# Existence of Herman Rings for Meromorphic Functions

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**Abstract**

We apply the Shishikura surgery construction to transcendental maps in order to obtain examples of meromorphic functions with Herman rings, in a variety of possible arrangements. We give a sharp bound on the maximum possible number of such rings that a meromorphic function may have, in terms of the number of poles. Finally we discuss the possibility of having "unbounded" Herman rings (i.e., with an essential singularity in the boundary), and give some examples of maps with this property.

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# 1. Introduction

Given a map $f$ defined on the complex plane $\mathbb{C}$, the sequence formed by its iterates is denoted by $f^0 := \mathrm{Id}$, $f^n := f \circ f^{n-1}$, $n \in \mathbb{N} = \{1, 2, \dots\}$. The grand orbit of a point $z$ under $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ consists of all points $z' \in \widehat{\mathbb{C}}$ whose orbits eventually intersect the orbit of $z$. It is well known (see e.g. [18]) that for a holomorphic map of the Riemann sphere a maximum of two points are allowed to have a finite grand orbit. Those points are called *exceptional values* and always belong to the stable or Fatou set (see below). We shall denote by $E(f)$ the set of exceptional values of $f$.

When one considers iteration of a holomorphic self map $f : X \to X$ where $X$ is a Riemann surface, the study makes sense and is non-trivial when $X$ is either the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the complex plane minus one point $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, sets that should be viewed as $\widehat{\mathbb{C}}$ minus one and two points (exceptional values) respectively, normalized to be $\infty$ and 0. If we allow the essential singularities to have preimages, i.e. if the map has poles, then we are in the class of meromorphic maps.

More precisely, in this paper we deal with the following classes of maps (partially following [4]).

$\mathcal{R} = \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f$ is rational of degree at least two$\}$.

$\mathcal{E} = \{f : \mathbb{C} \to \mathbb{C} \mid f$ is transcendental entire$\}$.

$\mathcal{P} = \{f : \mathbb{C}^* \to \mathbb{C}^* \mid f$ is transcendental holomorphic$\}$.

$\mathcal{M} = \{f : \mathbb{C} \to \widehat{\mathbb{C}} \mid f$ is transcendental meromorphic and has at least one pole which is not omitted$\}$. Functions in $\mathcal{M}$ have one single essential singularity. This class is usually called the general class of meromorphic functions (see [3]).

$\mathcal{K} = \{f : \widehat{\mathbb{C}} \setminus B \to \widehat{\mathbb{C}} \mid B$ is a compact countable set and $f$ is meromorphic$\}$. The set $B$ is formed by the essential singularities of $f$. We assume $B$ to have at least two elements and $f$ to have poles so that $\mathcal{K}$ and $\mathcal{P}$ are disjoint. Notice that if $f$ has more than two essential singularities, then it must have infinitely many poles.

We observe that the classes above are disjoint. Furthermore we want to distinguish two types of maps in the class $\mathcal{P}$. More precisely we write $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where

$$\begin{aligned} \mathcal{P}_1 &= \{f \in \mathcal{P} \mid \{0\} \text{ is a pole and an omitted value}\} \\ &= \{f(z) = \tfrac{e^{h(z)}}{z^n},\ h(z) \text{ entire non-constant}, n \in \mathbb{N}\}. \end{aligned}$$

$$\begin{aligned} \mathcal{P}_2 &= \{f \in \mathcal{P} \mid \{0\} \text{ is an essential singularity}\} \\ &= \{f(z) = z^n e^{g(z)+h(1/z)},\ g, h \text{ entire non-constant}, n \in \mathbb{Z}\}. \end{aligned}$$

Thus the subscripts 1 and 2 stand for one and two essential singularities respectively.

If $f$ is a map in any of the classes above and we denote by $X$ its domain of definition, the *Fatou set* $F(f)$ consists of all points $z \in X$ such that the sequence of iterates of $f$ is well defined and form a normal family in a neighbourhood of $z$. The *Julia set* is its complement, $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.

