian Systems and Differential Galois Groups of Higher Variational Equations

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Abstract

Given a complex analytical Hamiltonian system, we prove that a necessary condition for meromorphic complete integrability is that the identity component of the Galois group of each variational equation of arbitrary order along each integral curve must be commutative. This was conjectured by the first author based on a suggestion made by the third author due to numerical and analytical evidences concerning higher order variational equations. This non-integrability criterion extends to higher orders a non-integrability criterion (Morales-Ramis criterion), using only the first order variational equation, obtained by the first and the second author. Using our result (at order two, three or higher) it is possible to solve important open problems of integrability which escaped to Morales-Ramis criterion.

1 Introduction

The problem of *integrability by quadratures*, or in closed form, of *dynamical systems* is a very old, important and difficult problem. We know that, given an *algebraic* or *analytic* dynamical system, defined by ordinary differential equations, a solution always exists locally and sometimes we can prolong it for every time towards the past and the future. Then, as the mathematicians of XVIII-th century, we would like to find the general solution analytically in an "explicit" way. When this is possible we can say that the system is "integrable". Unfortunately (although a general definition of integrability for an arbitrary dynamical system is today missing) it is well-known that, "in general", dynamical systems are not integrable. In other words, the "majority" of dynamical systems are non-integrable (even one can suspect that integrability is a *codimension infinity* property in any reasonable sense) and it is impossible to find their general solution in closed form. (For some remarks about the meaning of integrability see [58].) The situation is similar to the problem of solvability by radicals of algebraic equations, and it is not surprising that our approach in this paper follows a Galoisian path.

In this paper we will only consider *analytical* dynamical systems over the *complex* field. (In the applications it is necessary to go back to the *real* field. Sometimes it is easy, sometimes it is delicate). Then there are at least two families of finite-dimensional complex-analytical dynamical systems for which the notion of integrability is well-defined: the Hamiltonian systems and the linear differential equations. For Hamiltonian systems the integrability is well-defined in the Liouville sense: the existence of a complete set of independent first integrals in involution. When this happens it is said that the Hamiltonian system is *completely integrable*; for simplicity we call it integrable. For linear ordinary differential equations, integrability is defined in the context of the differential Galois theory, also called the Picard-Vessiot theory.

We recall the precise definition of the complete integrability. A Hamiltonian system with Hamiltonian H defined over a symplectic analytical complex manifold M of (complex) dimension $2n$,

$$\dot{x} = X_H(x), \tag{1}$$

is integrable if there exist n first integrals $H = f_1, f_2, \ldots, f_n$ independent and in involution, $\{f_i, f_j\} = 0, i, j = 1, 2, \ldots, n$, being $\{\,,\}$ the Poisson bracket defined by the symplectic form.

In general, we will assume that the functions $f_1, f_2, \ldots, f_n$ are meromorphic, but some times we will have to impose that they are more regular, for instance also meromorphic at ∞, i.e., rational functions, if M is an open set (in the algebraic sense) of a complex projective space. For some specific facts about the integrability of complex Hamiltonian systems see [57], Chapter 3.

We assume that the reader is familiarised with the Picard-Vessiot theory from two approaches: the field algebraic approach and the geometric connection approach. For the necessary definitions and results the reader can look at [57], Chapter 2; for a more complete study, including detailed proofs and other references, a standard monograph is the recent book [77].

Given a system of linear ordinary differential equations

$$\dot{\xi} = A\xi, \tag{2}$$

with coefficients in a differential field K, $A \in Mat(m, K)$, we say that it is integrable if its general solution is obtained by a combination of quadratures, exponential of quadratures and algebraic functions. In other words, if $L := K(u_{ij})$ is the Picard-Vessiot extension of K, u_{ij} being a fundamental matrix of solutions of (2), then there exists a chain of differential extensions $K_1 := K \subset K_2 \subset \cdots \subset K_r := L$, where each extension is given by the adjunction of one element a, $K_i \subset K_{i+1} = K_i(a, a', a'', \ldots)$, such that a satisfies one of the following conditions:

(i) $a' \in K_i$,

(ii) $a' = ba$, $b \in K_i$,

(iii) a is algebraic over K_i

(the usual terminology is that the Picard-Vessiot extension L/K is Liouvillian). Then, it can be proved that *a linear differential equation is integrable if, and only if, the identity component G^0 of the Galois group G (which is algebraic over the constant field) of (2) is a solvable group.* In particular, if the identity component is commutative, then the equation is integrable. Along this paper we only consider the case of a differential coefficient field K which is the field of meromorphic functions over some suitable Riemann surface.

Given a complex analytical Hamiltonian system (1) defined over a symplectic manifold M of complex dimension $2n$, we can consider a particular solution, $\phi(x_0, t)$, being $\phi(x, t)$ the general solution, i.e., the flow of equation (1). If we assume that the particular solution $\phi(x_0, t)$ is not an equilibrium point, then it defines a Riemann surface Γ immersed in M. The first order variational equation VE_1 of (1) along Γ is given by

$$\frac{d}{dt}\frac{\partial \phi}{\partial x}(x_0, t) = \frac{\partial X_H}{\partial x}(\phi(x_0, t))\frac{\partial \phi}{\partial x}(x_0, t). \tag{3}$$

If we denote $\phi^{(1)} = \phi^{(1)}(t)$ the derivative of ϕ with respect to x at the point (x_0, t), then (3) can be written as

$$\dot{\phi}^{(1)} = \frac{\partial X_H}{\partial x}(\phi(x_0, t))\phi^{(1)}. \tag{4}$$

The solution of this equation gives us the linear part of the flow, $\phi(x, t)$ along Γ. Now we assume that we can complete the Riemann surface Γ to a Riemann surface $\overline{\Gamma}$ by adding some points: equilibrium points, singularities of the Hamiltonian field X_H and points at ∞, being the coefficient of (4) meromorphic at these points. Then the differential field of coefficients of the linear differential equation (4) is *by definition* the *field of meromorphic functions over* $\overline{\Gamma}$, see [60] (or [57]) for the details. Then we have [60]

Theorem 1 (Morales-Ramis) *If the Hamiltonian system* (1) *is completely integrable with meromorphic first integrals in a neighbourhood of* Γ*, not necessarily independent on* Γ *itself, then the identity component* G^0 *of the Galois group of the equation* (4) *is commutative.*

This result is a typical variant of several possible theorems in [60] when, instead of the Riemann surface Γ, we consider a Riemann surface $\overline{\Gamma}$ obtained from Γ by adding some points.

It is possible to give equivalent versions of Theorem 1:

... then the Lie algebra of the Galois group of the equation (4) *is abelian,*

or

... then the Galois group G of the equation (4) *is virtually commutative (i.e., it admits a commutative invariant subgroup H such that G/H is finite).*

In this last statement we can replace the differential Galois group G by the monodromy group (which is clearly a subgroup of G).

Corollary 1 *If the Hamiltonian system* (1) *is completely integrable with meromorphic first integrals in a neighbourhood of* Γ*, not necessarily independent on* Γ *itself, then the monodromy group of* (4) *is virtually commutative.*

Theorem 1 follows a tradition that goes back to Poincaré, who introduced the variational equations and found a relation between integrability and the monodromy matrix along *real* periodic orbits [69]. More recently Ziglin considered the monodromy group of the variational equations of (1) in the *complex analytical setting* in order to study necessary conditions for the existence of a complete set of independent first integrals but without any involution assumption [83]. For more information and precise statements about the history of the method of the variational equations in connection with the integrability problem of Hamiltonian systems, see [57]. Theorem 1 can be considered as a generalisation of the aforementioned result of Ziglin. In fact, it can be easily proved that Ziglin's theorem can be obtained from the results of [60] as a corollary (it follows easily from 1). It is clear that Theorem 1 (or the other versions of it in [60]) is a non-integrability criterion and since the end of the 90's it has been applied by several authors to the study of the non-integrability of a wide range of systems:

a) N-body problems, problems with homogeneous potentials and cosmological models [5, 6, 12, 13, 14, 16, 15, 36, 37, 44, 45, 47, 48, 51, 52, 53, 61, 62, 64, 67, 66, 75, 78, 81, 82].

b) Some physical problems [3, 4, 11, 25, 26, 27, 70].

c) Other mechanical problems (rigid body, spring–pendulum,...) [43, 46, 49, 50, 80].

d) Systems with some chaotic behaviour (splitting of asymptotic surfaces) [59, 79].

Moreover some surveys and general expository works have been also published [8, 9, 10, 19, 20].

So, using Morales-Ramis theorem and its variants, it is possible to solve a lot of long-time open problems of integrability, or to give simpler solutions of classical problems (as the heavy top problem [49]). However for some important systems, it is impossible, using only this theorem, to say if they are integrable or not, even if there is evidence of non-integrability from numerical experiments. One encounters such cases for some third order polynomial potentials (and more generally for some very degenerated situations in *parametrised* families of potentials). Considering this situation, the third author proposed to use *higher order variational equations* to solve such integrability problems.

2 Our main results

Beyond the first order variational equation (4), it is possible to consider the higher order variational equations VE_k along Γ, with $k > 1$. The "fundamental" solution of VE_k is given by $(\phi^{(1)}(t), \phi^{(2)}(t), \ldots, \phi^{(k)}(t))$, being

$$\phi(x,t) = \phi(x_0,t) + \phi^{(1)}(t)(x - x_0) + \ldots + \phi^{(k)}(t)(x - x_0)^k + \ldots$$

the Taylor series up to order k of the flow $\phi(x,t)$ with respect to the variable x at the point (x_0, t). That is, $\phi^{(k)}(t) = \dfrac{1}{k!}\dfrac{\partial^k}{\partial x_0^k}\phi(x_0,t)$. It is clear that the initial conditions are $\phi^{(1)}(0) = id$ and $\phi^{(j)}(0) = 0$ for all $j > 1$. We stress that, in contrast to some definitions, we do not consider as variational equation of order k the differential equations for $\phi^{(k)}$, but for $(\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(k)})$. For some examples on the use of higher order variational equations see, e.g., [72].

Although the variational equation VE_k *is not* a linear differential equation, it is in fact equivalent to a linear differential equation: there exits a linear differential equation

LVE_k with *coefficients in the field of meromorphic functions over* Γ *(resp.* $\overline{\Gamma}$) such that the differential extensions generated by the solutions of VE_k coincide with the Picard-Vessiot extensions of LVE_k. Then we can consider the Galois group G_k of VE_k, i.e., of the LVE_k, and it is natural to try to generalise Theorem 1 to the higher order variational equations. So in [57], Chapter 8, the first author conjectured that *a necessary condition for complete integrability of the Hamiltonian system* (1) *by means of meromorphic first integrals is that the identity component* $(G_k)^0$ *must be commutative for any* $k \geq 1$. This paper is devoted to prove this conjecture. This result was announced in [58] and our proof follows along essentially the same lines than for the first order variational equation in [60]. There is, however, a completely new argument: Artin's theorem is a central argument. It replaces Ziglin's lemma and in fact our proof gives a new proof of the first order case (Morales-Ramis theorem) without use of Ziglin's lemma.

We will now state our main results. The precise definitions and complete proofs will follow later.

As before we consider a non stationary particular solution Γ of (1) (for simplicity we "identify" the abstract Riemann surface Γ with its immersion $\iota\Gamma$ in M). For each point $m \in \Gamma \subset M$, there is a *natural faithful representation* of the Galois group G_k of the k-th variational equation VE_k in the group of k-jets at m of symplectic diffeomorphisms $Diff^k_{Sp}(M, m)$ fixing m. We will identify the groups G_k and their images.

There are natural group homomorphisms $G_{k+1} \to G_k$. These morphisms are *surjective* (this follows from differential Galois correspondence). We introduce the inverse limit $\hat{G} = \varprojlim_k G_k$ (it is a pro-algebraic group endowed with the Zariski topology). We can identify it with a subgroup of the group of formal jets at m of symplectic diffeomorphisms $\widehat{Diff}_{Sp}(M, m)$ fixing m. We have *surjective* morphisms $\hat{G} \to G_k$, for $k \in \mathbf{N}^*$.

Proposition 1 *Let* $k \in \mathbf{N}^*$*:*

(i) we have natural isomorphisms of finite groups

$$G_k/(G_k)^0 \to G_1/(G_1)^0,$$

$$\hat{G}/\hat{G}^0 \to G_1/(G_1)^0.$$

(ii) G_k *(resp.* $\hat{G}$*) is Zariski connected if and only if* G_1 *is Zariski connected;*

(iii) $(G_k)^0$ *(resp.* $\hat{G}^0$*) is solvable if and only if* $(G_1)^0$ *is solvable. In particular if* $(G_1)^0$ *is commutative, then* $(G_k)^0$ *and* $\hat{G}^0$ *are solvable.*

Theorem 2 *If the Hamiltonian system* (1) *is completely integrable with meromorphic first integrals in a neighbourhood of* Γ*, not necessarily independent on* Γ *itself, then:*

(i) for each $k \in \mathbf{N}^*$ *the identity component* $(G_k)^0$ *of the Galois group* G_k *of the* k*-th variational equation* VE_k *is commutative;*

(ii) for each $k \in \mathbf{N}^*$ *the Galois group of the equation* VE_k *is virtually commutative;*

(iii) for each $k \in \mathbf{N}^*$ *the Lie algebra* $\mathcal{G}_k$ *of the Galois group* G_k *of the* k*-th variational equation* VE_k *is abelian;*

(iv) the identity component $\hat{G}^0$ *of the group* $\hat{G}$ *is commutative;*

(v) the group $\hat{G}^0$ *is virtually commutative;*

(vi) the Lie algebra $\hat{\mathcal{G}}$ *of the group* $\hat{G}^0$ *is abelian.*

We remark that if there is no obstruction to integrability at the first level (the Morales-Ramis theorem fails), that is, if $(G_1)^0$ is *commutative*, then we can try to find a group ${G_k}^0$ $(k > 1)$ which is *non commutative.* However this group will automatically be *solvable.* Therefore in such a case, *non-integrability* in the Hamiltonian sense will correspond to *integrability* in Picard-Vessiot sense.

As it happens for Theorem 1, we have variants of our main theorem when we add to Γ some equilibrium points or points at infinity. In such situations we have the following result.

Proposition 2 *Let $k \in \mathbf{N}^*$. Then the k-th variational equation is regular singular if and only if the first variational equation is regular singular.*

If we do not add points to Γ then all the VE_k are regular singular (they correspond to *holomorphic* connections).

From the preceding facts, we can derive *purely topological* results on the dynamics of integrable Hamiltonian systems, extending Ziglin's results (cf. also [35]).

Let γ be a continuous closed loop of Γ at $m \in \Gamma$. The flow of the Hamiltonian system (1) near γ will give a germ $\psi_\gamma \in \mathit{Diff}_{Sp}(M, m)$ of the group of germs of analytic symplectic diffeomorphisms. We call it the holonomy of γ. Using the time parametrisation, we can interpret the time "along γ" as a time translation.

