A Nonperturbative Eliasson's Reducibility Theorem

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Abstract

This paper is concerned with discrete, one-dimensional Schrödinger operators with real analytic potentials and one Diophantine frequency. Using localization and duality we show that almost every point in the spectrum admits a quasi-periodic Bloch wave if the potential is smaller than a certain constant which does not depend on the precise Diophantine conditions. The associated first-order system, a quasi-periodic skew-product, is shown to be reducible for almost all values of the energy. This is a partial nonperturbative generalization of a reducibility theorem by Eliasson. We also extend nonperturbatively the genericity of Cantor spectrum for these Schrödinger operators. Finally we prove that in our setting, Cantor spectrum implies the existence of a G_δ-set of energies whose Schrödinger cocycle is not reducible to constant coefficients.

Keywords: Quasi-periodic Schrödinger operators, Harper-like equations, reducibility, Floquet theory, quasi-periodic cocycles, skew-product, Cantor spectrum, localization, irreducibility, Bloch waves.

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Contents

1 Introduction. Main results

Recently there has been substantial advance in the theory of quasi-periodic Schrödinger operators, both continuous and discrete, combining spectral and dynamical techniques. These operators arise naturally in many areas of physics and mathematics. They appear in the study of electronic properties of solids [AA80, Jan92, OA01], in the theory of KdV and related equations [JM82, Joh88, Chu89] or in Hamiltonian mechanics [Sin85]. Moreover, their eigenvalue equations are second order differential or difference linear equations with quasi-periodic coefficients like Hill's equation with quasi-periodic forcing [BPS03] or the Harper equation [KS97, DJKR04] which display a rich variety of dynamics ranging from quasi-periodicity to uniform and nonuniform hyperbolicity.

In this paper we pursue this fruitful combination of spectral and dynamical methods to study discrete, one-dimensional Schrödinger operators $H_{V,\omega,\phi}$,

$$(H_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z}, \tag{1}$$

where $V : \mathbb{T} \to \mathbb{R}$ is a real analytic function (the *potential*), $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is a *phase* and ω a *Diophantine frequency*. The latter means that there exist positive constants c and $\tau > 1$ such that the bound

$$|\sin 2\pi k\omega| > \frac{c}{|k|^{\tau}} \tag{2}$$

holds for any integer $k \neq 0$. This condition will be written as $\omega \in DC(c,\tau)$. In particular, this means that ω is *nonresonant*,that is

$$|\sin 2\pi k\omega| \neq 0$$

unless $k = 0$.

The best-studied example of quasi-periodic Schrödinger operator is the *Almost Mathieu operator* where $V(\theta) = b\cos\theta$, being b a real *coupling* parameter. A vast amount of literature is devoted to the study of the spectral properties of this operator (see Simon [Sim82, Sim00], Jitomirskaya [Jit95, Jit02] and Last [Las95] for surveys and references). In [Pui04a, Pui04b] (see also Avila & Jitomirskaya [AJ05] for the recent extension to the remaining frequencies) the Cantor structure of the spectrum of the Almost Mathieu operator was derived from the use of a localization result by Jitormirskaya [Jit99] and a dynamical analysis of its eigenvalue equation, the so-called *Harper equation.* This was a long-standing conjecture known as the "Ten Martini Problem". The combined approach there is not limited to the Almost Mathieu operator as we plan to make evident in this paper.

The eigenvalue equation of a quasi-periodic Schrödinger operator $H_{V,\omega,\phi}$ is the following Harper-like equation

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z}, \tag{3}$$

where $a \in \mathbb{R}$ is called the *energy* or *spectral parameter.* Since we want to study dynamical properties of this equation, it is better to transform it into a first-order system, obtaining the associated *quasi-periodic skew-product* on $\mathbb{R}^2 \times \mathbb{T}$,

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{4}$$

which can be seen as an iteration of the corresponding *Schrödinger cocycle*, $(A_{a,V}, \omega)$, on $SL(2, \mathbb{R}) \times \mathbb{T}$. Here $A_{a,V}$ denotes the matrix-valued function

$$A_{a,V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \qquad \theta \in \mathbb{T}. \tag{5}$$

The simplest class of Schrödinger cocycles occurs when $V = 0$ because in this case $A_{a,V}$ does not depend on θ (we will say that the corresponding cocycle is in *constant coefficients*). In analogy with periodic differential equations we may try to reduce a quasi-periodic Schrödinger cocycle to constant coefficients. Let us introduce first the notion of conjugation between cocycles, not necessarily of Schrödinger type. Here we restrict ourselves to the case of $SL(2, \mathbb{R})$-valued cocycles, although the notion of reducibility applies to more general cocycles (see [Pui04b] for an exposition). Two cocycles (A, ω) and (B, ω) on $SL(2, \mathbb{R}) \times \mathbb{T}$ are *conjugated* if there exists a continuous and non-singular *conjugation* $Z : \mathbb{T} \to SL(2, \mathbb{R})$ such that the relation

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B(\theta), \qquad \theta \in \mathbb{T}.$$

holds for all $\theta \in \mathbb{T}$. In this case the corresponding quasi-periodic skew-products

$$u_{n+1} = A(\theta_n)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

and

$$v_{n+1} = B(\theta_n)v_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

are conjugated through the change of variables $u = Zv$, so that they share the same dynamical properties.

A cocycle (A, ω) is *reducible to constant coefficients* if it is conjugated to a cocycle (B, ω) with B not depending on θ (i.e. with constant coefficients). In this case, B is called a *Floquet* matrix. Sometimes it may be necessary to "halve the frequency" if we do not want to complexify the system (although this case will not be treated in this paper). In contrast with the situation in the periodic case (when the nonresonance condition fails), quasi-periodic cocycles need not to be reducible to constant coefficients (see Theorem 5 and the following Remark 7).

The reducibility of a Schrödinger cocycle and the eigenvalues of the reduced Floquet matrix have implications for the spectrum of the corresponding Schrödinger operator. The spectrum of $H_{V,\omega,\phi}$ on $l^2(\mathbb{Z})$ is a compact subset of the real line which we denote by $\sigma(V, \omega)$ since it does not depend on ϕ. It is known that an energy a lies in the spectrum of a Schrödinger operator if, and only if, the corresponding skew-product has an *exponential dichotomy* (it is *uniformly hyperbolic*), see Johnson [Joh82]. Under our assumptions, V real analytic and ω Diophantine, this is equivalent to the reducibility of the Schrödinger cocycle to constant coefficients with a hyperbolic Floquet matrix (all its eigenvalues are outside the unit circle), see Johnson [Joh80].

For energies in the spectrum, the situation is much more involved. However, when in addition to the present hypothesis, the potential V is small (in some complex neighbourhood around $\mathbb{T}$ which depends on c and τ), then reducibility can be obtained by KAM methods for a set of energies in the spectrum of large measure, see Dinaburg & Sinai [DS75] and Moser & Pöschel [MP84].

A breakthrough in the KAM approach came with Eliasson [Eli92] who proved, among other statements, that reducibility to constant coefficients holds for almost every energy provided the potential is small enough and ω is Diophantine, $\omega \in DC(c, \tau)$ for some c and τ (see Section 2.3 for a more precise formulation. Like the results in the previous paragraph, the smallness condition here depends on the precise Diophantine conditions on ω. Eliasson's result, as well as the above KAM results presented above, holds for real analytic or C^∞ potentials $V : \mathbb{T} \to \mathbb{R}$ depending on several frequencies.

Our main result states that in the presence of only one frequency, $d = 1$ in the notation above, the smallness condition in Eliasson's theorem does not depend on the constants c and τ of the Diophantine condition as long as ω is Diophantine (that is, it is "nonperturbative" in some sense). To be more precise, we consider real analytic potentials $V : \mathbb{T} \to \mathbb{R}$ having an analytic extension to $|\mathrm{Im}\ \theta| < \rho$, for some $\rho > 0$ such that

$$|V|_\rho := \sup_{|\mathrm{Im}\ \theta|<\rho} |V(\theta)| < \infty,$$

(the set of such potentials will be denoted by $C^a_\rho(\mathbb{T}, \mathbb{R})$) and a Diophantine frequency $\omega \in DC(c, \tau)$ for some positive constants c and $\tau > 1$. Our extension of Eliasson's theorem reads as follows.

Theorem 1 *Let $\rho > 0$ be a positive number. Then, there is a constant $\varepsilon_0 = \varepsilon_0(\rho)$ such that, for any real analytic $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ with*

$$|V|_\rho < \varepsilon_0,$$

the Schrödinger cocycle $(A_{a,V}, \omega)$ is reducible to constant coefficients for every Diophantine frequency ω and almost all $a \in \mathbb{R}$ (with respect to Lebesgue measure).