Classes $\mathcal{R}$ and $\mathcal{E}$ are classical and were initially studied by P. Fatou and G. Julia, and later by many authors. Introductions to the subject are the books by Beardon [5], Carleson and Gamelin [10], Milnor [18] and Steinmetz [22].

Functions in $\mathcal{P}$ and $\mathcal{M}$ have been studied more recently. For a general survey including all the above classes we refer the readers to Bergweiler in [4]. Class $\mathcal{K}$ has been introduced by A. Bolsch in [6, 7]. Independently, broader generalizations have been given by M. Herring in [14].

Many properties of $J(f)$ and $F(f)$ are much the same for all classes above but different proofs are needed and some discrepancies arise. For any of these maps (except for class $\mathcal{K}$ for which we refer to [7]) we recall that the Fatou set $F(f)$ is open and therefore, by definition, the Julia set $J(f)$ is closed; the Julia set is perfect and non-empty; the sets $J(f)$ and $F(f)$ are completely invariant under $f$; for $z_0$ any non exceptional point, $J(f)$ coincides with the backward orbit of $z_0$; and finally the repelling periodic points are dense in $J(f)$.

The possible dynamics of a periodic connected component $U$ of the Fatou set of $f$ (i.e. $f^p(U) \subset U$, for some $p \geq 1$), are completely understood and classified. The cases that will be of interest in this paper will be the following.

1. There exists an analytic homeomorphism $\phi$: $U \to \mathbb{D}$ such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $U$ is called a Siegel disc.
2. There exists an analytic homeomorphism $\phi$: $U \to A_{1,r}$ where $A_{1,r}$ is an annulus $A_{1,r} = \{z : 1 < |z| < r\}, r > 1$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $U$ is called a Herman ring.

If $U$ is any invariant component of $F(f)$ (that is if $p = 1$), then $U$ must have connectivity 1,2, or $\infty$ [4]. If the connectivity is two, then $U$ is a Herman ring for all the classes except for the class $\mathcal{K}$ in which we can either have a Herman ring or a tongue over itself (see [8] for details).

In this paper, we deal with the existence of Herman rings for functions in $\mathcal{M}$ and $\mathcal{K}$. We first recall some related results concerning the previous classes.

It is well known that Herman rings do not exist if $f$ is a polynomial or if $f \in \mathcal{E}$ (this is a consequence of the maximum principle). Although Fatou and Julia originally conjectured that Herman rings did not exist for any rational map, this was proven to be false by Herman in [12] who gave an example by extending earlier work of Arnold in [2]. Later on, Shishikura in [21] used quasiconformal surgery to construct rational functions with Herman rings. We shall make extensive use of his construction all throughout the paper.

For functions in $\mathcal{P}_2$ it is known [3, 16] that $F(f)$ can have at most one multiply-connected component and the connectivity of such component is two. Examples of Herman rings for these maps were constructed by Herman in [13] and Baker in [3], by studying the complex Arnold family.

On the other hand, functions in $\mathcal{P}_1$ may not have Herman rings, as it was shown in [11] by applying the argument principle.

Although Herman rings have not been ruled out for functions in $\mathcal{M}$ or $\mathcal{K}$, no examples have been given until very recently in [15], where a meromorphic function with infinitely Herman rings was constructed (independently of this work).

Our first result gives a bound on the possible number of invariant Herman rings that a meromorphic map may have. More precisely we have the following statement.

For any doubly connected set $H \subset \mathbb{C}$ we denote by $B(H)$ the bounded component of $\widehat{\mathbb{C}} \setminus H$.