If we deform continuously the closed loop γ at m, then the germ ψ_γ will not change and we get an homomorphism of groups (the holonomy representation)

$$\rho : \pi_1(\Gamma, m) \to \mathit{Diff}_{Sp}(M, m).$$

(We must take the opposite group law on the fundamental group $\pi_1(\Gamma, m)$.)

We have natural maps

$$\mathit{Diff}_{Sp}(M, m) \to \widehat{\mathit{Diff}}_{Sp}(M, m).$$

Identifying $\mathit{Diff}_{Sp}(M, m)$ with a subgroup of $\widehat{\mathit{Diff}}_{Sp}(M, m)$, we get $\mathrm{Im}\ \rho \subset \hat{G}$ and the k-jets of the holonomies, ρ_k, satisfy $\mathrm{Im}\ \rho_k \subset G_k$.

The variants when we add to Γ equilibrium points and points at infinity do not change the holonomy groups, but the Galois groups can change.

Corollary 2 *If the Hamiltonian system* (1) *is completely integrable with meromorphic first integrals in a neighbourhood of Γ, not necessarily independent on Γ itself, then, denoting by* $\mathrm{Im}\ \rho$ *the holonomy group associated to a solution Γ:*

(i) $\mathrm{Im}\ \rho \subset \hat{G}$ *is virtually commutative;*

(ii) for $k \in \mathbf{N}^*$, the groups of k-jets $\mathrm{Im}\ \rho_k \subset G_k$ *are virtually commutative;*

(iii) $\mathrm{Im}\ \rho$ *is Zariski dense in $\hat{G}$, and for $k \in \mathbf{N}^*$, the group* $\mathrm{Im}\ \gamma_k$ *is Zariski dense in G_k.*

The statement (iii) remains true when we add to Γ some points if we suppose moreover that the meromorphic extension of the first variational equation to the extended curve is regular singular.

As for the case of the first variational equations, we can, in the case of the higher variational equations, "eliminate" the "trivial solutions". We can moreover restrict ourselves to the energy hypersurface M_0 containing our solution Γ. Then we get normal variational equations of higher order. Choosing a small transversal fibration to the flow M_1 in M_0, the corresponding Galois groups are subgroups of k-jets on M_1 at m. For these groups we have an evident version of our main theorem 2.

There is also a topological version: in Corollary 2, we can replace the holonomy group of the flow by the holonomy group of the corresponding one dimensional foliation, or, in order to keep the symplectic property, the holonomy of the one-dimensional foliation of the Hamiltonian system restricted to the energy hypersurface.

3 Jets and variational equations

3.1 Jets and jets groups

We will do an essential use of the jets formalism of C. Ehresmann. We recall here the basic definitions and results. Our references are [17] (Chapter 1, paragraph 3), [39], [38], [56] or the paper [74] where the reader can find more details. In general the jets formalism is described for $\mathcal{C}^\infty$ real functions. Here we will only use complex holomorphic functions.

As usual, given a manifold M and a point $p \in M$, we denote as (M, p) the germ of the manifold M at p.

Let $f, g : (\mathbf{C}, 0) \to \mathbf{C}$ be two germs of holomorphic functions at the origin and $k \in \mathbf{N}$. We suppose that $f(0) = g(0)$. We will say that f and g have the same k-jet at 0 if $f^{(j)}(0) = g^{(j)}(0)$ for $j \leq k$.

Now let M, N be complex analytic manifolds, and let $p \in M$, $q \in N$. Let $f : M \to N$ and $g : M \to N$ be holomorphic maps of M into N. We will say that f and g have the same k-jet at p whenever

- $f(p) = g(p) = q$,
- for all p-based parametrised analytic curves: $v : (\mathbf{C}, 0) \to (M, p)$ and all q-based complex valued holomorphic functions $u : N \to \mathbf{C}$, the holomorphic maps $u \circ f \circ v$ and $u \circ g \circ v$ have the same *k-jet*. It is an equivalence relation and the corresponding equivalence class will be denoted by $j^k_p(f)$. The point p is the *source* of $j^k_p(f)$ and the point q is its *target*.

We need the signification of the preceding definition in local coordinates.

Let U be an open neighbourhood of 0 in $\mathbf{C}^m$. Let f, g be two differentiable maps $f, g : U \to \mathbf{C}^n$, with $f(0) = g(0) = 0$. In coordinates $x = (x_1, \ldots, x_m)$, $f(x) = (f_1(x), \ldots, f_n(x))$, $g(x) = (g_1(x), \ldots, g_n(x))$. We will use the classical notations for partial derivatives: for a multi-index $\mu = (\mu_1, \ldots, \mu_m)$, $D^\mu_x = D^\mu = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}$. Then f and g have the same k-jet at the origin if and only if

$$D^\mu_x f_i(0) = D^\mu_x g_i(0), \quad 1 \leq i \leq n, \mid \mu \mid = \sum_{i=1}^m \mu_i \leq k.$$

Let $J^k_{p,q}(M, N)$ denote the set of all k-jets of maps from M to N of source p and target q. We define the set

$$J^k(M, N) = \bigcup_{p \in M, q \in N} J^k_{p,q}(M, N).$$

We have the classical *source* and *target* projections

$$\alpha : J^k(M, N) \to M, \qquad \beta : J^k(M, N) \to N,$$

defined by $\alpha(j^k_p(f)) = p$ and $\beta(j^k_p(f)) = f(p) = q$.

Let now $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be, respectively, open coordinates coverings of M and N. We get an open covering $\{W_{ij}\}_{i\in I, j\in J}$ of $J^k(M,N)$:

$$W_{ij} = \{j_p^k(f) \mid \alpha(j_p^k(f)) \in U_i, \beta(j_p^k(f)) \in V_j\}.$$

If $\{x_1, \dots x_m\}$ and $\{y_1, \dots y_n\}$ are respectively the coordinate functions on U_i and V_j, we may define coordinates functions (called *natural coordinates*) on W_{ij} by

$$(x_i(p), y_j(q), D_x^\mu(y_j \circ f)(p)), \quad 1 \le i \le m,\ 1 \le j \le n,\ 1 \le \mid \mu \mid \le k.$$

Holomorphic changes of local coordinates in U_i and V_j will induce an holomorphic change of coordinates in W_{ij}. Hence we have a complex analytic structure on $J^k(M,N)$. If dim $M = m$ and *dim* $N = n$, then

$$\dim\ J^k(M,N) = m + n \binom{m+k}{k}.$$

We have dim $J^k(M, \mathbf{C}) := \nu_{m,k} = m + \binom{m+k}{k}$. If m is already fixed, we set $\nu_{m,k} = \nu_k$.

Examples: The cotangent bundle $T^*(M)$ is identified with $J^1(M, \mathbf{C})$ and the tangent bundle $T(M)$ is identified with $J^1(\mathbf{C}, M)$.

For $r \le k$ there is a natural map $\pi_{k,r} : J_{p,q}^k(M,N) \to J_{p,q}^r(M,N)$.

Let M_1, M_2, M_3 be three complex analytic manifolds. Let $p_i \in M_i$ $(i = 1, 2, 3)$. The composition of applications induces an algebraic map

$$J_{p_2,p_3}^k(M_2, M_3) \times J_{p_1,p_2}^k(M_1, M_2) \to J_{p_1,p_3}^k(M_1, M_3).$$

This follows from chain's rule, which expresses the partial derivatives of a composition map $g \circ f$ as polynomials in the partial derivatives of g and those of f.

In the special case $M_1 = M_2 = M_3 = M$, $p_1 = p_2 = p_3 = p$, this composition map induces a product in $J_{p,p}^k(M,M)$. We will denote by $\mathit{Diff}^k(M,p)$ the subset of $J_{p,p}^k(M,M)$ of invertible elements. There is a natural isomorphism $\mathit{Diff}^1(M,p) \simeq GL(T_p(M))$. In particular, $\mathit{Diff}^1(\mathbf{C}^m, 0) \simeq GL(m; \mathbf{C})$.

We set $J_{p,0}^k(M, \mathbf{C}) = J^k(M,p)$. The group $\mathit{Diff}^k(M,p)$ acts linearly on $J^k(M,p)$ by composition on the right. The corresponding representation is faithful. In particular we get a linear action of $\mathit{Diff}^k(\mathbf{C}^m, 0)$ on $J^k(\mathbf{C}^m, 0)$ and a faithful representation of $\mathit{Diff}^k(\mathbf{C}^m, 0)$ (or more precisely of the *opposite group*) into $GL(J^k(\mathbf{C}^m, 0)$.

Using local coordinates we obtain the obvious identifications $J^k(M,p) \simeq J^k(\mathbf{C}^m, 0)$, $J_{p,p}^k(M,M) \simeq J_{0,0}^k(\mathbf{C}^m, \mathbf{C}^m)$, $\mathit{Diff}^k(M,p) \simeq \mathit{Diff}^k(\mathbf{C}^m, 0)$.

The $\mathbf{C}$-algebra structure of $\mathbf{C}$ gives a $\mathbf{C}$-algebra structure on $J^k(\mathbf{C}^m, 0)$ and the maps $\pi_{k,r}$ are surjective homomorphisms of $\mathbf{C}$-algebras. The linear action of $\mathit{Diff}^k(\mathbf{C}^m, 0)$ on $J^k(\mathbf{C}^m, 0)$ gives an automorphism of $\mathbf{C}$-algebras of $J^k(\mathbf{C}^m, 0)$. More precisely, we have the following result.

Proposition 3 *Let Φ be a linear endomorphism of $J^k(\mathbf{C}^m, 0)$. The following conditions are equivalent:*

(i) Φ is an homomorphism of $\mathbf{C}$-algebras.

(ii) There exists $\phi \in J_{0,0}^k(\mathbf{C}^m, \mathbf{C}^m)$ such that $\Phi(Y) = Y \circ \phi$ for all $Y \in J^k(\mathbf{C}^m, 0)$.

Moreover if these conditions are satisfied, then Φ is an automorphism of $\mathbf{C}$*-algebras if and only if* $\phi \in Diff^k(\mathbf{C}^m, 0)$

Proof. The implication *(ii)* $\Rightarrow$ *(i)* is trivial. It remains to prove *(i)* $\Rightarrow$ *(ii)*. Let $x_1, \ldots, x_m$ be the coordinate functions in $\mathbf{C}^m$. We denote by the same letters the corresponding jets in $J^k(\mathbf{C}^m, 0)$. Then we set $\phi_i = \Phi(x_i)$.

Let $Y \in J^k(\mathbf{C}^m, 0)$. We can write Y as an element of $\mathbf{C}[x_1, \ldots, x_m]$: $Y = P(x_1, \ldots, x_m)$. Then $\Phi(Y) = \Phi(P(x_1, \ldots, x_m)) = P(\Phi(x_1), \ldots, \Phi(x_m)) = P(\phi_1, \ldots, \phi_m) = P \circ \phi = Y \circ \phi$. □

The group $Diff^k(\mathbf{C}^m, 0)$ is a linear complex algebraic group. We have an exact sequence of algebraic groups:

$$\{id\} \to I^k(\mathbf{C}^m, 0) \to Diff^k(\mathbf{C}^m, 0) \xrightarrow{\pi_{k,1}} GL(m; \mathbf{C}) \to \{id\},$$

where $I^k(\mathbf{C}^m, 0)$ is the subgroup of $Diff^k(\mathbf{C}^m, 0)$ of germs tangent to identity. Using coordinates it is easy to build a section of the homomorphism $\pi_{k,1}$. Therefore $Diff^k(\mathbf{C}^m, 0)$ is a *semi-direct product* of $GL(m; \mathbf{C})$ by the *unipotent* group $I^k(\mathbf{C}^m, 0)$. More precisely we have exact sequences of algebraic groups

$$\{id\} \to I^{k+1,k}(\mathbf{C}^m, 0) \to I^{k+1}(\mathbf{C}^m, 0) \to I^k(\mathbf{C}^m, 0) \to \{id\},$$

being $I^{k+1,k}(\mathbf{C}^m, 0)$ the vector group in $I^{k+1}(\mathbf{C}^m, 0)$ of elements with only non-trivial contributions of order $k+1$, i.e., $I^{k+1}(\mathbf{C}^m, 0)$ is the *semi-direct product* of $I^k(\mathbf{C}^m, 0)$ by the additive group of a finite dimensional vector space. Therefore we get $Diff^k(\mathbf{C}^m, 0)$ from $GL(m; \mathbf{C})$ by a sequence of semi-direct products with additive groups of finite dimensional vector spaces [38, 74].

We need some symplectic variations. We denote $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbf{C}^{2n}$, and we set $\Omega = dq \wedge dp = \sum_{i=1}^{n} dq_i \wedge dp_i$. We will denote by $Diff^k_{Sp}(\mathbf{C}^{2n}, 0)$ the subgroup of jets of diffeomorphisms ϕ such that $\phi * \Omega = \Omega$ (in the evident sense). This subgroup is clearly an algebraic subgroup. It is easy to check that, for $k \geq r$, the homomorphisms of group $\pi_{k,r}$ induces a *surjective* homomorphism of groups

$$\pi_{k,r} : Diff^k_{Sp}(\mathbf{C}^{2n}, 0) \to Diff^r_{Sp}(\mathbf{C}^{2n}, 0).$$

We easily see that $Diff^k_{Sp}(\mathbf{C}^{2n}, 0)$ is the *semi-direct product* of $Sp(2n; \mathbf{C})$ by a *unipotent* group. More precisely we get $Diff^k_{Sp}(\mathbf{C}^{2n}, 0)$ from $Sp(2n; \mathbf{C})$ by a sequence of semi-direct products with additive groups of finite dimensional vector spaces.

We set $\sigma_{n,k} = \sigma_k = \dim\ Diff^k_{Sp}(\mathbf{C}^{2n}, 0)$.

It is easy to extend the preceding definitions when we replace $(\mathbf{C}^{2n}, 0)$ by a germ $(M, m) = ((M, \Omega), m)$ of complex symplectic manifold. We shall denote by $Diff^k_{Sp}(M, m)$ the subgroup of germs of symplectic diffeomorphisms in $Diff(M, m)$, etc.

3.2 Some properties of Poisson algebras

In this section we will expose some properties of the Poisson algebras of germs of meromorphic functions and of formal meromorphic functions at the origin of a complex symplectic vector space.

Let E be a complex symplectic vector space of complex dimension $2n$. The symplectic product is denoted by $\omega : (v,w) \mapsto \omega(v,w)$. Using the unique 2-form Ω invariant by translation such that "$\Omega(0) = \omega$" over E, we will consider E as a symplectic manifold. We can choose symplectic coordinates (q,p) over E and then $\Omega = \sum_{i=1,\dots n} dp_i \wedge dq_i$.

