

Lindstedt-Poincaré Solutions of Clohessy-Wiltshire Equations

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LINDSTEDT-POINCARÉ SOLUTIONS OF CLOHESSY-WILTSHIRE EQUATIONS

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The purpose of this paper is the semi-analytical computation of the bounded orbits of Hill's equations, describing the relative motion of two particles in their Keplerian motion around a central body. We have considered the case in which one of the particles moves along a circular reference orbit. The solutions obtained are the generalisation of the usual periodic orbits obtained for the linearised equations and also of the third order solution computed by D.L. Richardson and J.W. Mitchell. With the algorithm presented, those orbits can be computed in a fast and efficient way up to a high order in the in-plane and out-off-plane amplitudes.

INTRODUCTION

Hill's, or Clohessy–Wiltshire, equations describe the relative motion of two point-mass bodies under the gravitational influence of a point-mass central body. The bounded solutions of these equations are useful for several problems, such as the determination of formation flight suitable trajectories or the description of rendezvous manoeuvres. Usually, Hill's equations are considered in their linear approximation with one of the bodies moving along a circular orbit. For these linear equations is easy to get a bi-parametric family of periodic orbits. In a recent paper by Richardson and Mitchell⁴ the linear approximation is extended to a third order solution in the two parameters of the family (the in-plane and the out-off-plane amplitudes). In the present paper, using also the Lindstedt-Poincaré procedure for their determination, we extend the computations of Richardson and Mitchell to an arbitrary order, providing in this way more accurate results.

THE MODEL

As it has already been said, Hill's equations can be used to describe the relative motion of two spacecrafts subjected to the gravitational influence of a point-mass central body. We will assume that one of the spacecrafts, the leader, moves in a circular reference orbit of radius r_l about the central body and that the second spacecraft, the follower, moves close to the leader, at a distance δr from it. If \mathbf{r}_l and \mathbf{r}_f denote the inertial positions of the

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leader and the follower, respectively, with respect to the central body, then $\mathbf{r}_f = \mathbf{r}_l + \delta\mathbf{r}$ (see Fig. 1).

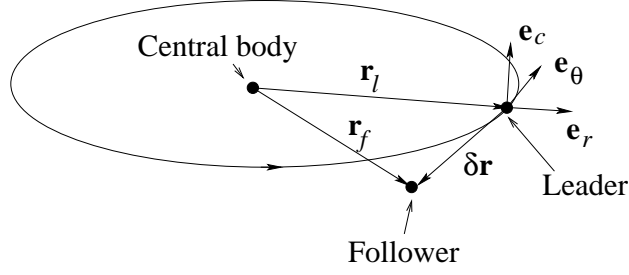


Figure 1 Reference system for the description of the relative motion of the follower spacecraft with respect to the leader.

For the description of the motion, is useful to introduce Hill's non-inertial reference system, $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_c$, centred at the leader and defined by

$$\mathbf{e}_r = \frac{\mathbf{r}_l}{r_l}, \quad \mathbf{e}_c = \frac{\mathbf{r}_l \wedge \mathbf{r}_l'}{\|\mathbf{r}_l \wedge \mathbf{r}_l'\|}, \quad \mathbf{e}_\theta = -\mathbf{e}_r \wedge \mathbf{e}_c,$$

where $r_l = \|\mathbf{r}_l\|$ and the prime denotes the time-derivative. To obtain the relative equations of motion in Hill's coordinates, we write the follower satellite position vector as

$$\mathbf{r}_f = \mathbf{r}_l + \delta\mathbf{r} = (X + r_l)\mathbf{e}_r + Y\mathbf{e}_\theta + Z\mathbf{e}_c.$$

Using that $\mathbf{e}_r' = n\mathbf{e}_\theta$, $\mathbf{e}_\theta' = -n\mathbf{e}_r$, where n is the mean motion of the leader on its circular orbit, we get that the follower acceleration is given by

$$\mathbf{r}_f'' = (X'' - 2nY' - n^2(X + r_l))\mathbf{e}_r + (Y'' + 2nX' - n^2Y)\mathbf{e}_\theta + Z''\mathbf{e}_c. \quad (1)$$

The follower equations of motion are given by

$$\mathbf{r}_f'' = -\frac{\mu}{r_f^3}\mathbf{r}_f = -\frac{\mu}{r_f^3} \begin{pmatrix} X + r_l \\ Y \\ Z \end{pmatrix}, \quad (2)$$

with

$$r_f = \sqrt{(X + r_l)^2 + Y^2 + Z^2}.$$

Equating (1) and (2) we get the nonlinear relative equations of motion

$$\begin{aligned} X'' - 2nY' - n^2(X + r_l) &= -\frac{\mu}{r_f^3}(X + r_l), \\ Y'' + 2nX' - n^2Y &= -\frac{\mu}{r_f^3}Y, \\ Z'' &= -\frac{\mu}{r_f^3}Z. \end{aligned}$$

Introducing dimensionless coordinates x, y, z , dimensionless time t via

$$X = r_l x, \quad Y = r_l y, \quad Z = r_l z, \quad dt = n d\tau,$$

and using that $\mu = n^2 r_l^3$, the relative equations of motion can be written as