The proof of this Theorem will be given in Section 3.

Remarks 2

1. *Recently Avila & Krikorian* [AK03] *proved Theorem 1 with more restrictive hypothesis on ω (although it is also a full measure condition). In fact, we will see that both results follow from a non-perturbative theorem on localization by Bourgain & Jitomirskaya* [BJ02a].
2. *When the potential is defined on a d-dimensional torus, $V : \mathbb{T}^d \to \mathbb{R}$, Eliasson's theorem holds, but a nonperturbative version like Theorem 1 cannot be true, as Bourgain showed in* [Bou02]. *Indeed, he proved that, if $V : \mathbb{T}^2 \to \mathbb{R}$ is a trigonometric polynomial with a nondegenerate maximum, there is a set of frequencies $\omega \in \mathbb{R}^2$, with positive Lebesgue measure, for which the operators $H_{V,\omega,\phi}$ have some point spectrum. This point spectrum is incompatible with reducibility to constant coefficients. See Bourgain* [Bou04b, Bou04a] *for the differences between the cases of one and several frequencies.*
3. *Let us stress that Theorem 1 is not a full nonperturbative version of Eliasson's theorem because the set of energies whose corresponding Schrödinger cocycle is reducible to constant coefficients is not explicitly characterized as it is in Eliasson's theorem (see Section 2.3).*
4. *The smallness condition in Theorem 1, $\varepsilon_0 = \varepsilon_0(\rho)$ is given by the localization result in* [BJ02a] *and can be explicitly given in terms of $\|V\|_1$, $\|V\|_2$, $\|V\|_\infty$ and ρ. If V is kept fixed, then $\varepsilon_0(\rho) = O(\rho)$ as $\rho \to 0$.*

Theorem 1 has some implications for the spectral properties of Schrödinger operators. The first one refers to the properties of solutions of the eigenvalue equation. An immediate application of Theorem 1 is the existence of analytic quasi-periodic *Bloch waves* for almost all a in the spectrum. An analytic quasi-periodic Bloch wave for a Harper-like equation (3) is a solution of the form

$$x_n(\phi) = e^{\mathrm{i}\varphi n} f\left(2\pi\omega n + \phi\right), \qquad n \in \mathbb{Z}, \tag{6}$$

where $\varphi \in [0, 2\pi)$ is called the *Floquet exponent* and $f : \mathbb{T} \to \mathbb{C}$ is a nontrivial analytic function. In Section 3 we will also prove the following about the existence of quasi-periodic Bloch waves.

Corollary 3 *Let ρ, ε_0 V and ω be as in Theorem 1. Then for (Lebesgue) almost all values of a in the spectrum $\sigma(V, \omega)$, the equation*

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n,$$

has analytic quasi-periodic Bloch waves.

Our second application deals with the structure of the spectrum of quasi-periodic Schrödinger operators. The Cantor structure of the spectrum is not specific of the Almost Mathieu operator. Indeed, although the proof of the "Ten Martini Problem" [Pui04a] is restricted to this model, Cantor spectrum is "generic" in our setting. More precisely, if we consider the set $C^a_\rho(\mathbb{T}, \mathbb{R})$ of real analytic functions furnished with the $|\cdot|_\rho$ norm, one has the following.

Theorem 4 *Let $\rho > 0$. Then there is a constant $\varepsilon = \varepsilon(\rho)$ such that for every Diophantine ω there is a generic set of real analytic potentials $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ with $|V|_\rho < \varepsilon$ such that the spectrum of the Schrödinger operator $H_{V,\omega,\phi}$ is a Cantor set.*

Our last application is concerned with the existence of Schrödinger cocycles which are not reducible to constant coefficients. We will see that Cantor spectrum and nonreducibility are related concepts in our setting.

Theorem 5 *Let $\rho > 0$. Then there is a constant $\varepsilon = \varepsilon(\rho)$ such that if ω is Diophantine and $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$, with $|V|_\rho < \varepsilon$, is such that $\sigma(V, \omega)$ is a Cantor set then for a G_δ-dense subset of energies in the spectrum the corresponding Schrödinger cocycle is not reducible to constant coefficients (by a continuous transformation).*

Remark 6 *According to Theorem 1, the G_δ-set above has zero Lebesgue measure.*

In particular, using Theorem 4 it is possible to give a nonperturbative version of a result in Eliasson [Eli92], namely, that the existence of a G_δ-subset of “nonreducible energies” is a generic property.

Remark 7 *This kind of nonreducibility holds for a zero measure subset of energies and the corresponding Schrödinger cocycle has zero Lyapunov exponent. Nonreducibility results can be obtained using, for instance, a result by Sorets & Spencer* [SS91]*, who prove that if the potential V is large enough then the Lyapunov exponent is positive for all energies in the spectrum and this prevents reducibility.*

Let us finally outline the contents of this paper. In Section 2 we introduce some of the preliminaries needed for the proof of the main theorem. In Section 3 this is used to prove 1 and 3 using a similar technique to the one used for the Almost Mathieu operator. The applications are included in Section 4.

2 Preliminaries

In this section we present some of the tools that will be needed in the proof of Theorem 1. As said in the introduction, we plan to extend some of the ideas in the proof of the “Ten Martini Problem” given in [Pui04a] with the aid of a nonperturbative localization result for long-range potentials by Bourgain & Jitomirskaya [BJ02b]. In Section 2.1 we introduce a convenient version of Aubry duality, which will lead us to consider certain long-range operators which are not of Schrödinger type. In Section 2.2 we give the definition and some properties of the integrated density of states IDS for these operators and its relation with Aubry duality. Finally, in Section 2.3 this IDS is linked to the fibered rotation number of quasi-periodic cocycles to give a more precise version of Eliasson’s result.

2.1 Aubry Duality

Aubry Duality [AA80] was originally introduced for the study of the Almost Mathieu operator but the same idea (which is Fourier transform) works for other potentials. Let us give first the heuristic approach and then a more rigorous one.

Assume that

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = a x_n,$$

has an analytic quasi-periodic Bloch wave,

$$x_n = e^{\mathrm{i}\varphi n}\tilde{\psi}\left(2\pi\omega n + \phi\right), \qquad (7)$$

being $\tilde{\psi} : \mathbb{T} \to \mathbb{C}$ analytic and $\varphi \in [0, 2\pi)$ the Floquet exponent. If $(\psi_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of $\tilde{\psi}$, a computation shows that they satisfy the following difference equation

$$\sum_{k\in\mathbb{Z}} V_k \psi_{n-k} + 2\cos(2\pi\omega n + \varphi)\,\psi_n = a\psi_n \qquad n \in \mathbb{Z},$$

where $(V_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of V,

$$V(\theta) = \sum_{k\in\mathbb{Z}} V_k e^{\mathrm{i}k\theta}.$$

This difference equation is the eigenvalue equation of the operator $L_{V,\omega,\varphi}$

$$(L_{V,\omega,\varphi}\psi)_n = \sum_{k\in\mathbb{Z}} V_k \psi_{n-k} + 2\cos(2\pi\omega n + \varphi)\,\psi_n$$

which we call a *dual operator* of $H_{V,\omega,\phi}$. This is a self-adjoint and bounded operator on $l^2(\mathbb{Z})$ (because V is real analytic) but it is not a Schrödinger operator unless V is exactly the cosine (this is what makes the Almost Mathieu operator so special). Such an operator will be called a *long-range (quasi-periodic) operator* even if it may be a finite-differences operator (if V is a trigonometric polynomial).

If ω is nonresonant, the spectrum of the long-range operators $L_{V,\omega,\varphi}$ does not depend on the chosen φ, so that one can write

$$\sigma^L(V,\omega) = \mathrm{Spec}\,(L_{V,\omega,\varphi})\,.$$

This naive approach to Aubry duality shows that whenever a is a value in the spectrum $\sigma^H(V,\omega)$ such that $(x_n)_{n\in\mathbb{Z}}$ is an analytic quasi-periodic Bloch wave with Floquet exponent φ, then a is a point eigenvalue of the dual operator $L_{V,\omega,\varphi}$ whose eigenvector decays exponentially and, thus, $a \in \sigma^L(V,\omega)$. The converse is also true: one can pass from exponentially decaying eigenvalues of $L_{V,\omega,\varphi}$ to quasi-periodic Bloch waves of $H_{V,\omega,\phi}$ with Floquet exponent φ.