We will consider the $\mathbf{C}$-algebras of polynomials, convergent series, formal series over E: $\mathbf{C}[E]$, $\mathbf{C}\{E\}, \mathbf{C}[[E]]$, and the corresponding fraction fields $\mathbf{C}(E)$, $\mathbf{C}(\{E\}), \mathbf{C}((E))$. On all these algebras we have a Poisson product

$$\{f,g\} = \Omega(df^\sharp, dg^\sharp). \tag{5}$$

We consider the spaces of germs at the origin of holomorphic (resp. meromorphic, formal, formal-meromorphic) vector fields $X = \sum_i a_i \frac{\partial}{\partial q_i} + \sum_i b_i \frac{\partial}{\partial p_i}$, $a_i, b_i \in \mathbf{C}\{E\}$ (resp. $\mathbf{C}(\{E\}), \mathbf{C}[[E]], \mathbf{C}((E))$).

The corresponding complex vector spaces are endowed with the Lie algebra structures defined by the usual bracket of vector fields. A germ of a holomorphic (resp. ...) vector field X is said to be *Hamiltonian* if there exists a germ of a holomorphic (resp. ...) function f such that

$$X = df^\sharp, \tag{6}$$

Then we will denote $X = X_f$.

The following results are well known [1] (3.36 proposition, page 189).

Proposition 4 *For a germ of holomorphic (resp. ...) vector field X the following conditions are equivalent:*

(i) X *is Hamiltonian;*

(ii) $d(X^\flat) = 0$;

(iii) $L_X\Omega = 0$;

(iv) $(exp\ X)^*\Omega = \Omega$.

Proposition 5 *The germs of holomorphic (resp. ...) Hamiltonian vector fields are a Lie subalgebra by defining*

$$[X_f, X_g] = X_{\{f,g\}}.$$

Let $((M,\Omega),m)$ be a germ of complex analytic symplectic manifold ($\dim_{\mathbf{C}} M = 2n$).

We recall that the natural maps

$$Diff^{k+1}(M,m) \to Diff^k(M,m),$$

$$Diff^{k+1}_{Sp}(M,m) \to Diff^k_{Sp}(M,m)$$

are surjective morphisms of algebraic groups.

By inverse limits we get the pro-algebraic group $\widehat{Diff}(M,m) = \lim\limits_{\overleftarrow{k}} Diff^k(M,m)$ and the pro-algebraic subgroup $\widehat{Diff}_{Sp}(M,m) = \lim\limits_{\overleftarrow{k}} Diff^k_{Sp}(M,m)$. These pro-algebraic groups are endowed with the *Zariski* topology. This topology is by definition the direct limit topology.

We have surjective morphisms

$$\widehat{Diff}(M,m) \to Diff^k(M,m),$$

$$\widehat{Diff}_{Sp}(M,m) \to Diff^k_{Sp}(M,m).$$

A k-jet at $m \in M$ of a vector field X vanishing at m acts linearly on the space $J^k(M,m)$ of k-jets of functions by truncation of the Lie derivative L_X. Then $exp\ X = exp\ L_X$ can be interpreted as an element of $Diff^k(M,m)$. It is easy to check that, using this remark, we can identify the space $\mathcal{L}^k(M,m)$ of k-jets of vector fields vanishing at m with the Lie algebra of the algebraic group $Diff^k(M,m)$. Then $\widehat{\mathcal{L}}(M,m) = \varprojlim_k \mathcal{L}^k(M,m)$ is identified with the Lie algebra of the pro-algebraic group $\widehat{Diff}(M,m)$. Be careful, the diffeomorphism $exp\ \hat{X}$ can be analytic, that is $exp\ \hat{X} \in Diff(M,m)$, even when $\hat{X}$ is *divergent*.

Similarly we can identify the space $\mathcal{L}^k_{sp}(M,m)$ of k-jets of Hamiltonian vector fields vanishing at m with the Lie algebra of the algebraic group $Diff^k_{Sp}(M,m)$. Then $\widehat{\mathcal{L}}_{sp}(M,m) = \varprojlim_k \mathcal{L}^k_{sp}(M,m)$ is identified with the Lie algebra of the pro-algebraic group $\widehat{Diff}_{Sp}(M,m)$. (There is a k-truncated version of Proposition 4. We leave the details to the reader.)

It is possible to define Poisson products on spaces of germs of k-jets $J^k(M,m)$:

$$J^k(M,m) \times J^k(M,m) \to J^{k-1}(M,m),\ k > 0,$$

$$J^{k,2}(M,m) \times J^{k,2}(M,m) \to J^{k,2}(M,m),\ k > 1,$$

where $J^{k,2}(M,m)$ is the space of jets vanishing at order one. We will say that an endomorphism of $J^k(M,m)$ conserves the Poisson product if it conserves these Poisson products.

There is a symplectic version of Proposition 3.

Proposition 6 *Let Φ be a linear endomorphism of $J^k(E,0)$. The following conditions are equivalent:*

(i) Φ is an homomorphism of $\mathbf{C}$-algebras and preserves the Poisson product;

(ii) There exists $\phi \in J^k_{Sp}(E,E)$ such that $\Phi(Y) = Y \circ \phi$ for all $Y \in J^k(E,0)$.

Moreover if these conditions are satisfied, then Φ is an automorphism of $\mathbf{C}$-algebras if and only if $\phi \in Diff^k_{Sp}(E,0)$.

We will state and prove now the main result of this section. It is the central tool in our obstruction theorem. As the result is local we can choose Darboux coordinates in a neighbourhood of m in M, and it is sufficient to study the case $(M,m) = (E,0)$. We will denote $Diff^k(E,0), \ldots$, the corresponding objects.

Let $f_1, \ldots, f_\ell$ be ℓ germs of meromorphic functions (resp. formal meromorphic functions) at $m \in M$. We will say that they are functionally independent *near* m if the germ at m of $df_1 \wedge \ldots \wedge df_\ell$ is not identically zero. In the meromorphic case we allow a singularity at m for some f_i's and we allow also that, even if the f_i's are holomorphic at m, then $df_1 \wedge \ldots \wedge df_\ell$ can vanish at m.

It is easy to prove the following result.

Lemma 1 *Let $f_1, \ldots, f_\ell$ be ℓ germs of meromorphic functions (resp. formal meromorphic functions) at $m \in M$ which are functionally independent near m. Let α be a germ of a meromorphic one form (resp. a formal meromorphic one form) at m such that $\alpha \wedge df_1 \wedge \ldots \wedge df_\ell \equiv 0$. Then there exists germs of meromorphic functions (resp. formal meromorphic functions) $\theta_1, \ldots, \theta_\ell$ such that $\alpha = \sum_{i=1,\ldots,\ell} \theta_i df_i$.*

The central result of this section is the following theorem. It will be the key of our main result. It is also the key of the analogue result for the case of the non-linear Galois theory, due to the second author (cf. Section 5 below).

Theorem 3 *Let E be a complex symplectic vector space of complex dimension $2n$. Let $f_1, \ldots, f_n$ be n germs at the origin of E of meromorphic functions, functionally independent near the origin (not necessarily at the origin itself). We suppose that these germs $f_1, \ldots, f_n$ are in involution. Let $\hat{\mathcal{L}}$ be a Lie algebra of Hamiltonian formal vector fields at the origin of E. We suppose that $f_1, \ldots, f_n$ are invariant by $\hat{\mathcal{L}}$. Then the Lie algebra $\hat{\mathcal{L}}$ is abelian.*

This theorem is a corollary of the following. (A particular case of Theorem 3 was suggested some years ago to the second author by L. Gavrilov in relation with Morales-Ramis theorem.)

Theorem 4 *Let E be a complex symplectic vector space of complex dimension $2n$. Let $f_1, \ldots, f_n$ be n germs at the origin of E of meromorphic functions, functionally independent near the origin (not necessarily at the origin itself). We suppose that these germs $f_1, \ldots, f_n$ are in involution. Let $\mathcal{A}$ be the $\mathbf{C}$-subalgebra of the field $\mathbf{C}((E))$ generated by $f_1, \ldots, f_n$. Then*

(i) $\mathcal{A}$ is involutive,

(ii) the orthogonal $\mathcal{A}^\perp$ of $\mathcal{A}$ in $\mathbf{C}((E))$ is an involutive $\mathbf{C}$-subalgebra of $\mathbf{C}((E))$.

Proof. We start with a preliminary result.

Lemma 2 *In the conditions of Theorem 4, let $\hat{\varphi} \in \mathcal{A}^\perp \subset \mathbf{C}((E))$. Then $\hat{\varphi}, f_1, \ldots, f_n$ are functionally dependent near the origin, that is*

$$d\hat{\varphi} \wedge df_1 \wedge \ldots \wedge df_n \equiv 0.$$

We will prove a slightly more general version.

Lemma 3 *In the conditions of Theorem 4, let $\hat{\alpha}$ be a meromorphic (resp. formal meromorphic) one form. We set $X = \hat{\alpha}^\flat$. We suppose that $df_i(X) = \iota_{X_{f_i}} \hat{\alpha} = \hat{\alpha}(X_{f_i}) = \omega(X_{f_i}, X) = 0$ for $i = 1, \ldots, n$. Then*

$$\hat{\alpha} \wedge df_1 \wedge \ldots \wedge df_n \equiv 0.$$

Proof.
First case. We suppose that $\hat{\alpha} = \alpha$ is meromorphic. Then, in each open neighbourhood of 0 in E, there exists a point x_0 such that $f_1, \ldots, f_n$ and α are holomorphic at x_0 and such that $df_1 \wedge \ldots \wedge df_n$ does not vanish at x_0. These properties remain true in an open neighbourhood U of x_0. Then, for $x_1 \in U$, the set $V = V_{x_1} = \{x \in U \mid f_1(x) = f_1(x_1), \ldots, f_n(x) = f_n(x_1)\}$ is an analytic submanifold of complex dimension n of U. We have $df_i(X) = 0,\ i = 1, \ldots, n$. Therefore the analytic vector field X is tangent to V. We have also $df_i(X_{f_j}) = 0,\ i, j = 1, \ldots, n$, and the analytic vector fields $X_{f_i},\ i = 1, \ldots, n$, are also tangent to V. As the

vectors $X_{f_1}(x), \ldots, X_{f_1}(x)$ are independent at each point of $x \in U$, they generate the tangent space to V at each point of V. We get a relation $X = \theta_1 X_{f_1} + \ldots + \theta_n X_{f_n}$ over U, where the θ_i's are holomorphic. The relation $\alpha = \theta_1 df_1 + \ldots + \theta_n df_n$ follows and $\alpha \wedge df_1 \wedge \ldots \wedge df_n \equiv 0$ over U. Then the similar relation holds for the germs at the origin by analytic continuation.

Remark. If we suppose that the one form α is closed, then $\{df_j, \alpha\}^\sharp := [X_{f_j}, X] = 0$. Now we consider the previous relation $X = \theta_1 X_{f_1} + \ldots + \theta_n X_{f_n}$ over U. We have

$$[X_{f_j}, X] = \sum_{i=1,\ldots,n} (L_{X_{f_j}} \theta_i) X_{f_i} = 0, \qquad j = 1, \ldots, n.$$

Therefore $L_{X_{f_j}} \theta_i = 0, \quad i, j = 1, \ldots, n$ and the analytic functions $\theta_i, \quad i = 1, \ldots, n$ are constant on each manifold V_{x_1}, $(x_1 \in U)$.

Second case. We suppose that $\hat{\alpha}$ is a formal one form (without singularity at the origin). We set $\hat{\alpha} = \sum_i y_i dp_i + \sum_i z_i dq_i$. Then we can interpret the system of equations

$$\hat{\alpha}(X_{f_i}) = 0, \ i = 1, \ldots n, \tag{7}$$

as a linear analytic (in the variable $x = (q_1, \ldots, q_n, p_1, \ldots, p_n)$) system of equations in the $2n$ unknowns (y, z).

Let $\hat{\mathcal{M}} \subset \mathbf{C}[[E]]$ be the maximal ideal, that is, the formal series in E without zero-order term. Then [7], for every $\mu \in \mathbf{N}^*$, there exists a germ $\beta_\mu = \sum_i \bar{y}_i dp_i + \sum_i \bar{z}_i dq_i$ of *analytic* 1-forms such that $(\bar{y}, \bar{z})$ satisfies the same analytic system, i.e., β_μ satisfies (7):

$$\beta_\mu(X_{f_i}) = 0, \quad i = 1, \ldots n, \tag{8}$$

and such that

$$\hat{\alpha} = \beta_\mu \pmod{\hat{\mathcal{M}}^\mu}.$$

Using the result for the first case, we get

$$\beta_\mu \wedge df_1 \wedge \ldots \wedge df_n \equiv 0$$

for every $\mu \in \mathbf{N}^*$. The formal relation

$$\hat{\alpha} \wedge df_1 \wedge \ldots \wedge df_n \equiv 0$$

follows easily.

General case. We suppose that $\hat{\alpha}$ is a formal meromorphic one form. Then there exists a formal one form $\hat{\zeta}$ and a *non zero* formal power series $\hat{g} \in \mathbf{C}[[E]]$ such that $\hat{g}\hat{\alpha} = \hat{\zeta}$. Then $\hat{\zeta}$ satisfies a system similar to (7). We can apply the result of the second case to $\hat{\zeta}$: $\hat{\zeta} \wedge df_1 \wedge \ldots \wedge df_n = \hat{g}\hat{\alpha} \wedge df_1 \wedge \ldots \wedge df_n \equiv 0$. The result for $\hat{\alpha}$ follows: $\hat{\alpha} \wedge df_1 \wedge \ldots \wedge df_n \equiv 0$.

This ends the proof of the lemmas and we can go back to the proof of Theorem 4. Let $\hat{\varphi}, \hat{\psi} \in \mathcal{A}^\perp$. From Lemmas 1 and 2 we get

$$d\hat{\varphi} = \sum_{i=1,\ldots,n} \hat{\theta}_i df_i,$$

with $\hat{\theta}_i \in \mathbf{C}((E))$. Therefore

$$\{\hat{\varphi}, \hat{\psi}\} = d\hat{\varphi}(X_{\hat{\psi}}) = \sum_{i=1,\ldots,n} \hat{\theta}_i df_i(X_{\hat{\psi}}) = \sum_{i=1,\ldots,n} \hat{\theta}_i \{f_i, \hat{\psi}\} = 0.$$

This ends the proof of Theorem 4.