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{3}$$

where

$$\Omega(x, y, z) = \frac{(x+1)^2 + y^2}{2} + \frac{1}{\sqrt{(x+1)^2 + y^2 + z^2}}.$$

In order to expand the nonlinear terms of (3), we use that

$$\frac{1}{\sqrt{(x-A)^2 + (y-B)^2 + (z-C)^2}} = \frac{1}{D} \sum_{n \geq 0} \left(\frac{\rho}{D}\right)^n P_n \left(\frac{Ax + By + Cz}{D\rho}\right),$$

where $D^2 = A^2 + B^2 + C^2$, $\rho^2 = x^2 + y^2 + z^2$ and P_n is the Legendre polynomial of degree n .¹ In this way, and since in our case $A = -1, B = C = 0$, equations (3) become

$$\begin{aligned} \ddot{x} - 2\dot{y} - 3x &= \frac{\partial}{\partial x} \sum_{n \geq 3} \rho^n P_n \left(-\frac{x}{\rho}\right), \\ \ddot{y} + 2\dot{x} &= \frac{\partial}{\partial y} \sum_{n \geq 3} \rho^n P_n \left(-\frac{x}{\rho}\right), \\ \ddot{z} + z &= \frac{\partial}{\partial z} \sum_{n \geq 3} \rho^n P_n \left(-\frac{x}{\rho}\right). \end{aligned} \tag{4}$$

The above equations are the starting point of our computations, which follow closely those done for the determination of Lissajous orbits around the collinear libration points of the Restricted Three Body Problem.^{2,3} It is convenient to write down the right-hand side of these equations using functions which can be computed (and stored) in a fast and easy way. This is done by introducing the functions

$$T_n(x, y, z) = \rho^n P_n \left(-\frac{x}{\rho}\right), \quad R_{n-1}(x, y, z) = \frac{1}{y} \frac{\partial T_{n+1}}{\partial y}.$$

Is easy to verify that T_n and R_n are polynomials of degree n satisfying

$$R_{n-1} = \frac{1}{z} \frac{\partial T_{n+1}}{\partial z}, \quad \frac{\partial T_{n+1}}{\partial x} = -(n+1)T_n.$$

Using these relations, equations (4) become

$$\begin{aligned}
\ddot{x} - 2\dot{y} - 3x &= - \sum_{n \geq 2} (n+1)T_n, \\
\ddot{y} + 2\dot{x} &= y \sum_{n \geq 2} R_{n-1}, \\
\ddot{z} + z &= z \sum_{n \geq 2} R_{n-1},
\end{aligned} \tag{5}$$

where T_n and R_n can be obtained using the recurrences that follow from the properties of the Legendre polynomials:

$$\begin{aligned}
T_n(x, y, z) &= \left(\frac{1}{n} - 2\right) x T_{n-1} - \left(1 - \frac{1}{n}\right) \rho^2 T_{n-2}, \\
R_n(x, y, z) &= -\frac{2n+3}{n+2} x R_{n-1} + \frac{2n+2}{n+2} T_n - \frac{n+1}{n+2} \rho^2 R_{n-2},
\end{aligned}$$

starting with: $T_0 = 1$, $T_1 = -x$, $R_0 = -1$, $R_1 = 3x$.

THE LINDSTEDT-POINCARÉ PROCEDURE

If we remove from (5) the non-linear right-hand side terms, the bounded solutions of the linear system obtained are 2π -periodic and can be written as

$$\begin{aligned}
x &= \alpha \cos(t + \phi_1), \\
y &= -2\alpha \sin(t + \phi_1), \\
z &= \beta \cos(t + \phi_2).
\end{aligned} \tag{6}$$

The parameters α , β are the in-plane and out-of-plane amplitudes of the orbit and ϕ_1 , ϕ_2 the phases. When the nonlinear terms of (5) are considered, one can look for formal series solutions expanded in powers of the amplitudes α and β , having (6) as first order terms, of the type

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sum_{i+j=1}^{\infty} \left(\sum_{|k| \leq i, |m| \leq j} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{ijkm} \begin{pmatrix} \cos \\ \sin \\ \cos \end{pmatrix} (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j, \tag{7}$$

where $\theta_1 = \omega t + \phi_1$, $\theta_2 = \omega t + \phi_2$. In order to avoid secular behaviours due to the nonlinear terms, the frequency ω cannot be kept equal to 1 and must be expanded also in powers of the amplitudes

$$\omega = 1 + \sum_{i+j \geq 1} \omega_{ij} \alpha^i \beta^j. \tag{8}$$

The goal is to recurrently compute the coefficients x_{ijkm} , y_{ijkm} , z_{ijkm} , and ω_{ij} up to an arbitrary finite order N , this is: for all the values of i and j such that $1 \leq i + j \leq N$.

It must be remarked that x and z have been both written as a cosine series and y as a sine one. This can be done due to the symmetries of the differential equations of

the problem and to the selected expression for the solution of the linear part. One can see also that if j is odd, the coefficients x_{ijkm} and y_{ijkm} are zero and if j is even, then the coefficients z_{ijkm} are zero. As is usual in the Lindstedt-Poincaré method, we only use coefficients with $|k| \leq i$ and $|m| \leq j$ and, moreover, k and m must be of the same parity as i and j , respectively. Due to the symmetries of the sinus and cosinus, we can assume that $k \geq 0$ and when $k = 0$ we can assume $m \geq 0$. Finally, in the expansion of the frequency ω only terms with both i and j even appear.