The argument given above heavily relies on the existence of quasi-periodic Bloch waves or, equivalently, exponentially localized eigenvectors. Nevertheless both operators can be related without the assumption of such point eigenvalues. This was done by Avron & Simon [AS83]. Here we will follow the idea by Gordon, Jitomirskaya, Last & Simon [GJLS97] (see also Chulaevsky & Delyon [CD89]), who studied duality for the Almost Mathieu operator, although it can be extended to the general case, see Bourgain & Jitomirskaya [BJ02b]. The idea is to shift to more general spaces where the extensions of quasi-periodic Schrödinger operators and their duals are unitarily equivalent. Note that it is not true that the operators $H_{V,\omega,\phi}$ and $L_{V,\omega,\varphi}$ are unitarily equivalent, since their spectral measures will, in general, be very different.

Let us consider the following Hilbert space,

$$\mathcal{H} = L^2 (\mathbb{T} \times \mathbb{Z}),$$

which consists of functions $\Psi = \Psi(\theta, n)$ satisfying

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{T}} |\Psi(\theta, n)|^2 \, d\theta < \infty.$$

The extensions of the Schrödinger operators H and their long-range duals L to $\mathcal{H}$ are given in terms of their *direct integrals*, which we now define. The *direct integral* of the Schrödinger operator $H_{V,\omega,\phi}$, is the operator $\tilde{H}_{V,\omega}$, defined as

$$\left(\tilde{H}_{V,\omega}\Psi\right)(\theta, n) = \Psi(\theta, n+1) + \Psi(\theta, n-1) + V(2\pi\omega n + \theta)\Psi(\theta, n),$$

and the direct integral of $L_{V,\omega,\varphi}$, denoted as $\tilde{L}_{V,\omega}$, is

$$\left(\tilde{L}_{V,\omega}\Psi\right)(\theta, n) = \sum_{k\in\mathbb{Z}} V_k \Psi(\theta, n-k) + 2\cos\left(2\pi\omega n + \theta\right)\Psi(\theta, n).$$

These two operators are bounded and self-adjoint in $\mathcal{H}$. Let us now see that, for any fixed real analytic V and nonresonant frequency ω, the direct integrals $\tilde{H}_{V,\omega}$ and $\tilde{L}_{V,\omega}$ are unitarily equivalent; i.e. there exists a unitary operator U on $\mathcal{H}$ such that the conjugation

$$\tilde{H}_{V,\omega} U = U \tilde{L}_{V,\omega}$$

holds. By analogy with the heuristic approach to Aubry duality in the beginning of this section, let U be the following operator on $\mathcal{H}$,

$$(U\Psi)(\theta, n) = \hat{\Psi}\left(n, \theta + 2\pi\omega n\right),$$

where $\hat{\Psi}$ is the Fourier transform. At a formal level this acts as

$$(U\Psi)(\theta, n) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \Psi(\phi, k) e^{-\mathrm{i}n\phi} e^{-\mathrm{i}(\theta + 2\pi\omega n)k} d\theta$$

if we disregard the convergence of the sum in k. The map U is unitary and satisfies

$$\tilde{H}_{V,\omega} U = U \tilde{L}_{V,\omega}$$

by construction of the dual long-range operators in terms of the Schrödinger operators. Therefore, the direct integrals $\tilde{H}_{V,\omega}$ and $\tilde{L}_{V,\omega}$ are unitarily equivalent and, in particular, their spectra are the same,

$$\sigma^H(V,\omega) = \bigcup_{\phi\in\mathbb{T}} \mathrm{Spec}\,(H_{V,\omega,\phi}) = \mathrm{Spec}\left(\tilde{H}_{V,\omega}\right) = \mathrm{Spec}\left(\tilde{L}_{V,\omega}\right) = \bigcup_{\varphi\in\mathbb{T}} \mathrm{Spec}\,(H_{V,\omega,\varphi}) = \sigma^L(V,\omega).$$

Hence, the spectrum of a quasi-periodic Schrödinger operator and its dual are the same. In the next section we will introduce the integrated density of states for Schrödinger operators (and their long-range duals) and we will see that this function is preserved by Aubry duality.

2.2 The integrated density of states and duality

The integrated density of states, IDS for short, is a very useful object for the description of the spectrum of quasi-periodic Schrödinger operators and more general quasi-periodic self-adjoint operators. Here we want to introduce it both for quasi-periodic Schrödinger operators and their long-range duals. In order to give a unified approach, let us consider a more general class of operators.

If $V, W : \mathbb{T} \to \mathbb{R}$ are real analytic functions, $(V_k)_k$ and $(W_k)_k$ are their Fourier coefficients, ω is a nonresonant frequency and $\phi \in \mathbb{T}$, let $K_{W,V,\omega,\phi}$ be the following operator

$$(K_{W,V,\omega,\phi}x)_n = \sum_{k\in\mathbb{Z}} W_k x_{n-k} + V(2\pi\omega n + \phi)x_n$$

acting on $l^2(\mathbb{Z})$, which is bounded and self-adjoint. The operators in the previous section occur as particular cases,

$$H_{V,\omega,\phi} = K_{2\cos,V,\omega,\phi} \quad \text{and} \quad L_{V,\omega,\phi} = K_{V,2\cos,\omega,\phi}.$$

Let us now define the IDS for the operators $K_{W,V,\omega,\phi}$. Take some integer $N > 0$ and consider $K^N_{W,V,\omega,\phi}$, the restriction of the operator $K_{W,V,\omega,\phi}$ to the interval $[-N,N]$ with zero boundary conditions. Let

$$k^N_{W,V,\omega,\phi}(a) = \frac{1}{2N+1}\#\left\{\text{eigenvalues} \le a \text{ of } K^N_{W,V,\omega,\phi}\right\}.$$

Then, due to the nonresonant character of ω, the limit

$$\lim_{N\to\infty} k^N_{W,V,\omega,\phi}(a)$$

exists, it is independent of ϕ and of the boundary conditions imposed above. It is called the *integrated density of states* of the operator $K_{W,V,\omega,\phi}$. We will write this as $k_{a,W,V,\omega}(a)$. The map

$$a \in \mathbb{R} \mapsto k_{a,W,V,\omega}(a) \tag{8}$$

is increasing and it is constant exactly at the open intervals in the resolvent set of the spectrum of $K_{W,V,\omega,\phi}$. It is the distribution function of a Borel measure $n_{W,V,\omega}$,

$$k_{a,W,V,\omega}(a) = \int_{-\infty}^{a} dn_{W,V,\omega}(\lambda)$$

called is the *density of states* of the operator $K_{W,V,\omega,\phi}$, which is supported on the spectrum of $K_{W,V,\omega,\phi}$. In the Schrödinger case we will use the notations

$$k^H_{V,\omega}(a) = k_{2\cos,V,\omega}(a), \quad n^H_{V,\omega} = n_{2\cos,V,\omega}$$

and

$$k^L_{V,\omega}(a) = k_{V,2\cos,\omega}(a), \quad n^L_{V,\omega} = n_{V,2\cos,\omega}.$$

for their long-range duals.

The IDS of the operators $K_{W,V,\omega,\phi}$ can be seen as an average in ϕ of the spectral measures of the operators (see Avron & Simon [AS83]). By the spectral theorem we know that there is a Borel measure μ_ϕ such that

$$\langle \delta_0, f\left(K_{W,V,\omega,\phi}\right)\delta_0\rangle_{l^2(\mathbb{Z})} = \int f(\lambda) d\mu_\phi(\lambda) \tag{9}$$

for every continuous function f, being δ_0 the delta function. The measures μ_ϕ are spectral measures in the sense that the spectral projection of $K_{W,V,\omega,\phi}$ over a certain subset A of the spectrum is zero if, and only if, $\mu_\phi(A) = 0$. Avron & Simon prove that, for any continuous function f

$$\int f(\lambda) dn^K_{W,V,\omega}(\lambda) = \int_{\mathbb{T}} d\phi \int f(\lambda) d\mu_\phi.$$

An approximation argument shows that, for any Borel subset of the spectrum, $A \subset \sigma(K_{W,V,\omega})$,

$$n^K_{W,V,\omega}(A) = \int_{\mathbb{T}} \mu_\phi(A) d\phi.$$

In particular, $n^K_{W,V,\omega}(A) = 0$ if $\mu_\phi(A) = 0$ for Lebesgue almost every $\phi \in \mathbb{T}$. Using this characterization of the IDS one can prove the following adaption of the duality of the IDS given in [GJLS97].