It remains to prove Theorem 3. Let $X \in \hat{\mathcal{L}}$. By definition we have $L_X(f_i) = 0$ $(i = 1, \dots, n)$. There exists a formal power series expansion $\hat{g} \in \mathbf{C}[[E]]$ such that $X^\flat = d\hat{g}$ (the formal field X is Hamiltonian). Then, with the notations of Theorem 3, we have $\hat{g} \in \mathcal{A}^\perp$. The result follows easily. □

3.3 Variational equations. Linearised variational equations

We will briefly recall the definition of the higher-order variational equations VE_k, $k > 1$. First we shall use local coordinates, giving the equations in a compact form. It is clear that these equations are non-linear. However if we start from the solutions of the equations of order $\leq k$, then it is possible to solve the equation of order $k+1$ by a quadrature. But it is not a priori evident that the theory of Picard-Vessiot extensions is applicable to the higher-order variational equations. Therefore it is necessary, for our purpose, to introduce an equivalent *linearised* version of these equations LVE_k; roughly speaking this is related to the fact that the jet groups are linear groups. The first author described in [57] how to linearise the second and third variational equations ([57], Section 8.3). Here this will be done in a systematic way. First a *local* version of this linearisation using coordinates is introduced. Later on we will use a geometric interpretation of the corresponding computations (based upon a *duality* trick) to derive a *global* version.

The global geometric version is indispensable for the proof. The coordinate version is not needed, strictly speaking, but it can be useful for explicit computations in the applications (in particular in order to use computer algebra or to do numerical checks using the variational equations) [48]. See also Appendix B. What follows will ensure users of our main theorem that it is in fact possible to perform these computations (without explicit need of the linearised variational equations) remaining automatically into a convenient Picard-Vessiot extension (depending on the order k) of a "field of rationality". Therefore, even if our main result seems abstract, one can *admit and use* it very easily for some practical application (forgetting about the proofs).

Let

$$\dot{x} = X(x) \tag{9}$$

be an analytic differential equation defined by an analytic vector field X over a complex connected manifold M of complex dimension m ($m = 2n$ in the symplectic case).

To a non stationary solution $\phi_t(x_0) := \phi(x_0, t)$, we associate an immersion $\iota : \Gamma \to M$, Γ being a connected Riemann surface. Consider, first, the local situation. Then, we can identify M with an open subset of $\mathbf{C}^m$, Γ with an open subset of $\mathbf{C}$ and suppose that ι is an embedding. The initial data are $t_0 = 0 \in \Gamma$ and $x_0 = (x_{0,1}, \dots, x_{0,m}) \in \mathbf{C}^m$. The components of the solution will be denoted as $\phi = (\phi_1, \dots, \phi_m)$. We can assume that this solution is maximal with the initial data, and it is defined for initial y_0 close enough to x_0.

We consider the germ of the flow ϕ along the graph $\Delta = \{(t, \iota(t) \mid t \in \Gamma\}$ in $\Gamma \times M$. We can interpret it as a family of germs of diffeomorphisms $\phi_t : (\mathbf{C}^m, x_0) \to (\mathbf{C}^m, \phi_t(x_0))$. Then we have the convergent power series

$$\phi_t(y_0) = \sum_{k \geq 0} \phi_t^{(k)}(x_0)(y_0 - x_0)^k, \tag{10}$$

where we introduce $\phi_t^{(0)}(x_0) := \phi_t(x_0)$ and $\phi_t^{(k)}(x_0) := D^k_{x_0}\phi_t/k!\,.$

Our immediate goal is to obtain the equations for the derivatives of ϕ_t with respect to x_0 and, therefore, for $\phi_t^{(k)}(x_0)$. Working first in coordinates, let X_i be the components of X and let us introduce the notation

$$D^s_{i_1,\ldots,i_s}\phi_j = \frac{\partial^s \phi_j}{\partial x_{0,i_1}\ldots\partial x_{0,i_s}}, \qquad D^r_{k_1,\ldots,k_r}X_i = \frac{\partial^r X_i}{\partial x_{k_1}\ldots\partial x_{k_r}}.$$

It will be also useful to introduce the power series expansion for X

$$X(y) = \sum_{k\geq 0} X^{(k)}(x)(y-x)^k,$$

with $X^{(0)}(x) := X(x)$, analogous to what was done before. It is clear that $\phi_t^{(k)}(x_0)$ and $X^{(k)}(x)$ are k-linear symmetric maps.

Then, by successive derivation of (9) with respect to the components of x_0 and exchange of the order of the derivations, we obtain the desired equations

$$\begin{array}{rcl}
\frac{d}{dt}D_k\phi_j &=& D_iX_jD_k\phi_i,\\
\frac{d}{dt}D^2_{k_1,k_2}\phi_j &=& D_iX_jD^2_{k_1,k_2}\phi_i + D^2_{i_1,i_2}X_jD_{k_1}\phi_{i_1}D_{k_2}\phi_{i_2},\\
\frac{d}{dt}D^3_{k_1,k_2,k_3}\phi_j &=& D_iX_jD^3_{k_1,k_2,k_3}\phi_i + D^2_{i_1,i_2}X_jD^2_{k_1,k_2}\phi_{i_1}D_{k_3}\phi_{i_2}+\\
&& D^2_{i_1,i_2}X_jD^2_{k_1,k_3}\phi_{i_1}D_{k_2}\phi_{i_2} + D^2_{i_1,i_2}X_jD_{k_1}\phi_{i_1}D^2_{k_2,k_3}\phi_{i_2}+\\
&& D^3_{i_1,i_2,i_3}X_jD_{k_1}\phi_{i_1}D_{k_2}\phi_{i_2}D_{k_3}\phi_{i_3},\\
\ldots &=& \ldots,
\end{array} \tag{11}$$

where, as usual, summation is done with respect to repeated indices. The first line in (11) gives the first variational equations VE_1, first two (resp. three) lines give the second (resp. third) variational equations VE_2 (resp. VE_3), etc. It is possible to write these equations in a general, more compact form, by making use of $\phi_t^{(k)}$ and $X^{(k)}$:

$$\begin{array}{rcl}
\dot{\phi}_t^{(1)} &=& X^{(1)}\phi_t^{(1)},\\
\dot{\phi}_t^{(2)} &=& X^{(1)}\phi_t^{(2)} + X^{(2)}(\phi_t^{(1)})^2,\\
\dot{\phi}_t^{(3)} &=& X^{(1)}\phi_t^{(3)} + 2X^{(2)}(\phi_t^{(2)},\phi_t^{(1)}) + X^{(3)}(\phi_t^{(1)})^3,\\
\ldots &=& \ldots,
\end{array} \tag{12}$$

or, in general,

$$\dot{\phi}_t^{(k)} = \sum \frac{j!}{m_1!\ldots m_s!} X^{(j)}\left((\phi_t^{(i_1)})^{m_1}, (\phi_t^{(i_2)})^{m_2}, \ldots, (\phi_t^{(i_s)})^{m_s}\right), \quad k\geq 1, \tag{13}$$

where the $X^{(k)}$ are evaluated at $\phi_t(x_0)$, the symmetry of the multilinear maps has been used and the composition of multilinear maps has the obvious meaning: a term in the right hand side of (13) acts on a string of k vectors $(u_1,\ldots,u_k)$ as

$$X^{(j)}\left(\phi_t^{(i_1)}(u_1,\ldots,u_{i_1}),\ldots,\phi_t^{(i_s)}(u_{k-i_s+1},\ldots,u_k)\right).$$

In (13) the summations are carried out for

$$1\leq j\leq k, \quad i_1>i_2>\ldots>i_s, \quad \sum_{r=1}^s m_r = j, \quad \sum_{r=1}^s m_r i_r = k.$$

The obvious initial conditions for $\phi_t^{(k)}(x_0)$ are $\phi_t^{(1)}(x_0) = id$ and $\phi_t^{(k)}(x_0) = 0$ for $k > 1$. The equations for VE_1 are linear homogeneous, while for the remaining VE_k they are non-homogeneous, the non-linear part depending on the previous $\phi_t^{(j)}, j < k$, as it is evident looking at (11) or (12). The linear part has the same form for all the equations. Hence, as said before, the solutions can be obtained in a recurrent way by quadrature using, for instance, the method of variation of the constants. More concretely: if we suppose that we know a solution $(\phi_t^{(1)}, \ldots, \phi_t^{(k)})$ of VE_k, then we can write the equation for the new terms which appear in VE_{k+1}:

$$\dot{\phi}_t^{(k+1)}(x_0) = X^{(1)}(\phi_t(x_0))\phi_t^{(k+1)}(x_0) + P(\phi_t^{(1)}, \ldots, \phi_t^{(k)}). \tag{14}$$

In equation (14) P denotes polynomial terms in the components of its arguments. The coefficients depend on t through $X^{(j)}(\phi_t(x_0))$.

The problem is now to find a system of *linear* equations for $(\phi_t^{(1)}(x_0), \ldots, \phi_t^{(k)}(x_0))$ equivalent to the system of higher variational equations. It is enough to write the equations satisfied by the monomials appearing in P. This is the content of next lemma. It is similar to typical procedures in automatic differentiation and Taylor integration routines.

Lemma 4 *Let $z \in \mathbf{C}^q$. Assume the components $(z_1, \ldots, z_q)$ of z satisfy linear homogeneous differential equations $\dot{z}_i = \sum_{j=1}^q a_{ij}(t) z_j$. Then the monomials z^k of order $|k|$ satisfy also a system of linear homogeneous differential equations.*

Proof. Let $k = (k_1, \ldots, k_q)$ a multiindex of non-negative integers. Then

$$\frac{d}{dt} z^k = \sum_{j=1}^q \left(k_j z_j^{k_j - 1} \sum_{r=1}^q a_{jr} z_r \prod_{i=1, i \neq j}^q z_i^{k_i} \right), \tag{15}$$

the right hand side being also homogeneous of degree $|k|$ in z. □

We observe that the above lemma is nothing else than the pull back to the symmetric fibre bundle, $S^k(\mathbf{C}^q)$, of the connection associated to the linear differential equations.

In our application to linearise the VE_k it is clear that the $a_{jr} = a_{jr}(t)$ depend on t through the components of the $X^{(i)}(\phi_t(x_0))$ for $1 \leq i \leq k$. We realize that after the last equation corresponding to VE_k we can supplement the system of linear differential equations with the equations for the components of $\left((\phi_t^{(i_1)})^{m_1}, (\phi_t^{(i_2)})^{m_2}, \ldots, (\phi_t^{(i_s)})^{m_s}\right)$. More concretely, after the first, second, ..., equations in (12) we must add equations of the form

$$\begin{array}{l} \frac{d}{dt}(\phi_t^{(1)})^2 = L_2(\phi_t^{(1)})^2, \\ \frac{d}{dt}(\phi_t^{(2)}, \phi_t^{(1)}) = L_{3,1}(\phi_t^{(2)}, \phi_t^{(1)}), \quad \frac{d}{dt}(\phi_t^{(1)})^3 = L_{3,2}(\phi_t^{(1)})^3, \\ \cdots, \end{array} \tag{16}$$

where the coefficients in the L's are obtained using (15). In this way we obtain recurrently the desired *linearised* version LVE_k.

It is easy to reformulate the construction of this linearisation in matrix form using a composition by an arbitrary scalar function (or jet) at the target.

At this point we need to consider the action of the flow on jets of functions.

We set $z = \phi_t(y_0) - \phi_t(x_0)$. Let $f : (M, \phi_t(x_0)) \to (\mathbf{C}, 0)$ be a germ of holomorphic function. We write its power series expansion:

$$f(z) = f^{(1)} z + f^{(2)}(z)^2 + \ldots . \tag{17}$$

We will suppose that, for each $k \in \mathbf{N}^*$, the multilinear symmetric map $f^{(k)}$ is *independent* of t. (Be careful, the source of the infinite jet of f depends on t: among the natural coordinates of this jet, *only* the source coordinates $\phi_t(x_0)$ depend on t.)

Then $f \mapsto f \circ \phi_t$ is a linear map. We set $f \circ \phi_t = \varphi_t(f^{(1)}, f^{(2)}, \ldots)$. Again $(f^{(1)}, f^{(2)}, \ldots) \mapsto \varphi_t(f^{(1)}, f^{(2)}, \ldots)$ is a linear map.

For each fixed value of $(f^{(1)}, f^{(2)}, \ldots)$ (independent of t), we can write a differential system (of infinite order) satisfied by $\varphi_t(f^{(1)}, f^{(2)}, \ldots)$. Then φ_t will appear as a *fundamental solution* of this linear system. Using infinite "matrix" form, we will obtain a linearisation LVE_ω of the higher variational equations all together. Finally we will get, for each k, the linearisation LVE_k of VE_k by truncation, i.e., replacing $(f^{(1)}, f^{(2)}, \ldots)$ by the k-jet $(f^{(1)}, f^{(2)}, \ldots, f^{(k)})$.

Then $(f^{(1)}, f^{(2)}, \ldots) \mapsto \varphi_t(f^{(1)}, f^{(2)}, \ldots)$ gives in matrix form:

$$F \mapsto F\Phi_t$$

where F is the infinite vector which contains all the components of the $f^{(k)}$ (supplemented at every order by the required products of lower order terms).

We see that $\varphi_t \to \Phi_t$ is a faithful representation of $\mathit{Diff}(\mathbf{C}^m, 0)$ in a group of invertible infinite dimensional matrices.

We have $\frac{d}{dt}(f \circ \phi_t) = df(X) \circ \phi_t$. In matrix form, the infinite matrix Φ_t satisfies a linear differential system

$$\dot{\Phi}_t = A_t \Phi_t, \tag{18}$$

where the infinite matrix A_t collects all the coefficients appearing in (11) and (16).

The LVE_k are obtained by truncation of *triangular* matrices:

$$\dot{j^k\Phi} = j^k A \, j^k\Phi.$$

The map $j^k\varphi \to j^k\Phi$ is a faithful representation of $\mathit{Diff}^k(\mathbf{C}^m, 0)$in a group of invertible upper triangular matrices. It follows hat j^kA takes its value in a corresponding Lie algebra, which is a faithful representation of $J^k(\mathbf{C}^m, 0)$.

Proposition 7 *The VE_k and the LVE_k are equivalent. More precisely, we can associate bijectively the solution $t \mapsto \phi_t$ of the VE_k such that $\phi_0 = id$ and the fundamental solution of the LVE_k which is equal to the identity for $t = 0$.*

Proof. It follows immediately from the preceding considerations. □

We have now the following problem: preceding computations rely on a coordinate choice, but in order to apply differential Galois theory we need a global geometric formulation in connections terms. Therefore we will give now such a geometric intrinsic formulation. The fundamental idea is to use *duality*. The starting point is to replace the family of diffeomorphisms $\phi_t : (\mathbf{C}^m, x_0) \to (\mathbf{C}^m, \phi_t(x_0))$ by the family of *inverse* diffeomorphisms $\psi_t = \phi_t^{-1} : (\mathbf{C}^m, \phi_t(x_0)) \to (\mathbf{C}^m, x_0)$. The key point is that now the *targets* are *independent* of t. We have $\phi_0 = id$, $\psi_0 = id$.