Denote by M, N, P the right-hand side terms of the first, second and third equation in (5), respectively, and assume that x, y, z have been obtained to order $n - 1$, so the series M, N and P are known up to order n . Some terms of the left-hand side of (5) are also known: for instance, writing $x = x_1 + x_2 + \dots + x_{n-1}$ and $\omega = 1 + \omega_2 + \omega_4 + \dots + \omega_{n-2}$, where now the subindex denotes the order ($i + j$) of the term, then in

$$\dot{x} = \omega \frac{\partial x}{\partial \theta_1} + \omega \frac{\partial x}{\partial \theta_2},$$

there are known n -order terms such as

$$\omega_2 \left(\frac{\partial x_{n-2}}{\partial \theta_1} + \frac{\partial x_{n-2}}{\partial \theta_2} \right) + \omega_4 \left(\frac{\partial x_{n-4}}{\partial \theta_1} + \frac{\partial x_{n-4}}{\partial \theta_2} \right) + \dots$$

Transferring to the right hand side of (5) the known terms of order n , computed when x, y, z are available at order $n - 1$, we get new right-hand side terms, whose components at order n are denoted by $\overline{M}_n, \overline{N}_n, \overline{P}_n$. In this way we have

$$\begin{aligned} \frac{\partial^2 x_n}{\partial \theta_1^2} + 2 \frac{\partial^2 x_n}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 x_n}{\partial \theta_2^2} - 2 \left(\frac{\partial y_n}{\partial \theta_1} + \frac{\partial y_n}{\partial \theta_2} \right) - 3x_n + \delta_n 2\omega_{n-1} \left(\frac{\partial^2 x_1}{\partial \theta_1^2} - \frac{\partial y_1}{\partial \theta_1} \right) &= \overline{M}_n, \\ \frac{\partial^2 y_n}{\partial \theta_1^2} + 2 \frac{\partial^2 y_n}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 y_n}{\partial \theta_2^2} + 2 \left(\frac{\partial x_n}{\partial \theta_1} + \frac{\partial x_n}{\partial \theta_2} \right) + \delta_n 2\omega_{n-1} \left(\frac{\partial^2 y_1}{\partial \theta_1^2} + \frac{\partial x_1}{\partial \theta_1} \right) &= \overline{N}_n, \\ \frac{\partial^2 z_n}{\partial \theta_1^2} + 2 \frac{\partial^2 z_n}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 z_n}{\partial \theta_2^2} + z_n + \delta_n 2\omega_{n-1} \frac{\partial^2 z_1}{\partial \theta_1^2} &= \overline{P}_n. \end{aligned}$$

with $\delta_n = 0$ if n is even and $\delta_n = 1$ if n is odd. So, the linear system of equations for the unknown terms of order n can be written as

$$\begin{aligned} - (3 + (k + m)^2) x_{ijkm} - 2(k + m) y_{ijkm} + 2\omega_{i-1,j} \delta_{1k} \delta_{0m} &= \overline{m}_{ijkm}, \\ -(k + m)^2 y_{ijkm} - 2(k + m) x_{ijkm} + 2\omega_{i-1,j} \delta_{1k} \delta_{0m} &= \overline{n}_{ijkm}, \\ (1 - (k + m)^2) z_{ijkm} - 2\omega_{i,j-1} \delta_{0k} \delta_{1m} &= \overline{p}_{ijkm}, \end{aligned} \tag{9}$$

where δ_{ij} stands for the Kronecker symbol and $\overline{m}_{ijkm}, \overline{n}_{ijkm}, \overline{p}_{ijkm}$ are the (known) coefficients of $\overline{M}_n, \overline{N}_n$ and \overline{P}_n , respectively.

To solve the above system we start with the last equation, for which we must consider the following two situations:

- If $(k, m) = (0, 1)$ the coefficient in z_{ij01} is zero and we can solve for $\omega_{i,j-1}$. In this way we can compute all the ω_{ij} (recall that i, j must be even). For these values of k and m we set $z_{ij01} = 0$.

- If $(k, m) \neq (0, 1)$ the coefficient of ω_n is zero. In this situation the equation is solved for z_{ijkm} except if $|k + m| = 1$. For this last situation we also set $z_{ijkm} = 0$.

The determination of x_{ijkm} and y_{ijkm} is done as follows. Since all the ω_n are known after solving the last equation, the system to be considered is

$$\begin{aligned} - (3 + (k + m)^2) x_{ijkm} - 2(k + m)y_{ijkm} &= \bar{m}_{ijkm} - 2w_{i-1,j}\delta_{1k}\delta_{0m}, \\ -2(k + m)x_{ijkm} - (k + m)^2 y_{ijkm} &= \bar{n}_{ijkm} - 2w_{i-1,j}\delta_{1k}\delta_{0m}. \end{aligned} \quad (10)$$

The determinant of this reduced linear system is

$$\left((k + m)^2 - 1 \right) (k + m)^2,$$

so we can solve (10) for x_{ijkm} and y_{ijkm} except if $k + m = 0$ or $|k + m| = 1$.