Theorem 8 ([GJLS97]) *Let $k^L_{V,\omega}$ and $k^H_{V,\omega}$ be the integrated density of states of $H_{V,\omega,\phi}$ and $L_{V,\omega,\varphi}$ respectively, for some real analytic $V : \mathbb{T} \to \mathbb{R}$ and nonresonant frequency ω. Then*

$$k^L_{V,\omega}(a) = k^H_{V,\omega}(a)$$

for all $a \in \mathbb{R}$.

Proof: Let

$$g(\theta, n) = \delta_{n,0}$$

which belongs to $\mathcal{H}$. Then $Ug = g$. Moreover by (9) and the unitary equivalence between $\tilde{H}_{V,\omega}$ and $\tilde{L}_{V,\omega}$ we have that, for any continuous f,

$$\langle g, f\left(\tilde{H}_{V,\omega}\right) g\rangle_{\mathcal{H}} = \langle Ug, Uf\left(\tilde{H}_{V,\omega}\right) g\rangle_{\mathcal{H}} = \\ \langle Ug, Uf\left(\tilde{H}_{V,\omega}\right) U^{-1}Ug\rangle_{\mathcal{H}} = \langle g, f\left(\tilde{L}_{V,\omega}\right) g\rangle_{\mathcal{H}}.$$

Therefore, since $n^H_{V,\omega}$ and $n^L_{V,\omega}$ are the Borel measures such that

$$\langle g, f\left(\tilde{H}_{V,\omega}\right) g\rangle_{\mathcal{H}} = \int f(\lambda) dn^H_{V,\omega}(\lambda)$$

and

$$\langle g, f\left(\tilde{L}_{V,\omega}\right) g\rangle_{\mathcal{H}} = \int f(\lambda) dn^L_{V,\omega}(\lambda)$$

for every continuous f the two measures must coincide (and also their distribution functions, $k^L_{V,\omega}$ and $k^H_{V,\omega}$). □

Let us end this section summing up some facts useful in the sequel.

Proposition 9 *Let V be real analytic, ω nonresonant and μ_ϕ a spectral measure of $L_{V,\omega,\phi}$. Assume that there is a measurable set A such that*

$$\mu_\phi(A) = 0$$

for almost every $\phi \in \mathbb{T}$. Then $n^L_{V,\omega}(A) = 0$ and $n^H_{V,\omega}(A) = 0$.

2.3 The rotation number and Eliasson's theorem revisited

We have seen in the previous section that it is possible to assign an IDS for quasi-periodic Schrödinger cocycles using its associated operator. Here we will see that it is possible to define an extension of this object, the fibered rotation number, for more general quasi-periodic cocycles. This object, introduced originally by Herman [Her83] in this discrete case (see also Johnson & Moser [JM82], Delyon & Souillard [DS83b]), allows us to give a version of Eliasson's theorem for these cocycles. Let us follow the presentation by Krikorian [Kri].

Let (A, ω) be a quasi-periodic cocycle on $SL(2, \mathbb{R}) \times \mathbb{T}$ which is *homotopic to the identity.* That is $A : \mathbb{T} \to SL(2, \mathbb{R})$ is a continuous map (although we will later assume that it is real analytic) that is homotopic to the identity (for example a Schrödinger cocycle). The fibered rotation number, which we now introduce measures how solutions wind around the origin in $\mathbb{R}^2$ in average.

Let $\mathbb{S}^1$ be the set of unit vectors of $\mathbb{R}^2$ and let us denote by $p : \mathbb{R} \to \mathbb{S}^1$ the projection given by the exponential $p(t) = e^{\mathrm{i}t}$, identifying $\mathbb{R}^2$ with $\mathbb{C}$. Because of the linear character of the cocycle and the fact that it is homotopic to the identity, the continuous map

$$\begin{array}{rccc} F: & \mathbb{S}^1 \times \mathbb{T} & \longrightarrow & \mathbb{S}^1 \times \mathbb{T} \\ & (v, \theta) & \mapsto & \left(\frac{A(\theta)v}{\|A(\theta)v\|}, \theta + 2\pi\omega \right) \end{array} \quad (10)$$

is also homotopic to the identity. Therefore, it admits a continuous lift $\tilde{F} : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ of the form:

$$\tilde{F}(t, \theta) = (t + f(\theta, t), \theta + 2\pi\omega)$$

such that

$$f(t + 2\pi, \theta + 2\pi) = f(t, \theta) \text{ and } p\,(t + f(t, \theta)) = \frac{A(\theta)p(t)}{\|A(\theta)p(t)\|}$$

for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}$. The map f is independent of the choice of $\tilde{F}$ up to the addition of a constant $2\pi k$, with $k \in \mathbb{Z}$. Since the iteration $\theta \mapsto \theta + 2\pi\omega$ is uniquely ergodic on $\mathbb{T}$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{T}$, one has that the limit

$$\lim_{N \to \infty} \frac{1}{2\pi N} \sum_{n=0}^{N-1} f\left(\tilde{F}^n(t, \theta)\right)$$

exists modulus $\mathbb{Z}$ and it is independent of (t, θ), see Herman [Her83]. This object is called the *fibered rotation number* of (A, ω), and it will be denoted by $\mathrm{rot}_f(A, \omega)$. The fibered rotation number of a Harper-like equation is defined as the fibered rotation number of the associated Schrödinger cocycle on $SL(2, \mathbb{R}) \times \mathbb{T}$ and will be denoted as $\mathrm{rot}_f(a, V, \omega)$.

The rotation number of a Harper-like equation can be linked to its IDS . Indeed, using a suspension argument (see Johnson [Joh83]) it can be seen that

$$\mathrm{rot}_f(a, V, \omega) = \frac{1}{2} k_{V,\omega}(a) \qquad (\text{mod. } \mathbb{Z}).$$

The rotation number is not invariant under conjugation, but one has the following.

Proposition 10 (cf. [Kri]**)** *Let ω be nonresonant and (A_1, ω) and (A_2, ω) be two quasi-periodic cocycles on $SL(2,\mathbb{R})\times\mathbb{T}$ homotopic to the identity. If they are conjugated for some continuous $Z : \mathbb{T} \to SL(2,\mathbb{R})$, then*

$$\mathrm{rot}_f(A_1, \omega) = \mathrm{rot}_f(A_2, \omega) + \langle \mathbf{k}, \omega\rangle \text{ modulus } \mathbb{Z},$$

where $\mathbf{k} \in \mathbb{Z}$ is the degree of the map $Z : \mathbb{T} \to SL(2,\mathbb{R})$. If the conjugation Z is not defined on $\mathbb{T}$ but on $(\mathbb{R}/(4\pi\mathbb{Z}))$ and it has degree $\mathbf{k} \in \mathbb{T}$, then

$$\mathrm{rot}_f(A_1, \omega) = \mathrm{rot}_f(A_2, \omega) + \frac{1}{2}\langle \mathbf{k}, \omega\rangle.$$

Keeping in mind this result, we can define two classes of rotation numbers which are preserved under conjugation. An important class is that of resonant rotation numbers. A number of the form

$$\alpha = \frac{1}{2}\langle \mathbf{k}, \omega\rangle, \qquad (\mathrm{mod}\ \frac{1}{2}\mathbb{Z})$$

for some $k \in \mathbb{Z}$ is called *resonant with respect to ω*. We can also define the class of fibered rotation numbers which are Diophantine with respect to ω. Its elements are the numbers α such that the bound

$$|\sin(\pi(2\alpha - \langle k, \omega\rangle))| \geq \frac{K}{|k|^\sigma},$$

holds for all $k \in \mathbb{Z} - \{0\}$ and suitable fixed positive constants K and σ. Both classes of rotation numbers are constant under conjugation.

With these definitions we can give a more precise version of Eliasson's reducibility theorem for general quasi-periodic cocycles on $SL(2,\mathbb{R})\times\mathbb{T}$ homotopic to the identity. Again the result is valid for more than one frequency, but we restrict ourselves to this one-dimensional case.

Theorem 11 **(**[Eli92]**)** *Let $\rho > 0$, $\omega \in DC(c, \tau)$ be Diophantine and A_0 be a matrix in $SL(2,\mathbb{R})$. Then there is a constant $C = C(c, \tau, \rho, |A_0|)$ such that, if $A \in C^a_\rho(\mathbb{T}, SL(2,\mathbb{R}))$ is real analytic with*

$$|A - A_0|_\rho < C$$

and the rotation number of the cocycle (A, ω) is either resonant or Diophantine with respect to ω, then (A, ω) is reducible to constant coefficients of a quasi-periodic (perhaps with frequency $\omega/2$) and analytic transformation.