THEOREM A. *Let $f \in \mathcal{M} \cup \mathcal{R}$ and $H$ be an invariant Herman ring of $f$. Suppose that $f$ has $n$ invariant Herman rings $H_1, \ldots, H_n$ in $B(H)$, where $n \in \mathbb{N}$ could possibly be 0. Then, the* *set* $B(H) \setminus \cup_{i=1}^{n}(H_n \cup B(H_n))$ *contains at least one pole of* $f$*. Equivalently, no two Herman rings of* $f$ *are homotopic in* $\mathbb{C} \setminus \{p \in \mathbb{C} : p$ *is a pole of* $f\}$*.*

COROLLARY. *If a map* $f \in \mathcal{M} \cup \mathcal{R}$ *has* $N > 0$ *poles, then* $f$ *may have at most* $N$ *Herman rings.*

Furthermore we give examples of maps that realize this maximum. More precisely, we adapt Shishikura's construction to transcendental maps to obtain the following example.

THEOREM B. *Given any* $N > 0$*, there exists* $f \in \mathcal{M}$ *with exactly* $N$ *poles and* $N$ *invariant Herman rings.*

We shall see that Theorem B follows from a straight forward application of Shishikura's construction, giving an example of a meromorphic map with $N$ Herman rings which are not nested. One could then ask if an example of $N$ nested Herman rings for a map with $N$ poles exists. It turns out that such an example may be constructed. More precisely

THEOREM C. *Given any* $N > 0$*, there exists* $f \in \mathcal{M}$ *with exactly* $N$ *poles and* $N$ *invariant nested Herman rings. Moreover, the rotation number of each of the rings may be any prescribed Brjuno number.*

In fact we will see that combining the proofs of both theorems, these rings could be arranged in any configuration.

Concerning maps in class $\mathcal{K}$, Theorem $A$ is still true if we add the possibility of having an essential singularity, instead of only a pole, "in between" Herman rings. About the existence of maps in $\mathcal{K}$ with Herman rings, it will become clear from the proof of Theorem B that such examples exist. We remark that if $f$ has three or more essential singularities we automatically must have infinitely many poles (or else we would have three exceptional values which is not possible). Then examples with infinitely many invariant Herman rings could be constructed.

A different set of questions is raised by studying the boundary of Herman rings for transcendental maps. It is well known (see [20]) that, for appropriate values of the parameter $\lambda = \mathrm{e}^{2\pi \mathrm{i}\theta}$, the maps $E_\lambda(z) = \lambda \mathrm{e}^z$ have a Siegel disc which is unbounded. This is quite remarkable since the boundary of such disc must then be highly non-locally connected, something never encountered (up to now) for rational maps. Furthermore, this happens for a set of angles $\theta$ of positive measure.

A natural question is to ask if the same situation may occur for Herman rings, that is, if there exist unbounded Herman rings. Since the difficulty lies on having an essential singularity in the boundary of the rotation domain (which does not need to be necessarily at infinity), the word "unbounded" must be understood correctly in this context. To be precise we give the following definitions.

DEFINITION. Let $f \in (\mathcal{P}_2 \cup \mathcal{M} \cup \mathcal{K})$ and assume $H \subset \mathbb{C}$ is a Herman ring of $F(f)$. We say that $H$ is *unbounded on one side* if an essential singularity of $f$ lies on one of the connected components of the boundary of $H$. If both connected components contain an essential singularity of $H$ we say that $H$ is *doubly unbounded.*

It is then clear that maps in $\mathcal{M}$ can only have rings unbounded on one side while maps in $\mathcal{P}_2$ or $\mathcal{K}$ could, a priori, have doubly unbounded Herman rings. Using a similar surgery construction as in the proofs of Theorems B and C we construct the following examples.

THEOREM D. *a) There exists* $f \in \mathcal{M}$ *such that* $F(f)$ *has a Herman ring which is unbounded on one side.*

*b) There exists $f \in \mathcal{K}$ such that $F(f)$ has a doubly unbounded Herman ring.*

We observe that all examples in Theorem D concern functions with poles. To our knowledge, there is no known example of an unbounded Herman ring (of either type) for a map without poles (i.e., a map in $\mathcal{P}_2$). This question is very related to the problem of finding an unbounded Siegel disc for a map in the family $\lambda z e^z$.

As a final application of Shishikura's surgery construction we give a simple proof of the following Proposition, which was left as an open question in [22, Chapter 4, p.112]. Here $\mathcal{B}$ denotes the set of Brjuno irrational numbers. (We refer to [17] for a precise definition, although we note that $\mathcal{B}$ contains all diophantine numbers).