- When $k + m = 0$, we normalise taking $y_{ijkm} = 0$ and solve the first equation for x_{ijkm} .
- When $k + m = -1$, we normalise taking $x_{ijkm} = 0$ and solve for y_{ijkm} .
- When $k + m = 1$:
 - If $(k, m) = (0, 1)$, we take $x_{ij01} = y_{ij01} = 0$.
 - In any other case, we normalise the solution taking $x_{ij10} = 0$ and solve the first equation for y_{ij10} .

The implementation of the above procedure allows a fast computation of the asymptotic Lindstedt-Poincaré series (less than 5 seconds with an Intel Mobile Centrino processor at 1.7 GHz, when the expansions are computed up to order 25). Once the series are available, the computation of a particular orbit for given values of the amplitudes α and β is a trivial fact. In the last section of the paper we will discuss the numerical results obtained.

A SECOND PROCEDURE FOR THE DETERMINATION OF THE LINDSTEDT-POINCARÉ SERIES

In order to avoid the recurrences required for the evaluation of the right hand side of the equations of motion, one can introduce an additional variable s through

$$1 + s = \frac{1}{\sqrt{(x + 1)^2 + y^2 + z^2}}.$$

This equation can also be written (gathering the different terms by orders) as

$$\begin{aligned} \underbrace{2(s + x)}_{\text{linear}} &+ \underbrace{x^2 + y^2 + z^2 + s^2 + 4xs}_{\text{quadratic}} \\ &+ \underbrace{2(x^2 + y^2 + z^2)s + 2xs^2}_{\text{cubic}} + \underbrace{(x^2 + y^2 + z^2)s^2}_{\text{quartic}} = 0. \end{aligned} \quad (11)$$

Using the new variable s , the equations of motion become

$$\begin{aligned}\ddot{x} - 2\dot{y} + 3s &= M, \\ \ddot{y} + 2\dot{x} &= N, \\ \ddot{z} &= P,\end{aligned}\tag{12}$$

to which we must add the algebraic relation (11). The right hand side terms, M , N and P , of (12) are polynomials in x , y , z and s of degree at least 2. Now, we look for the solution, at the different orders, of the system of equations (11) and (12). The type of the series expansions for the functions x , y , z and ω are the same ones as in the previous sections and for s is a serie in cosinus with the same simetries as x .

From the linear terms of equation (11), we find that

$$s_1 = -x_1.$$

Substituting this value in (12) and retaining only the linear terms, we get

$$\begin{aligned}\ddot{x}_1 - 2\dot{y}_1 - 3x_1 &= 0, \\ \ddot{y}_1 + 2\dot{x}_1 &= 0, \\ \ddot{z}_1 &= 0,\end{aligned}\tag{13}$$

whose solution is

$$\begin{aligned}x_1 &= \alpha \cos(t + \phi_1) = -s_1, \\ y_1 &= -2\alpha \sin(t + \phi_1), \\ z_1 &= \beta \cos(t + \phi_2),\end{aligned}\tag{14}$$

Once the linear part is solved, at each step we must solve for the unknown part of order n . Proceeding in an analogous way as in the previous section we get the linear system of equations,

$$\begin{aligned}-(k+m)^2 x_{ijkm} - 2(k+m)y_{ijkm} + 3s_{ijkm} + 2\omega_{i-1,j}\delta_{1k}\delta_{0m} &= \bar{m}_{ijkm}, \\ -(k+m)^2 y_{ijkm} - 2(k+m)x_{ijkm} + 2\omega_{i-1,j}\delta_{1k}\delta_{0m} &= \bar{n}_{ijkm}, \\ (1 - (k+m)^2) z_{ijkm} - 2\omega_{i,j-1}\delta_{0k}\delta_{1m} &= \bar{p}_{ijkm}, \\ x_{ijkm} + s_{ijkm} &= q_{ijkm},\end{aligned}\tag{15}$$

where \bar{m}_{ijkm} , \bar{n}_{ijkm} , \bar{p}_{ijkm} , q_{ijkm} are the (known) terms of the system.

Combining the first equation of (15) with the last one the system to be solved for x_{ijkm} , y_{ijkm} , z_{ijkm} and ω becomes

$$\begin{aligned}-(3 + (k+m)^2) x_{ijkm} - 2(k+m)y_{ijkm} + 2\omega_{i-1,j}\delta_{1k}\delta_{0m} &= \bar{m}_{ijkm} - 3q_{ijkm}, \\ -(k+m)^2 y_{ijkm} - 2(k+m)x_{ijkm} + 2\omega_{i-1,j}\delta_{1k}\delta_{0m} &= \bar{n}_{ijkm}, \\ (1 - (k+m)^2) z_{ijkm} - 2\omega_{i,j-1}\delta_{0k}\delta_{1m} &= \bar{p}_{ijkm},\end{aligned}\tag{16}$$

Note that the left hand side of these system is the same as in the previous section so the discussion about how to solve it follows in the same way. Once system (16) is solved we can set

$$s_{ijklm} = -x_{ijklm} + q_{ijklm}.$$

Of course, the values of the coefficients for the x , y and z series obtained with any of the two procedures described are identical, so there is no convergence improvement for any of both. The second method is much more faster than the first one, since all the recurrences for the Legendre functions are avoided.