We denote by f a germ of holomorphic function at x_0 on M, vanishing at x_0. The map $f \mapsto f \circ \psi_t$ is *linear.* We write it in matrix form

$$F \mapsto F\Psi_t.$$

We have clearly $\Psi_0 = \Phi_0 = I$ and $\Phi_t\Psi_t = I$. By derivation, we get

$$\dot{\Psi}_t = -\Phi_t^{-1}\dot{\Phi}_t\Phi_t^{-1}$$

and, using (18):

$$\dot{\Psi}_t = -\Psi_t A_t$$

and by transposition

$$^t\dot{\Psi}_t = (-^tA_t)^t\Psi_t. \tag{19}$$

Therefore $^t\Psi_t$ is the "fundamental solution" of the "dual" differential system of (18).

By truncation we get a *true duality*: $j^k(^t\Psi_t)$ is the fundamental solution of the dual system of $j^k\dot{\Phi}_t = j^kA_t\, j^k\Phi_t$. We denote this dual system by LVE_k^*.

Now we will see that these dual systems are associated to *natural connections* on fibre bundles admitting the groups $\mathit{Diff}^k(\mathbf{C}^m, 0)$ as structure groups.

A connection on a fibre bundle is the "same thing" that a *parallel transport* on the fibres and we will see that in our situation we have a natural parallel transport.

Let M be a connected complex manifold of dimension m. Let X be a holomorphic vector field on M. We choose a non stationary solution $\iota : \Gamma \to M$, Γ being a connected Riemann surface and ι an embedding. Let $\Delta \in \Gamma \times M$ be the graph of ι.

Using Δ, we can interpret the collection of germs of analytic manifold $\{(M,\xi) \mid \xi = \iota(\tau), \tau \in \Gamma\}$ along $\iota(\Gamma)$ as a non-linear locally trivial fibre bundle on Γ, the fibres being isomorphic to $(\mathbf{C}^m, 0)$ and the structure group being $\mathit{Diff}(\mathbf{C}^m, 0)$. Then we can see the flow of (9) along $\iota(\Gamma)$ as a parallel transport along this fibre bundle. Now if we replace the collection of germs of the analytic manifold $\{(M,\xi) \mid \xi = \iota(\tau), \tau \in \Gamma\}$ by the "dual" collection of germs of holomorphic functions $\{\mathcal{O}_\xi \mid \xi = \iota(\tau), \tau \in \Gamma\}$, we get a locally trivial infinite dimensional holomorphic linear fibre bundle $J^\omega_\Gamma M = \iota^* J^\omega M$ on Γ, the fibres being isomorphic to the complex vector spaces $(\mathbf{C}\{x_1, \ldots, x_m\}, 0)$ and the structure group being $\mathit{Diff}(\mathbf{C}^m, 0)$. The flow acting "dually" on the functions gives a linear parallel transport on this bundle corresponding to a "connection" ∇_ω. Replacing the spaces $\mathcal{O}_\xi$ of germs of holomorphic functions by the quotient spaces of k-jets $J^k(M, \xi)$, we get locally trivial finite dimensional holomorphic linear fibre bundles on Γ: $\iota^* J^k M = J^k_\Gamma M$. By quotients of our parallel transport we get holomorphic parallel transports on the bundles $J^k_\Gamma M$. These parallel transports define holomorphic connections ∇_k which are quotients of ∇_ω. We will see that $(J^k_\Gamma M, \nabla_k)$ corresponds to LVE_k^* if we introduce local coordinates.

It is possible to justify the preceding considerations using Grothendieck's definition of infinitesimal calculus [31] (infinitesimal neighbourhoods of the diagonal). Here we shall do the work explicitly and by elementary ways.

We will built each connection locally and after that check that our constructions glue together to give global connections. Hence we can suppose that Γ is a *simply connected* open subset of $\mathbf{C}$. Then for each non vanishing holomorphic vector field δ, we will define the covariant derivative ∇_δ associated to ∇. It is sufficient to do that for the vector field $\delta = \frac{d}{dt}$, t being a local coordinate.

We introduce the vector field $\widetilde{X} = \frac{d}{dt} + X$ on $\Gamma \times M$. The graph Δ is invariant by $\widetilde{X}$. By definition the horizontal sections of ∇_ω are the *first integrals* of $\widetilde{X}$. More precisely

Definition 1 *Let $t_0, t_1 \in \Gamma$ and be f_0, f_1 two germs of holomorphic functions on M, respectively at the points $\xi_0 = \iota(t_0)$ and $\xi_1 = \iota(t_1)$. We will say that we get f_1 from f_0 by parallel transport (from t_0 to t_1), if there exists a first integral $f : (\Gamma \times M, \Delta) \to \mathbf{C}$ of $\widetilde{X}$ holomorphic on an open neighbourhood of Δ in $\Gamma \times M$ such that $f(t_0, \xi) = f_0(\xi)$ and $f(t_1, \xi) = f_1(\xi)$. In this definition we allow to restrict Γ, this open set remaining simply connected and t_0, t_1 remaining fixed.*

The function f is a first integral of $\widetilde{X}$ if and only if

$$L_{\widetilde{X}} f(t,\xi) = L_{d/dt} f(t,\xi) + L_X f(t,\xi) = 0. \tag{20}$$

This is equivalent to the following condition:

$$\dot{f}(t,\xi) := \frac{\partial}{\partial t} f(t,\xi) = -\frac{\partial}{\partial \xi} f(t,\xi) X(\xi). \tag{21}$$

Lemma 5 *Let $f : (\Gamma \times M, \Delta) \to \mathbf{C}$ be an holomorphic function on a neighbourhood of Δ in $\Gamma \times M$. Let $t_0 \in \Gamma$ be a fixed point. The following conditions are equivalent:*

(i) f_t (where $f_t(\xi)) = f(t,\xi)$) comes by parallel transport from f_{t_0} for every $t \in \Gamma$.

(ii) There exists a unique family of germs of holomorphic diffeomorphisms $\phi_t : (M, \iota(t_0)) \to (M, \iota(t))$ such that $f_t \circ \phi_t = f_{t_0}$, for $t \in \Gamma$.

Moreover, if these conditions are satisfied, then:

(a) $\phi_{t_0}(\xi) = \phi(t_0,\xi) = \xi$ (i.e., $\phi_{t_0} = id$);

(b) $\dot{\phi}_t(\xi) = \frac{\partial}{\partial t}\phi(t,\xi) = X(\phi(t,\xi))$, that is ϕ is the flow of the field X, with initial conditions (t_0,ξ_0).

Proof. The proof is clear: conditions *(i)* and *(ii)* are obviously equivalent to say that $f(t,x)$ is constant along the flow curves. Later on we will need a similar lemma for k-jets. So we give another proof that we will use later in the jets case.

We suppose that *(ii)* is satisfied: $f_t \circ \phi_t = f_{t_0}$. Then, by t–derivation we get:

$$\dot{f}(t,\phi(t,\xi)) + D_x f(t,\phi(t,\xi))\dot{\phi}(t,\xi) = \dot{f}(t,\phi(t,\xi)) + D_x f(t,\phi(\xi)) X(\phi(\xi)) = 0. \tag{22}$$

We conclude using equivalence of *(i)* and (21). □

Let now $j^k f_t(\xi) = j^k f(t,\xi)$ be an holomorphic family of k-jets along Δ in $\Gamma \times M$ (an holomorphic function on the k-infinitesimal neighbourhood $\Delta^{(k)}$ of Δ in $\Gamma \times M$). For a fixed t consider $j^k f_t \in J^k(M, \iota(t))$. We will say that it is a first integral of $\widetilde{X}$ if and only if

$$(j^k f)^{\cdot}(t,\xi) = \frac{\partial}{\partial t} j^k f(t,\xi) = -\frac{\partial}{\partial \xi} j^k f(t,\xi) X(\xi), \tag{23}$$

in the evident jet sense.

If we interpret $j^k f$ as an holomorphic section of $J^k_\Gamma M$, (23) is a holomorphic differential system

$$\dot{Y} = j^k A_t Y. \tag{24}$$

The matrix function $t \mapsto A_t$ takes its value in $\mathcal{L}^k(\mathbf{C}^m, 0)$. Therefore (24) corresponds to a connection with structure group $\mathit{Diff}^k(\mathbf{C}^m, 0)$. Moreover, if we are in a symplectic situation (M symplectic, $m = 2n$, X Hamiltonian), then $A_t \in \mathcal{L}^k_{sp}(\mathbf{C}^{2n}, 0)$ and the structure group is $\mathit{Diff}^k_{Sp}(\mathbf{C}^{2n}, 0)$.

Now we can give an analog of Lemma 5 for k-jets.

Lemma 6 *Let $j^k f_t(\xi) = j^k f(t,\xi)$ be an holomorphic family of k-jets along Δ in $\Gamma \times M$. Let $t_0 \in \Gamma$ be a fixed point. The following conditions are equivalent:*

(i) $j^k f(t,\xi)$ is a first integral of $\widetilde{X}$ (in the k-jet sense).

(ii) There exists a unique family of germs of k-jets of holomorphic diffeomorphisms $j^k\phi_t : (M, \iota(t_0)) \to (M, \iota(t))$ such that $j^k f_t \circ \phi_t = j^k f_{t_0}$, for $t \in \Gamma$.

Moreover, if these conditions are satisfied, then:

(a) $j^k\phi_{t_0} = j^k id$.

(b) $(j^k\phi)^{\cdot}(t,\xi) = \frac{\partial}{\partial t} j^k\phi(t,\xi) = j^k(X(\phi(t,\xi)))$.

Proof. We prove the implication *(ii)* $\Rightarrow$ *(i)* by t–derivations in the functional case above. We will prove the implication *(i)* $\Rightarrow$ *(ii)*.

Assume that we have (23):

$$(j^k f)^{\cdot}(t,\xi) = \frac{\partial}{\partial t} j^k f(t,\xi) = -\frac{\partial}{\partial \xi} f(t,\xi)X(\xi).$$

We interpret this equation as a differential system (24)

$$\dot{Y} = j^k A_t Y,$$

where $j^k A_t \in \mathcal{L}^k(\mathbf{C}^m, 0)$ (resp. $j^k A_t \in \mathcal{L}^k_{sp}(\mathbf{C}^m, 0)$ in the symplectic case). Let $j^k\Phi_t$ be the unique holomorphic fundamental solution of this system such that $j^k\Phi_0 = id$. Then we have the following result.

Lemma 7 *We have* $j^k\Phi_t \in Diff^k(\mathbf{C}^m, 0)$ *(resp.* $j^k\Phi_t \in Diff^k_{Sp}(\mathbf{C}^{2n}, 0)$ *in the symplectic case).*

This follows from the following Lemma [22] (6.25 Lemme, p. 238).

Lemma 8 *Let G be a linear complex algebraic group and $\mathcal{G}$ its Lie algebra. Let $\dot{Y} = A_t Y$ be a holomorphic linear system in a neighbourhood of $t = 0$. We suppose that $A_t \in \mathcal{G}$. Then the unique holomorphic fundamental solution of this system takes its value in G (more precisely in the identity component G^0 of G).*

Now it is easy to end the proof of Lemma 6.

4 The differential Galois groups of the variational equations. The main theorem.

In this part we devote our attention to the Hamiltonian case.

Let

$$\dot{x} = X_H(x) \tag{25}$$

be the analytic differential equation defined by a *Hamiltonian* vector field X_H over a complex connected *symplectic* manifold M of complex dimension $2n$.

To a non stationary solution we associate an immersion $\iota : \Gamma \to M$, Γ being a connected Riemann surface.

We choose a non trivial derivation ∂ over the field k_Γ of meromorphic functions on Γ.

Let $m \in \Gamma$. We denote by $\mathcal{O}_m$ (resp. $\mathcal{M}_m$) the algebra of germs of holomorphic functions over Γ at m (resp. the field of germs of meromorphic functions over Γ at m). Here we will use the linear variational equations LVE_k of order k, which we defined in the preceding section. They are *holomorphic* connections ∇_k over the "restrictions" (more precisely pull backs by ι) $(J^k)^*_\Gamma M$ of the dual bundles $(J^k)^*M$ of the fibre bundles J^k_M of k-jets of scalar holomorphic functions on Γ. The structure groups of these connections are the groups of symplectic k-jets of diffeomorphisms $Diff^k_{Sp}(\mathbf{C}^{2n},0)$. We will also use the LVE^*_k which are holomorphic connections ∇^*_k on the bundles $J^k_\Gamma M$. The structure groups of these connections are also the groups of symplectic k-jets of diffeomorphisms $Diff^k_{Sp}(\mathbf{C}^{2n},0)$. For $k=1$ we get $(J^1)^*_\Gamma M = T_\Gamma M$ and $J^1_\Gamma M = T^*_\Gamma M$ and the structure group is $Sp(\mathbf{C}^{2n},0)$.

We recall that, for $r \leq k$, $(J^r)^*_\Gamma M$ (resp. $J^r_\Gamma M$) is a sub-bundle (resp. a quotient bundle) of $(J^k)^*_\Gamma M$ and that ∇_r (resp. ∇^*_r) is a subconnection (resp. a quotient) of ∇_k (resp. ∇^*_k).

We recall (cf. Appendix A) that each holomorphic bundle $J^k_\Gamma M$ or $(J^k)^*_\Gamma M$ is meromorphically trivialisable over Γ (as a bundle with structure group $Diff^k_{Sp}(\mathbf{C}^{2n},0)$) and we will suppose in this section that we have fixed a trivialisation for each k. Then we can write LVE_k or LVE^*_k as a differential system of order one:

$$\dot{Y} = A_k Y, \tag{26}$$

where A_k is a meromorphic function taking its values in the Lie algebra $\mathcal{L}^k_{sp}(\mathbf{C}^{2n},0)$.

This differential system can have singularities (depending on the choice of trivialisation); however these singularities are clearly *apparent singularities*. In the following we will work in general at a regular point, but the results extend immediately to an apparent singularity.

Let $m \in \Gamma$. Applying Cauchy theorem, we get a fundamental systems of solutions F_k whose entries belong to $\mathcal{M}_m$ (and the same happens for F^*_k) for the trivialisations of the LVE_k (and LVE^*_k). They are holomorphic, that is $\in \mathcal{O}_m$ if m is regular, meromorphic if m is an apparent singularity. We denote by L_k the sub-differential field of $\mathcal{M}_m$ generated by the entries of F_k over k_Γ. It is also the sub-differential field of $\mathcal{M}_m$ generated by the entries of F^*_k over k_Γ and it is independent of the choice of the trivialisation. Then L_k is a Picard-Vessiot extension of k_Γ associated to LVE_k and also to LVE^*_k. We have inclusions of differential fields $k_\Gamma \subset L_1 \subset \ldots \subset L_k \subset L_{k+1} \subset \ldots$. All the corresponding extensions of differential fields are *normal*.

By definition "the" differential Galois group $Gal\ \nabla_k$ of LVE_k (or LVE^*_k) is $Gal\ \nabla_k = Aut^\partial_{k_\Gamma} L_k$. (It depends up to a non natural isomorphism on the choice of a Picard-Vessiot extension, therefore here on the choice of m.)