Remark 12 *The proof of this theorem was originally given in* [Eli92] *in the continuous case and for Schrödinger operators (instead of cocycles), although it extends to the setting of Theorem 11.*

Applied to Schrödinger cocycles one obtains the perturbative version of Theorem 1 with the additional characterization of the set of reducible energies in terms of its rotation number. More precisely, the theorem above implies that the set of "reducible" rotation numbers is of full measure in $\mathbb{T}$. To obtain a full-measure condition on the energies it is necessary to use some facts on the growth of the rotation number at these reducible points which will be also used in Section 3 and which are due to Deift & Simon [DS83a].

3 Proof of Theorem 1

We are now ready to show that Theorem 1 is a direct consequence of the following result by Bourgain & Jitomirskaya [BJ02b], which we restate in a convenient way:

Theorem 13 ([BJ02b]) *Let $\rho > 0$ be a positive number. Then there is a constant $\varepsilon_0 = \varepsilon_0(\rho)$ such that, for any real analytic $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ with*

$$|V|_\rho < \varepsilon_0,$$

and Diophantine ω there is a set $\Phi \subset \mathbb{T}$, of zero (Lebesgue) measure such that, if $\phi \notin \Phi$, the operator $L_{V,\omega,\phi}$ has pure point spectrum with exponentially decaying eigenfunctions.

Remarks 14

1. *In* [BJ02a]*, the bound ε_0 depends on $\|V\|_1$, $\|V\|_2$, $\|V\|_\infty$ and ρ. Since V belongs to $C^a_\rho(\mathbb{T}, \mathbb{R})$, all these previous norms can be controlled by $|V|_\rho$.*
2. *The set Φ consists of those phases ϕ for which the relation*

$$|\sin(\phi + \pi k\omega)| < \exp\left(-|k|^{\frac{1}{2\tau}}\right) \tag{11}$$

holds for infinitely many values of k, where $\omega \in DC(c, \tau)$. For any Diophantine ω, this is a set of zero Lebesgue measure.

Our strategy to prove Theorem 1 will be, first of all, to show that Corollary 3 is a simple consequence of Theorem 13 and the duality of the IDS . Then, in Section 3.1 it will be shown that Corollary 3 actually implies Theorem 1.

Let $\rho > 0$ and V, ω and Φ be as in the Theorem 13. As a consequence of Proposition 9, the set

$$A = \sigma^L(V, \omega) \setminus \bigcup_{\phi \notin \Phi} \sigma^L_{pp}(V, \omega, \phi),$$

where $\sigma^L_{pp}(V, \omega, \phi)$ is the set of point eigenvalues of $L_{V,\omega,\phi}$ given by Theorem 13 satisfies that $n^L_{V,\omega}(A) = 0$. Indeed, according to Proposition 9 we only need to show that $\mu_\phi(A) = 0$ for all $\phi \notin \Phi$, where μ_ϕ are the spectral measures of the long-range operators $L_{V,\omega,\phi}$. This is a consequence of the fact that the spectral measures μ_ϕ, for $\phi \notin \Phi$ are supported on the set of point eigenvalues of the corresponding operator.

Therefore also $n^H_{V,\omega}(A) = 0$ due to Proposition 9. To prove Corollary 3 it only remains to show that also the Lebesgue measure of A is zero. To do so, one can invoke Deift & Simon [DS83a]. For almost periodic discrete Schrödinger operators they prove that for Lebesgue almost every a in the set where the Lyapunov exponent is zero, one has the inequality

$$2\pi \sin \pi k^H_{V,\omega}(a) \frac{dk^H_{V,\omega}}{da} \geq 1. \tag{12}$$

Thus, under the additional assumption that the Lyapunov exponent vanishes in the spectrum, the inequality (12) implies that if A is a subset of $\sigma^H(\omega, V)$ with $n^H(A) = 0$ then also the Lebesgue measure of A is zero.

As a consequence of Bourgain & Jitormirskaya [BJ02a, BJ02b], for any a in $\sigma^H(V, \omega)$, (with $|V|_\rho < \varepsilon$) the Lyapunov exponent is zero. Therefore, the set A has Lebesgue measure zero and for the values of a in its complement in the spectrum,

$$a \in \sigma^H_{V,\omega} \setminus A,$$

which is a total measure subset of $\sigma^H(V, \omega)$, the corresponding Harper-like equation

$$x_{n+1} + x_{n-1} + V(2\pi\omega n)x_n = ax_n, n \in \mathbb{Z} \tag{13}$$

has an analytic quasi-periodic Bloch wave, using the argument of duality in the beginning of Section 2.1. Indeed, we saw that if a is a point eigenvalue of the operator $L_{V,\omega,\phi}$ whose eigenfunction decays exponentially then the Harper-like equation (13) has an analytic quasi-periodic Bloch wave with Floquet exponent ϕ. This completes the proof of Corollary 3.

3.1 From Bloch waves to reducibility

In this section we will see how Corollary 3 (which we proved in the previous section) implies our main result, Theorem 1. By this corollary we know that if V, ω and Φ are as in Theorem 13 then, for almost all

$a \in \sigma^H(V,\omega)$, the equation (13) has an analytic quasi-periodic Bloch wave with Floquet exponent $\varphi \notin \Phi$. Since we only want to prove a result for almost every a, it is sufficient to show that if $\varphi \notin \Phi$ is such that

$$\varphi - \pi k\omega - \pi j \neq 0 \tag{14}$$

for all $k, j \in \mathbb{Z}$ and (13) has an analytic quasi-periodic Bloch wave with this Floquet exponent φ, then the corresponding Schrödinger cocycle $(A_{a,V}, \omega)$ is reducible to constant coefficients.

Remark 15 *If $\varphi/2\pi$ is resonant with respect to ω,*

$$\varphi = \pi k + \pi j\omega,$$

for some integers k, j, then one can also prove reducibility [Pui04a]. *In Section 4.3 we will consider the case of $\varphi = 2\pi k$ which will be used for the Cantor structure of the spectrum.*

The existence of a Bloch wave for Equation (13) implies that the Schrödinger cocycle has the following quasi-periodic solution

$$\begin{pmatrix} \tilde\psi(4\pi\omega+\theta) \\ e^{-i\varphi}\tilde\psi(2\pi\omega+\theta) \end{pmatrix} = e^{-i\varphi} \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde\psi(2\pi\omega+\theta) \\ e^{-i\varphi}\tilde\psi(\theta) \end{pmatrix} \tag{15}$$

for all $\theta \in \mathbb{T}$. Moreover, writing

$$v(\theta) = \left(\tilde\psi(\theta + 2\pi\omega), e^{-i\varphi}\tilde\psi(\theta)\right)^T \tag{16}$$

and

$$Y(\theta) = \begin{pmatrix} v_1(\theta) & \bar v_1(\theta) \\ v_2(\theta) & \bar v_2(\theta) \end{pmatrix}, \tag{17}$$

where the bar denotes complex conjugation, one always has the relation

$$A_{a,V}(\theta)Y(\theta) = Y(\theta + 2\pi\omega)\Lambda(\varphi), \tag{18}$$

where

$$\Lambda(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

Obviously, Y will only define a conjugation between the cocycles $(A_{a,V}, \omega)$ and $(\Lambda(\varphi), \omega)$ if it is nonsingular. Because of (18), the determinant of Y is constant as a function of θ and it is purely imaginary. In particular, $v(\theta)$ and $\bar v(\theta)$ are linearly independent for all θ if, and only if, they are independent for some θ. In the case that v and $\bar v$ are linearly independent, it is not difficult to prove reducibility to constant coefficients of the cocycle.

Lemma 16 *Let* $A : \mathbb{T} \to SL(2,\mathbb{R})$ *be a real analytic map and* ω *be nonresonant. Assume that there is an analytic map* $v : \mathbb{T} \to \mathbb{R}^2 \setminus \{0\}$, *with* v *and* $\bar{v}$ *linearly independent, such that*

$$v(\theta + 2\pi\omega) = e^{-\mathrm{i}\varphi} A(\theta) v(\theta)$$

holds for all $\theta \in \mathbb{T}$, *where* $\varphi \in [0, 2\pi)$. *Then the cocycle* (A, ω) *is reducible to constant coefficients by means of a real analytic transformation. Moreover, the Floquet matrix can be chosen to be of the form*

$$B = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{19}$$

Proof: Let $Z^1(\theta) = Y(\theta)$ as in (17), $B^1 = \Lambda(\varphi)$ and

$$d(\theta) = v_1(\theta)\bar{v}_2(\theta) - \bar{v}_1(\theta)v_2(\theta).$$

be the determinant of Z^1. Therefore Z^1 defines a conjugation between $(A_{a,V}, \omega)$ and (B^1, ω) because v and $\bar{v}$ are linearly independent, Z^1 is real analytic and, for every $\theta \in \mathbb{T}$, $Z^1(\theta)$ is nonsingular.