NUMERICAL RESULTS

Table 1 shows the non-zero numerical values obtained for the coefficients of the Lindstedt-Poincaré series of the coordinates up to the fourth order. Of course, up to the third order, these values agree with the ones computed by Richardson and Mitchell.⁴

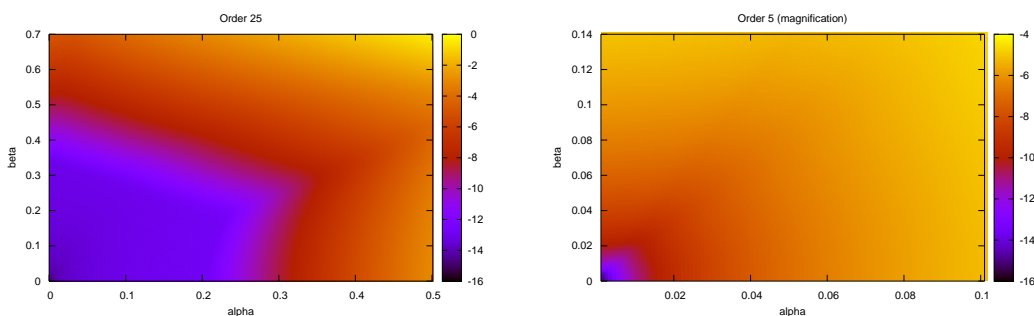


Figure 2 Convergence test results in the α - β plane for the orders 25 (left) and 5 expansions (right).

It has been found that all the coefficients of the frequency series (ω_{20} , ω_{02} , ω_{40} , ω_{22} and ω_{04}) are zero. The same result has been obtained for all the orders up to the maximum one explored ($N = i + j = 35$). This means that the orbits of the two-parametric family obtained are 2π -periodic and, as a consequence, the orbits of the leader and the follower have the same semi-major axis. We have not been able to prove that $\omega_{ij} = 0$ for all i, j .

Some computations have been done to test the convergence of the formal series solutions. Essentially, the tests consist in comparing the numerically integrated trajectories against the series expansions evaluated during a certain time span, varying the values of the amplitudes α and β and the phases ϕ_1 and ϕ_2 . For fixed values of α and β , the convergence results are almost the same for any values of the phases, so here we will show only those corresponding to $\phi_1 = \phi_2 = 0$. In Fig. 2 we give the results obtained for the maximum difference between the numerical integration and the values obtained using the expansions for $t \in [0, 2\pi]$. This error is displayed as a function of the α and β amplitudes. In the figure we display the results for both the order 5 and 25 solutions. Clearly, there is a great improvement in the convergence region of the order 25 solution in front of the order 5.

Figures 3 to 6 show the orbits computed for different values of the amplitudes and the phases. For these representations we have used both the inertial reference system,

in which the orbits are elliptic Keplerian orbits, and Hill's reference frame, which is the system where the orbits have been computed. Of course, the expansions can be evaluated for values of the amplitudes very far from the practical convergence region (see Fig. 6), in this case the shape of the orbit is far from a Keplerian one.

Table 1 Values of the coefficients of the Lindstedt-Poincaré series of the coordinates, up to order four. The values that do not appear in the table are all equal to zero.

i	j	k	m	x_{ijklm}	y_{ijklm}	z_{ijklm}
1	0	1	0	1.000000	-2.000000	
0	1	0	1			1.000000
2	0	0	0	-0.500000	0.000000	
2	0	2	0	0.500000	0.250000	
0	2	0	0	-0.250000	0.000000	
0	2	0	2	-0.250000	0.250000	
1	1	1	-1			1.500000
1	1	1	1			-0.500000
3	0	1	0	0.000000	1.125000	
3	0	3	0	-0.375000	-0.291666	
1	2	1	-2	0.000000	0.375000	
1	2	1	0	0.000000	-0.000000	
1	2	1	2	0.125000	-0.125000	
2	1	0	1			0.000000
2	1	2	-1			0.000000
2	1	2	1			0.375000
0	3	0	1			0.000000
0	3	0	3			-0.000000
4	0	0	0	0.359375	0.000000	
4	0	2	0	-0.708333	-0.604166	
4	0	4	0	0.348958	0.302083	
2	2	0	0	-0.374999	0.000000	
2	2	0	2	0.250000	0.3.1250	
2	2	2	-2	-0.343750	0.000000	
2	2	2	0	0.187500	0.000000	
2	2	2	2	-0.093749	0.093749	
0	4	0	0	-0.062500	0.000000	
0	4	0	2	-0.062500	0.062500	
0	4	0	4	0.000000	0.000000	
3	1	1	-1			0.000000
3	1	1	1			0.312500
3	1	3	-1			0.020833
3	1	3	1			-0.333333
1	3	1	-3			-0.187499
1	3	1	-1			0.937499
1	3	1	1			-0.124999
1	3	1	3			0.000000

CONCLUSIONS

In this paper we have presented two different procedures, based on Lindstedt- Poincaré method, in order to obtain high order expansions of the bounded solutions of the nonlinear Hill-Clohessy-Wiltshire equations. Both procedures allows to compute them in a fast way