Using the differential Galois correspondence, we get short exact sequences

$$\{e\} \to Aut^\partial_{L_k} L_{k+1} \to Gal\ \nabla_{k+1} \to Gal\ \nabla_k \to \{e\}. \tag{27}$$

We denote by Sol_k (resp. Sol^*_k) the linear complex space generated by the entries of F_k (resp. F^*_k). From each of these spaces we get easily the solutions of the (non-linear) classical k-variational equation VE_k.

The differential Galois group $Gal\ \nabla_k = Aut^\partial_{k_\Gamma} L_k$ acts naturally on the linear spaces Sol_k and Sol^*_k and we get natural faithful representations of $Gal\ \nabla_k$ in $GL(Sol_k)$ and $GL(Sol^*_k)$. We will use these representations to built a natural homomorphism of algebraic groups

$$Gal\ \nabla_K \to Diff^k_{Sp}(M,m).$$

(We denote $Diff^k_{Sp}(M,m)$ the pull back of $Diff^k_{Sp}(M,\iota(m))$ by the map ι.)

We will see that this homomorphism is injective. *By definition* its image will be "the" differential Galois group of the higher variational equation VE_k.

Our construction will be local. We will use coordinates. (It is easy to check that the results are independent of the choice of coordinates.) Near $\iota(m)$ we can choose Darboux coordinates on the symplectic manifold M centred at $\iota(m)$. Then we can use these coordinates to trivialise locally our scalar jets bundles $(J^k)^*_\Gamma M$ and $J^k_\Gamma M$: we write them as products of the basis by $J^k(\mathbf{C}^{2n},0)$ and $(J^k)^*(\mathbf{C}^{2n},0)$, using the standard coordinates on the jets. (We identify $(M,\iota(m))$ with $(\mathbf{C}^{2n},0)$ with its standard symplectic structure of linear space.) Then we can interpret an element P of the fibre at $\iota(m)$ of J^k_M as an element of $J^k(\mathbf{C}^{2n},0)$ and this last element as a polynomial in y with values in $\mathbf{C}$. On Γ we can use the temporal parametrisation t ($t=0$ at $m=x_0$).

We will do an essential use of the following result [60] (Theorem 14, Appendix C, page 92).

Proposition 8 *Let G be a complex linear algebraic group. Let ∇ be a G-meromorphic connection on a trivial G-bundle over a connected Riemann surface Γ. Then its differential Galois group "is" a Zariski closed subgroup of G.*

As before, we denote by ϕ_t the flow map ($\phi_0 = id$) and we set $\psi_t = \phi_t^{-1}$; ψ_t^k is the k-jet of ψ_t. Its source is $\phi(x_0)$ and is target is $x_0 = m$.

Let $f \in J^k(\mathbf{C}^{2n},0)$. We consider it as an initial condition for LVE^*_k. The corresponding solution is (locally) $f(t) = f \circ \psi_t^k$ ($f(0) = f$). We have $f(t) \in Sol^*_k$ and $\sigma \in Gal\ \nabla_k$ transforms $f(t)$ into another solution $g(t) \in Sol^*_k$. We set $g(0) = g$. Then $g(t) = g \circ \psi_t^k$. We get a faithful representation of $Gal\ \nabla_k = Aut^{\partial}_{k_\Gamma} L_k$ in the *linear space* $J^k(\mathbf{C}^{2n},0)$ (the differential Galois group acts on the initial conditions):

$$\rho_k : Gal\ \nabla_k \to GL(J^k(\mathbf{C}^{2n},0))$$

$$\sigma \to (f \mapsto g).$$

This action is clearly compatible with the *ordinary* product of scalar jets, therefore if $\sigma \in Gal\ \nabla_k$, then $\rho_k(\sigma)$ is an automorphism of the $\mathbf{C}$-algebra $J^k(\mathbf{C}^{2n},0)$. Using Proposition 3, we can interpret $\rho_k(\sigma)$ as an element of $J^k_{0,0}(\mathbf{C}^{2n}) = Diff^k(\mathbf{C}^{2n},0)$. We get a faithful representation (we do not change the name):

$$\rho_k : Gal\ \nabla_k \to Diff^k(\mathbf{C}^{2n},0).$$

It remains to prove that $\rho_k(\sigma) \in Diff^k(\mathbf{C}^{2n},0)$ is a *symplectic* jet. This is the delicate point.

We recall that we choose our trivialisation of LVE^*_k without changing the structure group $Diff^k_{Sp}(\mathbf{C}^{2n},0)$: the matrix A of the corresponding differential system takes its values in the Lie algebra $\mathcal{L}^k_{sp}(\mathbf{C}^{2n},0)$. Now we can apply Proposition 8: the image of ρ_k in $Diff^k(\mathbf{C}^{2n},0)$ is contained in the algebraic subgroup $Diff^k_{Sp}(\mathbf{C}^{2n},0)$ (the structure group of our system). We get a faithful representation (we do not change the name):

$$\rho_k : Gal\ \nabla_k \to Diff^k_{Sp}(\mathbf{C}^{2n},0).$$

It is easy to check that this representation is independent of the choices of trivialisation and local coordinates (it depends only on the point m) and we get a faithful natural representation (we do not change the name):

$$\rho_k : Gal\ \nabla_k \to Diff^k_{Sp}(M,m).$$

By definition the image of the homomorphism ρ_k in $Diff^k_{Sp}(M,m)$ is "the" differential Galois group of the higher variational equation VE_k. We will denote it G_k, or $G_{m,k}$ if necessary. If γ is a continuous path from m_1 to m_2 on Γ, then the flow from m_1 to m_2 along γ (more precisely its k-jet) induces an isomorphism between $G_{m_1,k}$ and $G_{m_2,k}$

Proposition 9 *The natural homomorphism of algebraic groups Gal* $\nabla_k \to \mathit{Diff}^k_{Sp}(M,m)$ *is a morphism of algebraic groups and an injection. Therefore Gal* $\nabla_k \to G_k$ *is an isomorphism of algebraic groups. (The differential Galois groups of the* LVE_k *and of the* VE_k *are isomorphic.)*

We have commutative diagrams ($G\nabla_k = Gal\ \nabla_k$))

$$\begin{array}{ccc} G\nabla_{k+1} & \xrightarrow{\rho_{k+1}} & G_{k+1} \\ \downarrow & & \downarrow \\ G\nabla_k & \xrightarrow{\rho_k} & G_k \end{array}$$

As we saw above (see (27)), the natural maps $G\nabla_{k+1} \to G\nabla_k$ are surjective morphisms of algebraic groups. Therefore the natural maps $G_{k+1} \to G_k$ are also *surjective* homomorphisms of algebraic groups. This is an essential result in our approach.

Proposition 10 *The natural maps* $G_{k+1} \to G_k$ *are surjective homomorphisms of algebraic groups.*

We denote by $\mathcal{G}_k$ the Lie algebra of G_k. It is identified with a Lie subalgebra of $\mathcal{L}^k_{sp}(M,m)$.

By definition the *formal differential group* of the Hamiltonian system (1) along $\iota(\Gamma)$ is the pro-algebraic group $\hat{G} = \lim_{\substack{\leftarrow \\ k}} G_k \subset \mathit{Diff}^{\infty}(M,m)$.

The Lie algebra $\hat{\mathcal{G}}$ of $\hat{G}$ is $\hat{\mathcal{G}} = \lim_{\substack{\leftarrow \\ k}} \mathcal{G}^k$. It is identified with a Lie subalgebra of the Lie algebra of *formal* Hamiltonian vector fields $\hat{\mathcal{L}}^k_{sp}(M,m)$. Then we get

Proposition 11

(i) The natural maps $\mathcal{G}^{k+1} \to \mathcal{G}^k$ *are surjective homomorphisms of Lie algebras.*

(ii) The natural maps $\hat{\mathcal{G}} \to \mathcal{G}^k$ *are surjective homomorphisms of Lie algebras.*

(iii) The natural maps $\hat{G} \to G_k$ *are surjective homomorphisms of algebraic groups.*

Remark. $\hat{G}$ (resp G_k, $k \geq 1$) is Zariski connected if and only if G_1 (that is the differential Galois group of the VE_1) is Zariski connected: the successive extensions $G_{k+1} \to G_k$ ($k \geq 1$) are extensions by *unipotent* algebraic groups. This follows from the recursive integration of the VE_k ($k \geq 2$) by the method of variations of constants, of if one prefers from the structure of the groups Diff^k.

The main result of this paper is the following.

Theorem 5 *(Main theorem) If the Hamiltonian system (1) is completely integrable with meromorphic first integrals along* $\iota(\Gamma)$*, then*

(i) the identity component $\hat{G}^0$ *of the formal differential Galois group along* $\iota(\Gamma)$ *is commutative,*

(ii) the identity components $(G_k)^0$ *of the differential Galois groups of the variational equations along* $\iota(\Gamma)$ *are commutative* ($k \in \mathbf{N}^*$).

We have a preliminary result.

Proposition 12 *If* f *is a meromorphic first integral, then* *(i) its germ f_m at m is invariant by the formal differential Galois group $\hat{G}$,*

(ii) the germ f_m is orthogonal to the Lie algebra $\hat{\mathcal{G}}$.

Here orthogonality means, by definition, that if a formal vector field $\hat{X}$ belongs to $\hat{\mathcal{G}}$, then $L_{\hat{X}} f_m = df_m(\hat{X}) = 0$.

Proof. Claim (ii) follows easily from (i), similarly to the proof given in [60] for the first order variational equations. We will prove (i).

First case. We will suppose firstly that the first integral f is *holomorphic.*

Let $m \in \Gamma$. We pull back at m the germ of manifold $(M, \iota(m))$ and denote it by (M, m). We choose Darboux coordinates $\xi = (\xi_1, \dots, \xi_{2n})$ on M at m, and the time parametrisation t on Γ, with $t_0 = 0$.

To a germ of holomorphic function $f = f_m = f(\xi)$, we associate the holomorphic family (in t) of germs of holomorphic function $f_t = f \circ \psi_t$, where $\psi_t = \phi_t^{-1}$ as before. We set $f_t(\xi) = f(t, \xi)$. We have $f_0 = f$.

The function f is a first integral of $X = X_H$ if and only if $f(t, \xi) = f(\xi)$, that is if $f_t = f_0 = f$.

Let $k \in \mathbf{N}$. We consider the family (in t) of scalar jets $j^k f(t, \xi)$. We can write them using standard coordinates, that is as a polynomial in ξ whose coefficients a_p are holomorphic functions of t (p is a multi-index). These coefficients belong to the Picard-Vessiot extension L_k. Therefore an element $\sigma \in Gal\ \nabla_k = Aut^{\partial}_{k_\Gamma} L_k$ acts on the coefficients a_p and we get a "new" family of jets $j^k g_t = \sigma j^k f_t$. If f is a *first integral*, then the coefficients a_p are *scalars* (that is complex numbers); therefore they are invariant by σ and $j^k g_t = j^k f_t$, $j^k g = j^k g_0 = j^k f$. The k-jets of first integrals are invariant by $\sigma \in Gal\ \nabla_k$, or equivalently by G_k. Claim (i) follows (increasing k).

Second case. We will suppose now that the first integral $f = f(\xi)$ is *meromorphic.* We write $f(\xi) = F(\xi)/G(\xi)$, where F, G are analytic and where the quotient is irreducible up to invertible elements.

As above, we write $f(t, \xi) = F(t, \xi)/G(t, \xi)$. If f is a first integral, then $f(\xi) = f(t, \xi) = F(t, \xi)/G(t, \xi)$. Let $\sigma \in Gal\ \nabla_k = Aut^{\partial}_{k_\Gamma} L_k$. We have $F(t, \xi)G(0, \xi) - F(0, \xi)G(t, \xi) = 0$ and this element is invariant by σ. Therefore $F(t, \xi)/G(t, \xi) = F(0, \xi)/G(0, \xi)$ is invariant by σ and we can concluse as in the first case.

If $\hat{X} \in \hat{\mathcal{G}}$, then $L_{\hat{X}} f(\xi) = 0$ formally. If moreover $\hat{X} = X$ is *convergent*, then $L_X f = 0$ *meromorphically.* □

Now we can go back to the proof of our main theorem. The Lie algebra $\hat{\mathcal{G}}$ is symplectic and orthogonal to the germs $f_1 \dots, f_m$. Therefore it is abelian (see Theorem 3). □

In the main theorem, as in the case of the first variational equation we can add to Γ some equilibrium points and some points at infinity. We replace Γ by $\bar{\Gamma}$, the differential field k_Γ by the a priori smaller field $k_{\bar{\Gamma}}$ and $Aut^{\partial}_{k_\Gamma} L_k$ by the a priori bigger differential Galois group $Aut^{\partial}_{k_{\bar{\Gamma}}} L_k$. If we add only some equilibrium points, we have a similar statement, mutatis mutandis. If we add some points at infinity, we need to be careful: if the extended connection is not regular singular at the corresponding points (a property that we can check on the first variational equation), it is necessary to suppose (as in the first variation case) that the first integral used in the definition of integrability are *meromorphic at infinity.*

We can also get a similar result by considering the local Galois group around a singular point $m \in \bar{\Gamma} \cup \{$*points at the infinity*$\}$, i.e., we take as coefficient field the differential field $\mathcal{M}_m$ and the non-commutativity of the identity component of the local Galois gives us an obstruction to the meromorphic integrability of the Hamiltonian system in a neighbourhood of the point $\iota(m)$. Of course, the interesting case appears when m is an irregular singular point.

5 Obstructions to integrability and non-linear Galois theory.

When one wants to prove the non-integrability of some Hamiltonian system, it sounds a priori quite strange to have to choose a "nice" solution of the system (a solution that we can parametrise using some "special functions"). It seems on the contrary reasonable to think that a solution of a non-integrable system is "very transcendental". In fact in *all* the applications done up to now of our theory, this is not the case: the "generic solution" of the system is presumably highly transcendental (there are few precise results in this direction), but there are *invariant integrable subsystems* giving interesting "nice" solutions. Perhaps this is due to the fact that applications are done to simple enough systems.

However, from a theoretical point of view, it remains interesting to try to avoid the choice of a particular solution Γ. In order to do this, a natural tool is the *non-linear differential Galois theory*. This theory was introduced by J. Drach [23, 24], and developped later by J. Drach and E. Vessiot. J. Drach studied some applications of his theory to various questions of geometry and mechanics (as the spinning top). P. Painlevé gave a "proof" of the fact that Painlevé transcendants are "new transcendants" using Drach's theory. Unfortunately there are important gaps in the foundations of Drach's theory, and in its applications. Therefore, when we began to work on the present paper some years ago, no satisfying non-linear Galois theory was available. Today the siuation is completely different: we have two such theories due respectively to H. Umemura [76] and B. Malgrange [54, 55]. We will explain the analogue of our main theorem using Malgrange's approach (the similar result must be true with Umemura's approach, but it is conjectural). This analogue is due to the second author (unpublished). We will be very sketchy. The interested reader will find some details about this result, and the necessary definitions and theorems in [54, 55, 18] (cf. in particular [18] 5.4).