Moreover, from the conjugacy (18) and the nonresonance of ω, $d(\theta)$ is constant as a function of θ. By the linearity of our system, we choose this constant value to be $-\mathrm{i}/2$ (recall that, due to the form of Z_1, its determinant must be purely imaginary).

To obtain the real rotation consider the composition

$$Z(\theta) = Z^1(\theta) Z^2$$

where Z^2 is the constant matrix

$$Z^2 = \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix},$$

Then Z satisfies the desired conjugation

$$A(\theta) Z(\theta) = Z(\theta + 2\pi\omega) B$$

being B the rotation of angle φ given by (19). Thanks to the construction Z is real and with determinant one. $\square$

To complete the proof of Theorem 1, it only remains to rule out the possibility that φ satisfies (14) and v and $\bar{v}$ are linearly dependent at the same time. Recall that these two vectors are linearly independent for all θ if, and only if, they are linearly independent for some θ. Note that both $v(\theta)$ and $\bar{v}(\theta)$ are different from zero for all $\theta \in \mathbb{T}$ by construction. Assume that $v(\theta)$ and $\bar{v}(\theta)$ were linearly dependent for all θ. Since these vectors depend analytically on θ, there would exist an analytic $h : \mathbb{T} \to \mathbb{R}$ and an integer $k \in \mathbb{Z}$ such that

$$\bar{v}(t) = e^{\mathrm{i}(h(t)+kt)} v(t)$$

for all $t \in \mathbb{R}$. Using that v and $\bar{v}$ are quasi-periodic solutions of (A, ω), this would imply that

$$e^{\mathrm{i}(h(t)+kt)} e^{\mathrm{i}\varphi} = e^{\mathrm{i}(h(t+2\pi\omega)+kt+2\pi k\omega)} e^{-\mathrm{i}\varphi}.$$

Therefore, h must satisfy the following small divisors equation

$$h(\theta + 2\pi\omega) - h(\theta) = 2\varphi - 2\pi k\omega - 2\pi j$$

for all $\theta \in \mathbb{T}$, where j is some fixed integer. Clearly, such analytic h cannot exist unless

$$\varphi = \pi (j + k\omega),$$

which is a contradiction with the nonresonance condition (14). This ends the proof of Theorem 1. □

4 Applications

In this section we will prove several consequences of the main theorem which are summarized in theorems 4 and 5. In Section 4.1 we will present the setting of this section. In 4.2 we adapt Moser-Pöschel perturbation arguments to the discrete case. This is applied in Section 4.3 to the proof of nonperturbative genericity of Cantor spectrum. Finally, in Section 4.4 we prove that, in our situation, Cantor spectrum implies nonreducibility for a G_δ-set of energies.

4.1 Reducibility at gap edges

In previous sections, we discussed the reducibility of a quasi-periodic Schrödinger cocycle $(A_{a,V}, \omega)$ when a is a point eigenvalue of the dual operator $L_{V,\omega,\phi}$ and ϕ satisfies a nonresonance condition of the form (14), which was enough to prove the main result. The "resonant" values of ϕ:

$$\phi = \pi j + \pi\omega k, \qquad j, k \in \mathbb{Z} \tag{20}$$

are particularly important for the description of the spectrum of these operators because the corresponding point eigenvalues lie at endpoints of spectral gaps. Let us prove the reducibility at these endpoints. What follows mimics the proof of the "Ten Martini Problem" given in [Pui04a].

Bourgain & Jitomirskaya [BJ02a] also prove that, provided $|V(\theta)| < \varepsilon$ and ω is Diophantine $L_{V,\omega,\phi}$ has pure-point spectrum with exponentially localized eigenfunctions if ϕ is of the form (20), see Remark 8.2 after Theorem 7 in [BJ02a]. Taking into account the symmetries of the operators, the study reduces to the four cases $\phi = 0, \pi, \pi\omega, \pi\omega + \pi$. For the sake of simplicity we consider here $\phi = 0$. As a direct consequence of Aubry duality and Bourgain-Jitomirskaya result for the dual operators $L_{V,\omega,0}$, the set of pure-point eigenvalues $\sigma^L_{pp}(V,\omega,0)$ is a dense subset of $\sigma^H(V,\omega)$ and any energy $a \in \sigma^L_{pp}(V,\omega,0)$ has a quasi-periodic Bloch wave with 0 as a Floquet exponent for the dual eigenvalue equation. Let a be one of these eigenvalues. The condition above means that there is an analytic map $\tilde{\psi} : \mathbb{T} \to \mathbb{C}$ such that

$$x = (x_n)_{n\in\mathbb{Z}} = \left(\tilde{\psi}\left(2\pi\omega n + \theta\right)\right)_{n\in\mathbb{Z}}$$

is a nonzero solution of $H_{V,\omega,\theta}x = ax$. Clearly, due to the symmetry of the eigenvalue equation for $L_{V,\omega,0}$, the function $\tilde{\psi}$ can be chosen real analytic. In terms of the cocycle we have that the relation

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega + \theta) \\ \tilde{\psi}(2\pi\omega + \theta) \end{pmatrix} = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega + \theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

holds for all $\theta \in \mathbb{T}$. Instead of Lemma 16 we now have the following (see [Pui04a] for the proof, which is a simple triangularization and averaging argument).

Lemma 17 *Let $A \in C^a_\delta(\mathbb{T}, SL(2,\mathbb{R}))$ be a real analytic map and ω be Diophantine. Assume that there is a nonzero real analytic map $v \in C^a_\delta(\mathbb{T}, \mathbb{R}^2)$ such that the relation*

$$v(\theta + 2\pi\omega) = A(\theta)v(\theta)$$

holds for all $\theta \in \mathbb{T}$. Then the quasi-periodic cocycle (A,ω) is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|Im\ \theta| < \delta$. Moreover the Floquet matrix can be chosen to be of the form

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{21}$$

for some $c \in \mathbb{R}$.

In the Almost Mathieu case Ince's argument [Inc44, Pui04a] shows that $c \neq 0$. Otherwise the dual model (which is also a Schrödinger operator) would have a point eigenvalue with two linearly independent eigenvectors in $l^2(\mathbb{Z})$, and this is a contradiction with the limit-point character of Schrödinger operators (or just the preservation of the Wronskian in this discrete case). The fact that $c \neq 0$ is important for the description of the spectrum, because if a Schrödinger cocycle is reducible to a Floquet matrix of the form (21) with $c \neq 0$ then the corresponding energy is at the endpoint of an open gap in the spectrum, as it will be seen in the next section.

For general potentials V, however, we cannot use Ince's argument and it may happen that some of these are collapsed. In fact, there are examples of quasi-periodic Schrödinger operators (with V small, ω Diophantine) for which some c are zero [BPS03] or even do not display Cantor spectrum (see De Concini & Johnson [DCJ87]).

Nevertheless, even if c can be zero, Moser & Pöschel [MP84] showed that, in this reducible setting, a closed gap can be opened by means of an arbitrarily small and generic real analytic perturbation of the potential. In the next section we give an adaption of their proof to the discrete case together with some extra properties which will be needed later.

4.2 Moser-Pöschel perturbation argument

In this section we prove the following adaption of Moser-Pöschel argument to the discrete case, which deals with cocycles which are perturbations of constant matrices of the form (21).