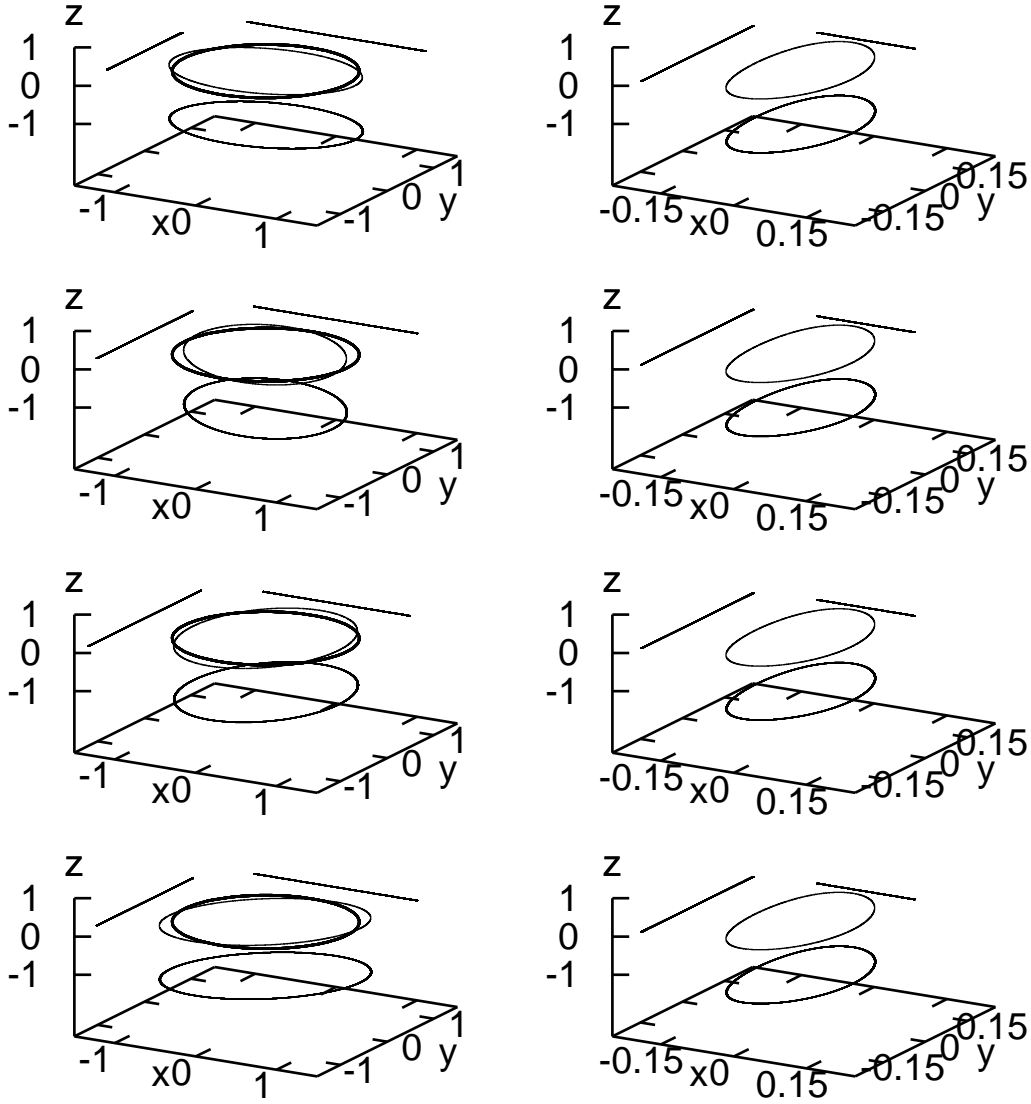


Figure 3 From top to bottom, 3D representation and coordinate planes projections of orbits with the same amplitudes, $\alpha = 0.1$, $\beta = 0$, and different phases: $(\phi_1, \phi_2) = (0, 0)$, $(\pi/2, 0)$, $(\pi, 0)$, and $((3/4)\pi, 0)$. The orbits on the left column are represented in the inertial reference frame and the ones on the right column in the Hill's reference frame. In the inertial frame representations, the circular reference orbit of radius $r_l = 1$ is also displayed. As is clearly seen in the projections, the orbit of the follower is coplanar with the one of the leader, since $\beta = 0$ in all the cases.

for a wide range of amplitudes. It has been found that all the orbits obtained have the same period so the periodic solutions of the equations of relative motion are elliptical orbits with the same semi-major axis as the reference reference orbit.