The main tool of B. Malgrange is the notion of Lie $\mathcal{D}$-groupoïd on a complex analytic manifold M. Roughly speaking it is a sub-groupoïd of the groupoïd $\mathcal{M}$ of germs of analytic diffeomorphisms of M defined by analytic PDE. We recall that a groupoïd is a (small) category whose all the morphisms are *isomorphisms*. Here we start with the groupoïd $\mathcal{M}$ whose objects are the points $(a, b, \ldots)$ of M and the morphisms are the invertible germs g of analytic maps $g : (M, a) \to (M, b)$ $(f(a) = b)$. A $\mathcal{D}$-groupoïd has a *$\mathcal{D}$-Lie algebra* defined by PDE. B. Malgrange defines the Galois $\mathcal{D}$-groupoïd or $\mathcal{D}$-hull of an analytic dynamical system (differential equation, foliation, etc) as the *smallest* $\mathcal{D}$-groupoïd such that "its Lie algebra contains the infinitesimal transformations of the dynamics" (more precisely these transformations must be *solutions* of the *$\mathcal{D}$-Lie algebra*). The (very) difficult point is that such a smallest groupoïd *exists*. This definition is (a posteriori) very natural: as B. Malgrange says it is "what algebra sees from the dynamics". It is related to the idea that there is no "Lie third theorem" for Lie $\mathcal{D}$-groupoïds.

For an autonomous system we get the smallest $\mathcal{D}$-groupoïd such that its "Lie algebra contains the corresponding vector field".

The computation of the Galois $\mathcal{D}$-groupoïd seems extremely difficult (even for apparently "simple" cases as Painlevé equations). There are only partial theoretical results (cf. in particular [18]) and nothing is effective (there are no algorithms). Therefore the following result is, for the moment, nice but theoretical.

Like for the computation of differential Galois groups in the linear case, a natural idea is to try to reduce the problem using "majorants" of the Galois $\mathcal{D}$-groupoïd, i.e. $\mathcal{D}$-groupoïds containing the Galois $\mathcal{D}$-groupoïd.

Theorem 6 *Let (M, ω) be a connected symplectic analytic manifold of dimension $2n$. Let $H : M \to \mathbf{C}$ be an analytic function, the Hamiltonian. We suppose that the corresponding Hamiltonian system is completely meromorphically integrable. Then the $\mathcal{D}$-Lie algebra of the Galois $\mathcal{D}$-groupoïd of the system (i.e., the $\mathcal{D}$-hull of the vector field X_H) is abelian.*

An evident majorant of our Galois $\mathcal{D}$-groupoïd is the $\mathcal{D}$-groupoïd defined by the analytic system of PDE:

$$g^* f_i = f_i, \quad i = 1, \ldots, n, \qquad g^* \omega = \omega. \tag{28}$$

Its linearisation is

$$L_X f_i = 0, \quad i = 1, \ldots, n, \qquad L_X \omega = 0. \tag{29}$$

This *$\mathcal{D}$-Lie algebra* is clearly *abelian*. This follows from Theorem 3. (We need only the simple *meromorphic* version, not the formal version). Then Theorem 6 follows immediately.

One works with $\mathcal{D}$-groupoïds systematically on the "complementary of an hypersurface" (in some "algebraic" delicate sense). On the contrary our theory (more precisely its applications) is (are) centred on the choice of non-generic solutions Γ, which in general will live into the exceptional hypersurfaces. Therefore a comparison between the two approaches seems difficult. We will only suggest an analogy: in terms of topology of foliations, Galois $\mathcal{D}$-groupoïds will correspond to *holonomy groupoïds*, our Γ (the "interesting" one) will correspond to *holonomy carriers* (exceptional leaves).

6 On the applications

From now on we identify the linearised variational equation with the variational equation VE_k (abuse of notation).

We will give some indications and references about some interesting applications of our main result (Theorem 5).

A typical situation [57] is the following:

- the Riemann surface Γ is a punctured elliptic curve, the corresponding elliptic curve being denoted as $\overline{\Gamma} = \Gamma \cup \{\infty\}$;
- the extension to $\overline{\Gamma}$ of the first variational equation is regular singular.

In the classical applications there appear parametrised families of Hamiltonian systems. With the above hypotheses we get parametrised families of regular singular equations on $\overline{\Gamma}$ (the corresponding $VE = VE_1$), and in many cases we get obstructions to integrability for all the values of the parameter except for a finite or discrete subset using Morales-Ramis theory (that is the first VE). The exceptional values of the parameter correspond typically to direct sum of *Lamé-Hermite* equations. In such a situation the identity component of the differential Galois group G_1 of VE_1 is commutative (isomorphic to an additive group $\mathbf{C}^p$) and there is no obstruction.

We denote by $\wp$ the Weierstrass function corresponding to a double pole at the origin of $\mathbf{C}$ ($\wp(0) = \infty \in \overline{\Gamma}$). Then a Lamé équation is a linear ODE

$$\frac{d^2\xi}{d\,t^2} = (A\,\wp(t) + B)\xi, \tag{30}$$

The unique singular point is the origin modulo periods of the Weierstrass $\wp$ function, that is $\infty \in \overline{\Gamma}$. The equation is regular singular. The classical notation is $A = n(n+1)$.

The differential Galois group of (30) is *commutative* if and only if $n \in \mathbf{Z}$. It is the Lamé-Hermite case. In that case there exists a meromorphic solution (an elliptic solution) and the differential Galois group is triangular. Its elements are represented by unipotent triangular 2×2 matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}. \tag{31}$$

It is trivial or isomorphic to the additive group $\mathbf{C}$. Therefore it is commutative and connected.

We suppose now that the VE is a direct sum of Lamé-Hermite equations. For the sake of simplicity we can consider the case of two such equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \\ \alpha & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & \beta & 1 & \\ & & & & \ddots \end{pmatrix}. \tag{32}$$

The 2×2 unipotent matrices in the elements of G_1 give us the Galois group of each of the Lamé equations (30). It is either trivial or the additive group $\mathbf{C}$. As a consequence G_1 is a *commutative connected group.*

The variational equation VE_1 is regular singular. Therefore all the higher order variational equations VE_k (more precisely their linear counterparts LVE_k) are also regular singular (for $k \geq 1$) and their Galois groups G_k are given by the Zariski closure of their monodromy groups. Furthermore, G_k is connected because G_1 is connected. We have the following lemma.

Lemma 9 *Assume that the first order variational equation* VE_1 *decomposes in a direct sum of Lamé-Hermite type equations. Then* G_k *is commutative if and only if the solutions of* VE_k *are meromorphic functions on the covering* $\mathbf{C}$ *of* $\bar{\Gamma}$.

The proof is easy. The monodromy group of each of the variational equations VE_k is a linear representation of the fundamental group of $\Gamma = \overline{\Gamma} \setminus \{\infty\}$ (the point ∞ is represented in the Weierstrass parametrisation by the origin modulo periods) and this fundamental group is a free non-commutative group generated by two generators (the translation along the periods). The commutator of these two generators is represented by a simple loop around the singular point ∞. Hence, a monodromy group is commutative if and only if the monodromy associated to this simple loop is trivial. By Zariski closure, a differential Galois group G_k is commutative if and only if the corresponding monodromy subgroup is commutative. Therefore we can check the commutativity of G_k *locally* at ∞: recursively, by local power series expansions of the solutions (at 0 in Weierstrass parametrisation) of

VE_{k-1} and quadratures it is easy to know if the variational equation VE_k have ramified solutions with ramification around 0: we only need to check the existence of a residue different from zero, which will give rise by integration to a local logarithm in the expansion of the solution.

In [48] the authors solved completely the problem of meromorphic integrability of two-degrees of freedom Hamiltonian systems with homogeneous potentials of degree three. After some reductions the problem is restricted to the study of the 2-parameter family of potentials

$$V(x_1, x_2) = \frac{1}{3}ax_1^3 + \frac{1}{2}x_1^2 x_2 + \frac{1}{3}cx_2^3, a, b \in \mathbf{C}. \tag{33}$$

The authors made a large use of Morales-Ramis theorem (see their Introduction) about the commutativity of $(G_1)^0$ (in fact, they used a corollary of it stated in [62]). Moreover, several of the subfamilies for which no obstruction to integrability is obtained from the first order analysis are well-known integrable systems, except the 1-parametric subfamily with potential $c = 1$ and arbitrary a in (33). In order to study the meromorphic integrability of this subfamily, the authors applied the main theorem of the present paper, Theorem 5, and checked the commutativity of the identity component of the Galois groups of the higher order variational equations: if one of them is non commutative, the system is not integrable with meromorphic first integrals. It is interesting to point out that the problem of the integrability of a member of this subfamily (i.e., for $a = 0$) was considered as an open problem in the recent monograph [2] (p. 180).

They used Lamé-Hermite approach. Coming back to the Hamiltonian system with potential (33), the first order variational equations decompose in two Lamé equations with Galois group G_1 given by 4×4 matrices like (32). In [48] a residue different from zero is obtained for the integrand of a solution of the second order order variational equation if $a \neq 0$ (resp. of the third order variational equation if $a = 0$). Then $G_2 = (G_2)^0$ (resp. $G_3 = (G_3)^0$) is non-commutative and this family is non-integrable.

Along the same lines it is possible to prove that the two degrees of freedom Hamiltonian system defined by the cubic Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{3}x_1^3 + \frac{1}{2}x_1 x_2^2. \tag{34}$$

is non-integrable. This system is one of the Hénon-Heiles family of one-parameter Hamiltonians considered by Ito [34]. By means of the first order variational equation it was proved that for all except *four* of the values of the parameter, the systems in this family are non-integrable, see [34, 63, 57]. *Three* of these remaining cases are *trivially integrable.* The fourth case is (34) and its non-integrability was conjectured from numerical experiences. But all the attempts to give a rigorous proof of this fact during the last years were unsuccessful. Now we can prove the non-integrability of this last case, using our main theorem and the Lamé-Hermite approach. We get an obstruction (a non trivial residue) for the third variational equation. We give the details in Appendix B.

In this way it is possible to close the problem of integrability for the Hénon-Heiles family of Ito. This example was in fact the motivation for the Section 8.3.2 in [57] and it also was the initial motivation for the present paper, following an idea of the third author.

7 Open problems

The main theorem in this paper is a necessary condition for integrability of Hamiltonian systems by meromorphic first integrals: the identity components of all the Galois groups G_m of the higher order variational equations, $m \geq 1$, must be commutative. This gives us an *infinite number* of conditions to be satisfied. Then it is very natural to ask if this *necessary* condition for meromorphic integrability is also a *sufficient* condition. This problem was already formulated in [57], p. 146, where the following problem was stated. Assume that the identity components of the Galois groups G_k of all the variational equations VE_k of order $k \geq 1$ (or to be more precise their linear counterpart LVE_k) are commutative, then:

Is the Hamiltonian system X_H completely integrable with meromorphic first integrals in some neighbourhood of the completed integral curve represented by the Riemann surface $\overline{\Gamma}$?

We remark that, without any further additional assumption about X_H (or about Γ), the answer to this problem is *negative.*

An example of a non-integrable system with an integral curve such that all the groups G_k are commutative, is the planar three-body problem along the parabolic solutions of Lagrange, see [13, 75]. When the angular momentum of the bodies is zero, the Galois groups of all the variational equations VE_k are commutative. The fundamental group of the Riemann surface Γ defined by such a solution is commutative: Γ is the Riemann sphere with two points deleted. The variational equations VE_k are of Fuchs class and their monodromy groups, which are representations of this fundamental group, are also commutative. As the Galois groups G_k are given by the Zariski adherences of the corresponding monodromy groups, they are also commutative.

Another candidate to a counter-example is given by a particular case of a generalised family of spring-pendulum systems studied in [50]. This family depends essentially on two parameters, a, k. Using the first order variational equation, the authors proved that this family is non-integrable when $a + k \neq 0$. For $a + k = 0$ they studied the variational equations up to order seven and they did not obtain any obstruction to integrability, i.e., the identity components of all the Galois groups G_k are commutative for $k \leq 7$. However, there are numerical evidences of the non-integrability of the system. So, the problem is to understand if it is possible to impose some natural generic conditions in order to get a positive answer to our question:

Problem. Let X_H be a complex analytical Hamiltonian system defined over a complex analytical symplectic manifold and let Γ be the immersed Riemann surface defined by a particular integral curve of X_H which is not reduced to an equilibrium point.

Under which conditions is the Hamiltonian system X_H completely integrable with meromorphic first integrals in some neighbourhood of the completed integral curve represented by the Riemann surface $\overline{\Gamma}$, provided the identity components of the Galois groups G_k are commutative for arbitrary k?

Another problem concerns the *dynamical implications of non-integrability.* In systems with two degrees of freedom a typical effect of non-integrability is the existence of *transversal homoclinic orbits*. It is well known that this kind of orbits prevents from the existence of analytic first integrals in a vicinity of the orbit. See [65] for a general exposition. For early applications to the three-body problem see [40, 41, 42]. In [63] the non-integrability of some systems is illustrated by the existence of hyperbolic periodic orbits whose invariant manifolds have transversal intersection. In this example there is a strong numerical evidence that such orbits do not exist in real phase space, For this reason the search for homoclinic orbits was done in the complex phase space. In systems with a larger number of degrees of freedom the problem is even more subtle.

It seems natural to ask for the following question:

Problem. Assume a complex analytical Hamiltonian is proved to be non-integrable by the methods presented in this paper.

Is it true that some transversal homoclinic orbit to an invariant object exists?

Finally there is another kind of related problems, worth to be clarified. In the seminal work [32] not only numerical evidence of non-integrability was give, but also some *quantitative information* on the lack of integrability was given. This has a primordial relevance for physical applications. The method used was the computation of an indicator, analogous to the maximal Lyapunov exponent, such that it takes the value zero in ordered orbits and is positive on chaotic orbits. This allows to define a *fraction of integrability* on selected levels of the energy. For instance, for the Hénon-Heiles system the system can be considered as integrable for any practical purpose for energies in the range $[0, 0.05]$.

For small perturbations of an integrable system (e.g., near a totally elliptic fixed point in a general system, where the integrable approximation is a Birkhoff normal form) it is well known that the lack of integrability is exponentially small in the small parameter. See [68] for upper bounds using averaging theory and [28, 29] for upper bounds of the related splitting. Equivalent to a quantitative measure of the lack of integrability we can look for the existence of *quasi-integrals*, which are approximately preserved in the real phase space. In [33] is was proved that the J_2 problem (motion of a satellite around an oblate planet) is non-integrable. But, as it was proved in [73], this lack of integrability can be completely neglected in the case of artificial Earth satellites for any practical application (the width of the largest chaotic zone created by the non-integrability, in the real phase space, is less that 10^{-500} using the Earth's radius as unit).