Proposition 18 *Let V be real analytic, ω Diophantine and $(A_{a,V}, \omega)$, for some $a \in \mathbb{R}$ be a quasi-periodic Schrödinger cocycle. Assume that $(A_{a,V}, \omega)$ is analytically reducible to the constant coefficients cocycle (B, ω) with*

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

for some $c \in \mathbb{R}$. Let $W : \mathbb{T} \to \mathbb{R}$ be real analytic and α real. If $Z : \mathbb{T} \to SL(2, \mathbb{R})$ is the real analytic reducing matrix and the conditions

$$c \neq 0 \quad \textit{and} \quad [W z_{11}^2] \neq 0 \tag{22}$$

or

$$c = 0 \quad \textit{and} \quad -[W z_{11} z_{12}]^2 + [W z_{12}^2][W z_{11}^2] > 0 \tag{23}$$

are satisfied, then the quasi-periodic cocycle $(A_{a,V+\alpha W}, \omega)$ has an exponential dichotomy provided $|\alpha| > 0$ is small enough and

$$c\alpha[W z_{11}^2] < 0 \textit{ if (22) holds.} \tag{24}$$

Moreover in the case $c \neq 0$, the Lyapunov exponent of $(A_{a,V+\alpha W},\omega)$, $\gamma(a, V+\alpha W,\omega)$, *and its rotation number,* $\mathrm{rot}_f(a,V+\alpha W,\omega)$, *satisfy*

$$\lim_{\substack{\alpha\to 0,\\ c\alpha[Wz_{11}^2]>0}} \frac{|\gamma(\alpha)|}{|\alpha|^{1/2}} = c[Wz_{11}^2] = \lim_{\substack{\alpha\to 0,\\ c\alpha[Wz_{11}^2]<0}} \frac{|\mathrm{rot}_f(\alpha)-\mathrm{rot}_f(0)|}{|\alpha|^{1/2}} \tag{25}$$

Remark 19 *This type of perturbation arguments have been used in a variety of contexts, c.f. Moser & Pöschel* [Mos81, MP84]*, Johnson* [Joh91]*, Núñez* [Nn95]*, Broer, Puig & Simó* [BPS03] *and Puig & Simó* [PS04]*.*

Proof: Since Z is the reducing transformation of $(A_{a,V},\omega)$ to (B,ω), it also renders the perturbed cocycle $(A_{a,V+\alpha W},\omega)$ to $(B+\alpha WP,\omega)$ where

$$P(\theta) = \begin{pmatrix} z_{11}z_{12}-cz_{11}^2 & -cz_{11}z_{12}+z_{12}^2 \\ -z_{11}^2 & -z_{11}z_{12} \end{pmatrix}.$$

After one step of averaging this cocycle can be analytically conjugated to

$$\left(B+\alpha[WP]+\alpha^2R_2,\omega\right)$$

where $[\cdot]$ denotes the average of a quasi-periodic function and R_2 depends analytically on α and θ in some open neighbourhoods of 0 and $\mathbb{T}$. Moreover, a computation shows that

$$B+\alpha[WP]+\alpha^2R_2 = \exp\left(\tilde{B}_0+\alpha\tilde{B}_1+\alpha^2\tilde{R}_2\right),$$

being

$$\tilde{B}_0 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix} [Wz_{11}z_{12}]-\frac{c}{2}[Wz_{11}^2] & -c[Wz_{11}z_{12}]+[Wz_{12}^2] \\ -[Wz_{11}^2] & -[Wz_{11}z_{12}]+\frac{c}{2}[Wz_{11}^2] \end{pmatrix}$$

and $\tilde{R_2}\in sl(2,\mathbb{R})$ depending analytically on α and θ. Let

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & -d_1 \end{pmatrix} = \tilde{B}_0+\alpha\tilde{B}_1,$$

whose determinant is $d=-d_1^2-d_2d_3$. Now let us distinguish between the cases $c\neq 0$ and $c=0$.

If (22) holds then the expression for the determinant becomes

$$d = c\alpha[Wz_{11}^2]+O(\alpha^2)$$

so that it is negative if, in addition (24) holds. In this case, the matrix

$$Q = \begin{pmatrix} d_2 & d_2 \\ -d_1+\sqrt{-d} & -d_1-\sqrt{-d} \end{pmatrix},$$

which is well-defined, has determinant

$$2c\sqrt{-c\alpha[Wz_{11}^2]} + O(\alpha).$$

and satisfies $DQ = Q\Delta$, where

$$\Delta = \begin{pmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d}. \end{pmatrix}.$$

Therefore the change of variables defined by Q transforms the cocycle

$$\left(\exp\left(\tilde{B}_0 + \alpha\tilde{B}_1 + \alpha^2\tilde{R}_2\right), \omega\right) \tag{26}$$

into

$$\left(\exp\left(\left(\Delta + \tilde{S}_2\right)\right), \omega\right) \tag{27}$$

where

$$\tilde{S}_2(\alpha,\theta) = \alpha^2 Q^{-1} R_2(\alpha,\theta) Q$$

which is $O(|\alpha|^{3/2})$ uniformly in θ. Note that

$$\Delta + \tilde{S}_2 = \sqrt{-c\alpha[Wz_{11}^2]}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + O(|\alpha|)\right).$$

so that if (24) holds and $|\alpha| > 0$ is small enough the cocycle $(A_{a,V+\alpha W}, \omega)$ has an exponential dichotomy and the Lyapunov exponent satisfies

$$\frac{1}{\sqrt{|\alpha|}}\left|\gamma\left(A_{a,V+\alpha W}, \omega\right)\right| \to \sqrt{|c[Wz_{11}^2]|},$$

see Coppel [Cop78]. To obtain the asymptotics of the rotation number, we can consider the transformation Q defined for $c\alpha[Wz_{11}^2] > 0$. This, although complex, is a well-defined conjugation between D and Δ, which is now a complex rotation of angle $\sqrt{|d|}$. Therefore (26) is conjugated to (27), a perturbation of a complex rotation. Using the definition of the fibered rotation number given in Section 2.3 the result follows.

Let us now consider the situation when (23) holds. In this case, the matrix D becomes

$$D = \alpha\begin{pmatrix} [Wz_{11}z_{12}] & [Wz_{12}^2] \\ -[Wz_{11}^2] & -[Wz_{11}z_{12}] \end{pmatrix} =: \alpha\tilde{D}$$

Condition (23) is equivalent to the hyperbolicity of $\tilde{D}$ whose determinant is

$$\tilde{d} = -[Wz_{11}z_{12}]^2 + [Wz_{12}^2][Wz_{11}^2].$$

Therefore there is a change of variables Q, independent of α and θ, which renders it to a diagonal form $\tilde{\Delta}$ with $\sqrt{-d}$ and $-\sqrt{-d}$ as diagonal entries. This conjugation transforms the cocycle

$$\left(\exp\left(\alpha\tilde{D}+\alpha^2\tilde{R}_2\right),\omega\right)$$

into

$$\left(\exp\left(\left(\alpha\tilde{\Delta}+\alpha^2\tilde{S}_2\right)\right),\omega\right)$$

where

$$\tilde{S}_2(\alpha,\theta)=Q^{-1}\tilde{R}_2(\alpha,\theta)Q.$$

Since,

$$\alpha\Delta+\alpha^2\tilde{S}_2=\alpha\sqrt{-\tilde{d}}\left(\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)+\frac{\alpha}{\sqrt{-\tilde{d}}}\tilde{S}_2\right).$$

the cocycle $(A_{a,V+\alpha W},\omega)$ has an exponential dichotomy when $\alpha\neq 0$ is small enough (see, again Coppel [Cop78]). □

The perturbation argument in the previous proposition can applied to the reducible cocycles at endpoints of gaps as we do next.

Corollary 20 *Let V, a and ω as in Proposition 18 and assume that $c\neq 0$. Then a is at the endpoint of a noncollapsed spectral gap I of $\sigma(V,\omega)$ (the right one if $c>0$ and the left one if $c<0$). Moreover, the limits*

$$\lim_{\substack{\alpha\to 0,\\ a+\alpha\in I}}\frac{\gamma(a+\alpha,V,\omega)}{\sqrt{|\alpha|}}=\lim_{\substack{\alpha\to 0,\\ a+\alpha\in I}}\frac{|\mathrm{rot}_f(a+\alpha,V,\omega)-\mathrm{rot}_f(a,V,\omega)|}{\sqrt{|\alpha|}} \tag{28}$$

exist and are different from zero.

Proof: Take $W=1$ in Proposition 18. Then, the cocycle $(A_{a+\alpha,V},\omega)$ has an exponential dichotomy if $c\alpha<0$ and $|\alpha|$ is small enough. This means that there is an open spectral gap besides a (to the left if $c>0$ and to the right otherwise). Moreover the asymptotics of formula (25) imply (28). □

Finally we consider the variation of the rotation number in the case $c=0$ in a more general setting which will be needed in the next section.

Proposition 21 *Let V be continuous and ω nonresonant. Assume that the Schrödinger cocycle $(A_{a_0,V},\omega)$ is reducible to the cocycle (B,ω) with $B\in SO(2,\mathbb{R})$ a constant matrix. Then the map*

$$a\in\mathbb{R}\mapsto\mathrm{rot}_f(a,V,\omega)$$

is differentiable at a_0.