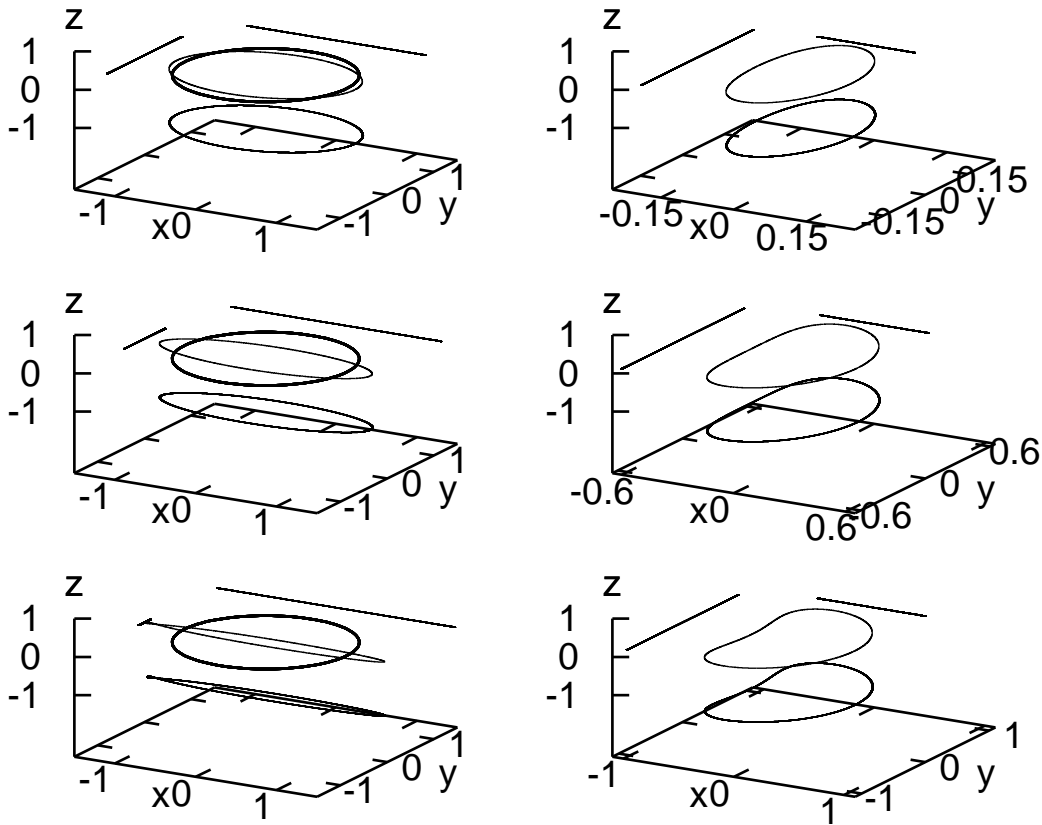


Figure 4 From top to bottom, 3D representation and coordinate planes projections of orbits with $(\alpha, \beta) = (0.1, 0)$, $(0.3, 0)$, $(0.5, 0)$, and $(\phi_1, \phi_2) = (0, 0)$. The orbits on the left column are represented in the inertial reference frame and the ones on the right column in the Hill's reference frame. In the inertial frame representations, the circular reference orbit of radius $r_l = 1$ is also displayed. Note that the orbit with $(\alpha, \beta) = (0.5, 0)$ is not in the convergence domain of the expansions, but is rather close to a Keplerian orbit.