On the other hand, most of the proofs of non-integrability make use of special orbits with singularities for some $t \in \mathbf{C}$. The measures of splitting use also, typically, the behaviour of invariant manifolds in a neighbourhood of these singularities. Then, the next question, which is certainly related to [79], seems relevant:

Problem. Assume some singularities are used to detect non-integrability by applying the main Theorem of this paper.

Which kind of information is also needed to produce quantitative estimates of the lack of integrability? (e.g., giving estimates of a suitable splitting, or of the measure of the domain with positive maximal Lyapunov exponent, or of the metric entropy, in the real phase space).

Appendix A: A trivialisation theorem

In order to apply differential Galois theory to the higher variational equations LVE_k, we need to replace these connections by ordinary differential systems, that is to *trivialise* our jet bundles. We can avoid this problem if we use the Tannakian approach of the differential Galois theory [21].

Let $k, n \in \mathbf{N}$. Let $E = \mathbf{C}^{2n}$. We set $Diff^k_{Sp}(2n; \mathbf{C}) = Diff^k_{Sp}(E)$. For $k = 1$, we have $Diff^1_{Sp}(2n; \mathbf{C}) = Sp(2n; \mathbf{C})$.

We recall that $Diff^1_{Sp}(2n; \mathbf{C})$ is the semi-direct product of $Sp(2n; \mathbf{C})$ and a unipotent linear algebraic group $U^k(2n; \mathbf{C}) = U^k(E)$.

Let X be a connected Riemann surface. Let G be a complex linear algebraic group. In [60], Appendix A, we defined locally trivial meromorphic bundles over X admitting G as structure group.

Proposition 13 *Let X be a complex connected, non compact Riemann surface and let $(\mathcal{F}, p, X)$ be a locally trivial holomorphic vector bundle over X having* $\mathrm{Diff}^k_{Sp}(2n; \mathbf{C})$ *as structure group. Then $\mathcal{F}$ is holomorphically trivial.*

The algebraic group $Diff^k_{Sp}(2n; \mathbf{C})$ is connected for the ordinary topology ($Sp(2n; \mathbf{C})$ and $U^k(2n; \mathbf{C})$ are connected for this topology). Then the proposition follows from a theorem of Grauert ([60], Appendix A, Theorem A.1).

Proposition 14 *Let X be a complex connected compact Riemann surface. Let $(\mathcal{F}, p, X)$ be a locally trivial holomorphic vector bundle over X with structure group* $\mathrm{Diff}^k_{Sp}(2n; \mathbf{C})$. *Then $\mathcal{F}$ is meromorphically trivial.*

We have the following result.

Proposition 15 *Let X be a compact connected Riemann surface. Let $(\mathcal{F}, p, X)$ be a locally trivial holomorphic vector bundle over X with structure group G. Then $\mathcal{F}$ is meromorphically trivial in the following cases:*

(i) $G = Sp(2n; \mathbf{C})$,

(ii) G is unipotent.

This result follows from the "GAGA" paper of J.P. Serre [71]. Claim (i) is proved in [60] Appendix A (Proposition A.2). We can prove (ii) along the same lines: due to a result of Rosenlicht: denoting by $\mathbf{G}$ the sheaf of regular maps from X to the algebraic complex group G, and by $\mathbf{G}^{an}$ the sheaf of complex analytic maps from X to the analytic complex group defined by G, the natural map

$$L : H^1(X; \mathbf{G}) \to H^1(X; \mathbf{G}^{an})$$

is a bijection when G is a unipotent group.

Then Proposition 15 follows easily from the following result.

Lemma 10 *Let X be a compact connected Riemann surface. Let*

$$\{e'\} \to G' \to G \to G'' \to \{e''\}$$

be an exact sequence of complex linear algebraic groups. We denote by $\mathbf{G}^{me}$ the sheaf of meromorphic maps from X to the algebraic complex group G. Suppose that $H^1(X; \mathbf{G}'^{me})$ and $H^1(X; \mathbf{G}''^{me})$ are trivial. Then $H^1(X; \mathbf{G}^{me})$ is also trivial.

Proof. We will use definitions and results of [30]. On the cohomology sets $H^1(X;\mathbf{G}^{an})$ we do not have in general a natural structure of group (G is in general non-commutative). However we have a natural structure of pointed set (there exists a privileged element ε). We can define exact sequences of pointed sets ([30] 5. Définition 5.1, page 153) and we have such a sequence ([30], 6. Théorème I.2, page 156):

$$H^1(X;\mathbf{G}'^{me}) \xrightarrow{i} H^1(X;\mathbf{G}^{me}) \xrightarrow{p} H^1(X;\mathbf{G}''^{me}).$$

Then

$$H^1(X;\mathbf{G}'^{me})=\{\varepsilon'\},\quad H^1(X;\mathbf{G}''^{me})=\{\varepsilon''\},\quad H^1(X;\mathbf{G}'^{me})=p^{-1}(\{\varepsilon''\})=i(\{\varepsilon'\})=\{\varepsilon\}.$$

Appendix B: Proof of non-integrability of system (34)

This special case of the Hénon-Heiles problem has only two fixed points. One of them, totally elliptic, located at the origin. The other fixed point, P_{hp}, located at $x_1=-1, x_2=y_1=y_2=0$ on the level $H=h^*=1/6$, is of hyperbolic-parabolic type. We note that the plane $x_2=y_2=0$ is invariant. On that plane and on the level $H=h^*$ the system has a separatrix, tending to P_{hp} for $t\to\pm\infty$ on the real phase space. We shall make use of this special solution. The fact that the point P_{hp} has a degeneracy is certainly related to the difficulties in proving non-integrability for this system.

Working on the real phase space, a Poincaré section through $x_2=0$ on the bounded component of the level $H=h$ for $h\in(0,h^*)$, displays only a tiny amount of chaoticity. However, for energies $h>h^*$ the chaotic domain is clearly visible. Figure 1 shows an illustration for $h=1/5$. The point marked as Q corresponds to an hyperbolic periodic orbit. It belongs to a family born at $h=h^*$.

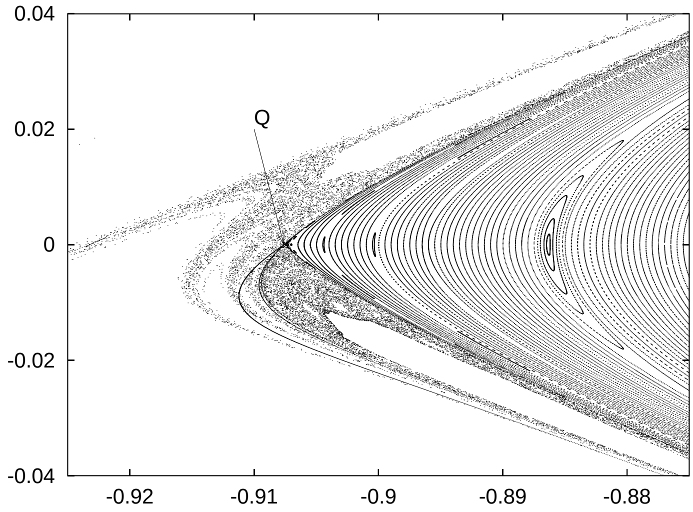

Figure 1: Part of a Poincaré section of (34) through $x_2=0$ in the energy level $h=1/5$, displayed in the (x_1,y_1) variables. Initial points are taken on $y_1=0.$ and $20,000$ iterates are computed for each initial point, keeping only the ones in the window shown.

The special solution Γ is given by

$$x_1(t)=\frac{3/2}{\cosh^2(t/2)}-1,\qquad y_1(t)=\frac{-(3/2)\sinh(t/2)}{\cosh^3(t/2)},\qquad x_2=y_2=0,\qquad (35)$$

where the origin of time has been taken on $y_1 = 0$, to have a symmetric expression. It has singularities for $\cosh(t/2) = 0$, i.e., for $t = (2k+1)\pi\,\mathrm{i}$, with $k \in \mathbf{Z}$. Our approach starts by computing the *third order monodromy*, that is, the solution of the variational equations up to order three, starting at the point $(x_1(0), y_1(0))$, along a path $\gamma \subset \Gamma$ which encloses only the singularity $t = \pi\,\mathrm{i}$ and having index 1 with respect to it. It is clear that the result is independent on the path. For concreteness we introduce the following notation (see (11))

(i) $x_3 = y_1$, $x_4 = y_2$,

(ii) The components $D_k x_j$ will be denoted as $x_{j,k}$. In a similar way, the components $D^2_{k_1,k_2} x_j, D^3_{k_1,k_2,k_3} x_j$ will be denoted as $x_{j,k_1k_2}, x_{j,k_1k_2k_3}$, respectively.

We recall that one should take $x_{j,k} = \delta_{jk}, x_{j,k_1k_2} = 0, x_{j,k_1k_2,k_3} = 0$ as initial conditions for $t = 0$. In principle, first, second and third variational equations give rise to $4^2, 4^3$ and 4^4 equations, respectively. But, from one side, these equations have the symmetries of the differential operators D^2 and D^3. From the other, due to the special solution chosen and to the simplicity of (34), it is immediate to prove the following lemma.

Lemma 11

1. *The $x_{j,k}$ with j and k of different parity are identically zero.*
2. *For x_{j,k_1k_2} and $x_{j,k_1k_2k_3}$, if the cardinal of the set of indices k which have parity different from the one of j, has the parity of j, then these elements are identically zero.*

Before starting the analytic computations it is worth to know about the expected results. To this end we have integrated numerically the required variational equations along paths γ as described before. The results show that, when returning to the initial point, the final values of the elements $x_{j,k}$ coincide again with δ_{jk}, the ones of x_{j,k_1k_2} are again zero and the ones of $x_{j,k_1k_2k_3}$ are also zero with the following exceptions:

$$x_{2,222}, \quad x_{2,224}, \quad x_{2,244}, \quad x_{2,444}, \quad x_{4,224}, \quad x_{4,244}, \quad x_{4,444}.$$

Taking a different initial point one can also have $x_{4,222} \neq 0$ after closing the loop. For our purposes it is enough to show that some of the final elements $x_{j,k_1k_2k_3}$ is different from 0. We select the $x_{2,222}$, whose numerically computed value is $\approx 90.477868423386\,\mathrm{i}$.

Let $a_{31} = -1 - 2x_1$, $a_{42} = -1 - x_1$ the only non-zero and non-trivial elements in DX along Γ. To obtain $x_{2,222}$ we only need to integrate the following systems

$$\begin{pmatrix} \dot{x}_{2,2} \\ \dot{x}_{4,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{42} & 0 \end{pmatrix} \begin{pmatrix} x_{2,2} \\ x_{4,2} \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_{1,22} \\ \dot{x}_{3,22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{31} & 0 \end{pmatrix} \begin{pmatrix} x_{1,22} \\ x_{3,22} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{2,2}^2 \end{pmatrix} \tag{36}$$

and

$$\begin{pmatrix} \dot{x}_{2,222} \\ \dot{x}_{4,222} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{42} & 0 \end{pmatrix} \begin{pmatrix} x_{2,222} \\ x_{4,222} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{2,2}x_{1,22} \end{pmatrix}. \tag{37}$$

It is clear that to integrate (36) and (37) we need to solve the first order variational equations, both tangential (in the (x_1, x_3) variables) and normal (in (x_2, x_4)), which are uncoupled. The solutions can be written explicitly. To shorten the notation we introduce

$c := \cosh(t/2)$ and $s := \sinh(t/2)$. Then

$$
\begin{aligned}
x_{1,1} &= -\frac{15ts}{16c^3}+\frac{15}{8c^2}-\frac{5}{8}-\frac{c^2}{4}, & x_{1,3} &= -\frac{4}{3}y_1,\\
x_{3,1} &= -\frac{15t(3-2c^2)}{32c^4}-\frac{45s}{16c^3}-\frac{sc}{4}, & x_{3,3} &= \frac{4}{3}(x_1+x_1^2),\\
x_{2,2} &= 2x_1, & x_{2,4} &= \frac{tx_1}{2}+\frac{3s}{2c},\\
x_{4,2} &= 2y_1, & x_{4,4} &= \frac{x_1}{2}+\frac{ty_1}{2}+\frac{3}{4c^2}.
\end{aligned}
\tag{38}
$$

Furthermore

$$
x_{1,22} = x_{1,1}\left(\frac{2}{8}-\frac{16}{9}x_1^3\right)+x_{1,3}K(t),
\tag{39}
$$

where

$$
K(t) = t\left(-\frac{45}{16c^6}+\frac{45}{8c^4}-\frac{15}{4c^2}\right)+s\left(-\frac{45}{8c^5}+\frac{15}{2c^3}-\frac{3}{c}+c\right).
$$

We remark that one of the columns of the fundamental matrix of the normal variational equations coincides (except by a factor of 2) with (35). This is true for any h because (x_1, y_1) are solutions of the first equation in (36).

Having (38) and (39) we are ready to solve (37). As the homogeneous part coincides with the first order normal variational equation, the solution, after closing the loop is given by

$$
\begin{pmatrix} x_{2,222} \\ x_{4,222} \end{pmatrix} = \begin{pmatrix} x_{2,2} & x_{2,4} \\ x_{4,2} & x_{4,4} \end{pmatrix} \int_\gamma \begin{pmatrix} -x_{2,4}R \\ x_{2,2}R \end{pmatrix}, \quad \text{where} \quad R(t) = -3x_{2,2}x_{1,22}.
\tag{40}
$$

It is readily checked that the residues inside the integral are $72/5$ and 0, respectively. Hence, the final value of $x_{2,222}$ after the loop is $\frac{72}{5}2\pi\,\mathrm{i}$, which coincides with the value given above in all the digits shown. This computation also explains why the final value of $x_{4,222}$ is zero.

Now we are ready to prove the desired non-integrability of (34). We remark that in all the computations of order less than three no residue appears.

Proposition 16 *The system* (34) *in non-integrable in a vicinity of the solution given by* (35).

Proof. One can take as Γ a solution on the invariant plane $x_2 = y_2 = 0$ and an energy level $h < h^*$ close to h^*. Then the solution is given by elliptic functions with a parallelogram of periods and a double pole. When $h \to h^*$ the periods tend to $\pi\,\mathrm{i}$ and ∞. Take paths γ_1 and γ_2 along the generators. To compute the commutator it is enough to carry out the integration along a path γ of index 1 around the pole.

As all the solutions are obtained by quadratures, we have $(G_k)^0 = G_k$ for all k. Hence, commutativity of $(G_k)^0$ implies commutativity at the (kth order) monodromy level.

The integrals along γ are continuous as functions of h. As for $h = h^*$ there are integrals different from zero, the same happens for nearby values $h < h^*$. This implies $(G_3)^0$ non-commutative. □

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