Proof: Let ρ be the angle of the rotation,

$$B = \begin{pmatrix} \cos\rho & \sin\rho \\ -\sin\rho & \cos\rho \end{pmatrix}.$$

The cocycle $(A_{a,V},\omega)$ is conjugated to $(B+\alpha R,\omega)$ where

$$R(\theta) = Z(\theta+2\pi\omega)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z(\theta)$$

and $\alpha = a - a_0$. The cocycle $(B+\alpha R,\omega)$ induces a lift $\tilde{F}$ from $\mathbb{R}\times\mathbb{T}$ to itself of the form

$$\tilde{F}(t,\theta) = (t+\rho+\alpha f(t,\theta,\alpha), \theta+2\pi\omega)\,,$$

where f is continuous and 2π-periodic in both t and θ. Therefore,

$$\mathrm{rot}_f(B+\alpha R,\omega) - \mathrm{rot}_f(B,\omega) = \lim_{N\to\infty}\frac{1}{2\pi N}\sum_{n=0}^{N-1}\left(\alpha f\left(\tilde{F}^n(t,\theta)\right)\right) = O(\alpha)$$

as we wanted to show. □

Remarks 22

1. *A computation shows that the derivative of the rotation number above is nonzero. In particular, when a Schrödinger cocycle is reducible to the identity, the corresponding energy lies at the endpoint of a collapsed gap.*
2. *Similar results have been obtained when $Z, Z^{-1} : \mathbb{T}\to SL(2,\mathbb{R})$ are square integrable and $B : \mathbb{T}\to SO(2,\mathbb{R})$ is measurable, compare with Moser* [Mos81] *and Deift & Simon* [DS83a].

4.3 Genericity of Cantor spectrum

In the previous section we have seen that if a Schrödinger cocycle is reducible to a matrix with trace 2 then the corresponding energy is at the endpoint of a spectral gap which is collapsed if the Floquet matrix is the identity. The next consequence of Proposition 18 is that when the Floquet matrix is the identity (a similar statement holds for minus the identity) one can "open up" the collapsed gap by means of a generic perturbation.

Corollary 23 *Let V, a, ω and Z be as in Proposition 18 and assume that $c = 0$. If W is a generic real analytic potential then for $|\beta|\neq 0$ small enough the spectrum $\sigma(V+\beta W,\omega)$ has an open spectral gap with* IDS *$k(a,V,\omega)$.*

Proof: In Proposition 18 we proved that for a perturbation $\tilde{W}$ satisfying (23)

$$-[\tilde{W} z_{11} z_{12}]^2 + [\tilde{W} z_{12}^2][\tilde{W} z_{11}^2] < 0$$

the cocycle $(A_{a,V+\beta\tilde{W}}, \omega)$ has an exponential dichotomy if $|\beta| > 0$ small enough. This means that a lies in a spectral gap of $\sigma(V+\beta\tilde{W}, \omega)$ which, by continuity must satisfy that

$$k(a, V + \beta\tilde{W}, \omega) = k(a, V, \omega)$$

for $|\beta|$ small enough. Let us now show that if W is a generic potential, then for every $|\beta| \neq 0$ small enough there is a value of α such that $a + \alpha$ lies in a spectral gap of $\sigma(V + \beta W, \omega)$ with

$$k(a + \alpha, V + \beta W, \omega) = k(a, V, \omega).$$

Note that the condition (23) can be rewritten as

$$[\tilde{W} y_1]^2 - [\tilde{W} y_2]^2 - [\tilde{W} y_3]^2 < 0$$

where

$$y_1 = \frac{1}{2}\left(z_{11}^2 + z_{12}^2\right), \quad y_2 = \frac{1}{2}\left(z_{11}^2 - z_{12}^2\right), \quad y_3 = z_{11} z_{12}.$$

Let α be such that

$$[(\alpha + W) y_1] = \alpha[y_1] + [W y_1] = 0,$$

(this determines α since $[y_1] \neq 0$). Then the shifted perturbation $\alpha + W$ satisfies condition (23) unless

$$-[W y_1][y_2] + [W y_2][y_1] = 0 \quad \text{and} \quad -[W y_1][y_3] + [W y_3][y_1] = 0,$$

which is clearly a generic condition. Then, if $|\beta| > 0$ is small enough, the spectrum $\sigma(V + \beta W, \omega)$ has an open gap with

$$k(a + \alpha\beta, V + \beta W, \omega) = k(a, V, \omega)$$

as we wanted to show. □

Remark 24 *As Moser & Pöschel show, when $c = 0$ it is always possible to choose the reducing transformation such that $[z_{11}^2] = [z_{12}^2] = 1$ and $[z_{11} z_{12}] = 0$ so that $[y_1] = 1$, $[y_2] = 0$, $[y_3] = 0$ and the generic W must satisfy*

$$[W(z_{11} + z_{12})^2] \neq 0 \qquad \textit{or} \qquad [W(z_{11} - z_{12})^2] \neq 0.$$

Let us now summarize the situation. Using the two past sections we have seen that if V is a real analytic potential on $C^a_\rho(\mathbb{T},\mathbb{R})$, with $|V|_\rho < \varepsilon$ and ω is Diophantine, there is a countable dense subset of energies in the spectrum where the system is reducible to a Floquet matrix with trace 2. These lie at endpoints of gaps. Although these can be collapsed, a generic and arbitrarily small perturbation opens them as Corollary 23 says. Since there is a countable number of gaps Theorem 4 follows.

4.4 Cantor spectrum implies nonreducibility

In [Eli92] it was seen that, for a generic real analytic Schrödinger cocycle (with Diophantine frequencies) besides the almost everywhere reducibility there was a set of zero measure of energies for which the cocycle was not reducible to constant coefficients. The proof relies on the KAM procedure developed there, but the Cantor structure of the spectrum is seen to play a key role. In this section we prove irreducibility for a G_δ-set of energies assuming only Cantor structure of the spectrum and Theorem 1. This argument is reminiscent of some techniques in circle maps, see Arnol'd [Arn61]. We state here a slightly more general version than that of Theorem 5. More applications will be given elsewhere.

Theorem 25 *Let $\rho > 0$. There is a constant $\varepsilon > 0$ such that if $V \in C^a_\rho(\mathbb{T},\mathbb{R})$ is real analytic with $|V|_\rho < \varepsilon$, ω is Diophantine and I is an open interval such that*

$$K = \sigma(V,\omega) \cap \bar{I}$$

is a nonvoid Cantor set, then there is a G_δ-dense set of energies in K for which the corresponding Schrödinger cocycle is not reducible to constant coefficients by means of a continuous transformation.

Proof: Consider, for any $a_1, a_2 \in K$ with $a_1 \neq a_2$,

$$\delta(a_1,a_2) = \left| \frac{k(a_1,V,\omega) - k(a_2,V,\omega)}{a_1 - a_2} \right|.$$

Now, for any $a \in \mathbb{R}$ we can define

$$m(a) = \sup_{\lambda \neq a, \lambda \in K} \delta(a,\lambda)$$

which is either a positive real number or $+\infty$.

If $a \in \sigma(V,\omega)$ is reducible to constant coefficients then we have two situations. Either the Floquet matrix B has trace ± 2, in which case

$m(a) = \infty$ (see Corollary 20) or $B \in SO(2, \mathbb{R})$ and then $m(a) < \infty$ (see Proposition 21). Due to the fact that $|V|_\rho < \varepsilon$, ω is Diophantine and the Cantor structure of the spectrum there is a dense set of endpoints of gaps, $\mathcal{G}_K$, where the system is reducible to constant coefficients because of Eliasson's Theorem 11.

We will show that the set where $m(a) = \infty$ is a G_δ-dense subset of K. Excluding the endpoints of gaps where there is reducibility to a Floquet matrix with trace ± 2 (which are at most countable) we will still have a G_δ-dense subset of energies in K whose corresponding cocycle cannot be reducible to constant coefficients.

Let, for any $n \in \mathbb{N} \cup \{0\}$ and $a_0 \in K$,

$$U(a_0, n) = \{a \in K; \delta(a, a_0) > n\}$$

and

$$U(n) = \bigcup_{a_0 \in K} U(a_0, n).$$

The sets $U(n)$ are open in K because of the continuity of the rotation number. Moreover they are dense in K because they contain $\mathcal{G}_K$, which is dense in K. Therefore

$$U(\infty) \bigcap_{n > 0} U(n) = \{a \in K; m(a) = \infty\},$$

is a G_δ-dense subset of K. If we exclude the endpoints of open gaps the remaining energies, which still form a G_δ-dense subset of K, cannot be reducible to constant coefficients by means of a continuous transformation. This proves 25 and also 5. $\square$

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