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REFERENCES

1. M. Abramowitz, I.A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., 1972.

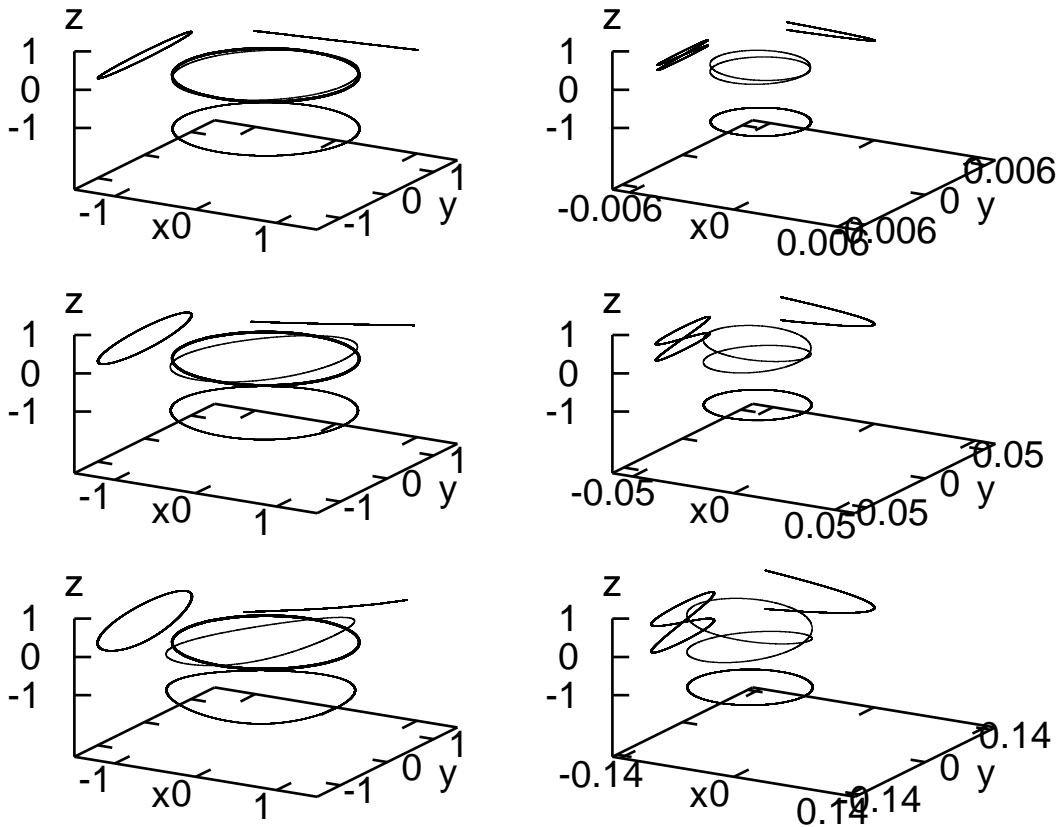


Figure 5 From top to bottom, 3D representation and coordinate planes projections of orbits with $(\alpha, \beta) = (0.0, 0.1)$, $(0.0, 0.3)$, $(0.0, 0.5)$, and $(\phi_1, \phi_2) = (0, 0)$. The orbits on the left column are represented in the inertial reference frame and the ones on the right column in the Hill's reference frame. In the inertial frame representations, the circular reference orbit of radius $r_l = 1$ is also displayed.

2. G. Gómez, À. Jorba, J.J. Masdemont, and C. Simó. *Dynamics and Mission Design Near Libration Point Orbits – Volume 3: Advanced Methods for Collinear Points*. World Scientific, 2001.
3. J.J. Masdemont. “High Order Expansions of Invariant manifolds of Libration Point Orbits with Applications to Mission Design”. *Dynamical Systems: An International Journal*, 20 (1) 59–113, 2005.
4. D.L. Richardson and J.W. Mitchell. “A third-Order Analytical Solution for Relative Motion with a Circular Reference Orbit”. *The Journal of the Astronautical Sciences*, 51 (1) 1–12, 2003.
5. H. Schaub and J.L. Junkins. *Analytical Mechanics of Aerospace Systems*. AIAA Education Series, 2003.

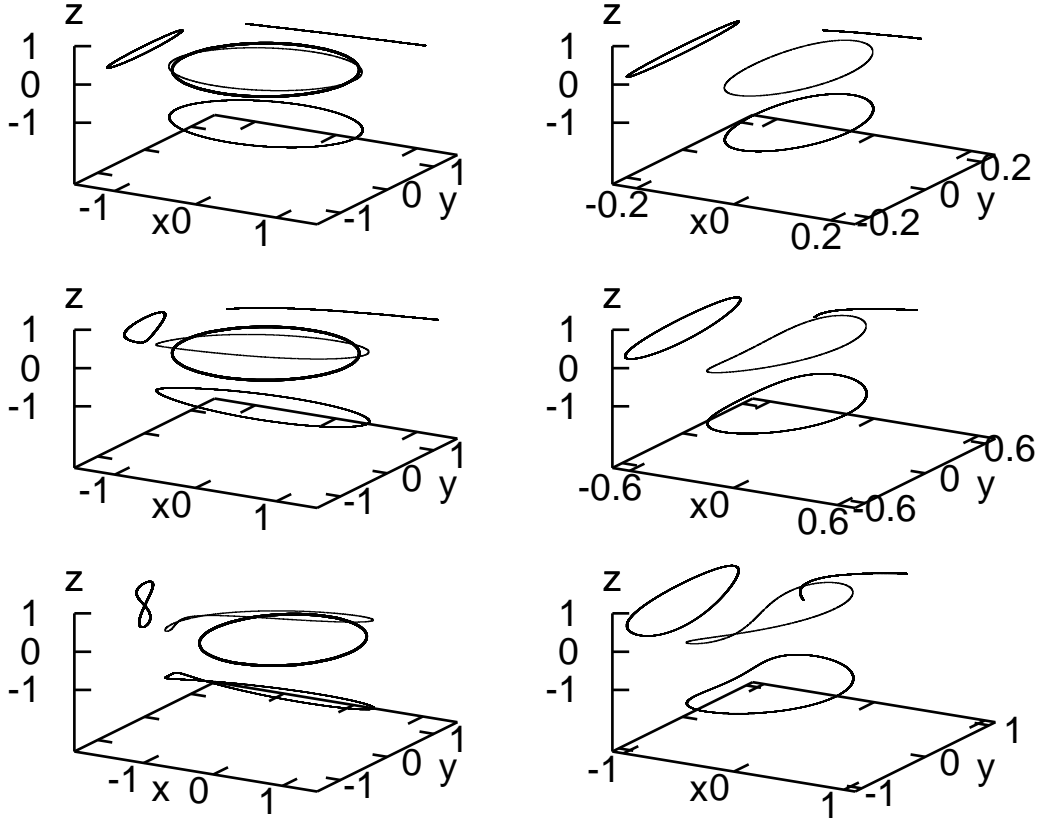


Figure 6 From top to bottom and left to right, 3D representation and coordinate planes projections of orbits with $(\alpha, \beta) = (0.1, 0.1)$, $(0.3, 0.3)$, $(0.5, 0.5)$ and $(\phi_1, \phi_2) = (0, 0)$. The circular reference orbit of radius $r_l = 1$ is also displayed. The orbits on the left column are represented in the inertial reference frame and the ones on the right column in the Hill's reference frame. Clearly, the orbit computed for $(\alpha, \beta) = (0.5, 0.5)$ is not a Keplerian orbit, since for these values of the amplitudes the expansions are far from the practical convergence region.