PROPOSITION E. *Let $F_{\alpha,a,b}(z) = e^{i\alpha} z \frac{z-a}{1-\overline{a}z} \frac{z-b}{1-\overline{b}z}$. Then, for any $\rho \in \mathcal{B}$, the parameters $\alpha$, $a$ and $b$ can be adjusted so that $F$ has two Siegel discs and one Herman ring, all of rotation number $\rho$.*

M. Herman in [12] proved the existence of parameters for which $F_{\alpha,a,b}(z)$ has two Siegel discs and a Herman ring. His proof did not involve surgery and it is not clear to us that the rotation number of all domains is the same one.

The paper is organized as follows. In Section 2 we explain the surgery construction (following [21]) that we shall use to prove Theorems A to D. We write it in a general setting of two arbitrary meromorphic maps. In Section 3 we apply this construction to prove Theorems A, B and C. Section 4 refers to maps in class $\mathcal{K}$, Section 5 contains the proof of Theorem D and Section 6 has the proof of Proposition E.

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## 2. The main construction

Our goal in this section is to explain the surgery construction which shall be used in the proofs of our results. The construction is basically in [21], where Shishikura constructs rational functions with Herman rings, by combining several rational functions with Siegel discs. We will adapt it to meromorphic maps, trying to be more precise where the differences arise and also at points that were not obvious to us initially.

We start with any two maps $f$ and $\widetilde{f}$ which may belong to any of the classes $\mathcal{R}$, $\mathcal{E}$, $\mathcal{M}$ or $\mathcal{P}_1$, that is, maps that have at most one essential singularity. We suppose that $z = 0$ is a fixed point of $f$ and $\widetilde{f}$, linearizable in both cases and with opposite rotation numbers. That is to say that $f$ has an invariant Siegel disc $S$ around 0 of rotation number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\widetilde{f}$ has an invariant Siegel disc $\widetilde{S}$ around 0 of rotation number $-\theta$.

The idea of the construction is as follows. We define two sets $B \subset S$ and $\widetilde{B} \subset \widetilde{S}$ to be open sets around 0, bounded by invariant curves $\gamma \subset S$ and $\widetilde{\gamma} \subset \widetilde{S}$ respectively. We start in the dynamical plane of $f$ and define a new map $g$ which equals the old map $f$ everywhere except in $B$. In the set $B$ we define $g$ to have the dynamics of $\widetilde{f}$ in $\widehat{\mathbb{C}} \setminus \widetilde{B}$, and we do it by means of a quasi-conformal "gluing" map $\Psi$. The new map $g$ has a ring $R$ of invariant curves formed by the two leftover pieces of $S$ and $\widetilde{S}$ glued together. Furthermore, $g$ reflects the dynamics of $f$ in the unbounded component of $\mathbb{C} \setminus R$ and the dynamics of $\widetilde{f}$ in the bounded component of $\mathbb{C} \setminus R$. But the combination is even richer than that since, for example, $g$ has poles at those points where $\widetilde{f}$ has zeros. This ends the topological part of the surgery construction. As usual, $g$ is not holomorphic but only quasi-regular. We proceed to make $g$ meromorphic by constructing a g-invariant, bounded, almost complex structure $\sigma$. We then apply the Measurable Riemann Mapping Theorem to obtain a quasi-conformal homeomorphism $\varphi$ such that the map $F := \varphi \circ g \circ \varphi^{-1}$ is meromorphic, and $F$ has the desired dynamics.

One should think of the map $F$ as the two spheres (one for $f$, say $S_f$, and one for $\widetilde{f}$, say $S_{\widetilde{f}}$) glued or melted together through their respective Siegel discs, to form one unique sphere, say $S_F$ where the map $F$ is defined (see Figure 1). In particular, the point at infinity in $S_{\widetilde{f}}$ corresponds to the point at zero in the new glued sphere, while $\infty$ on $S_f$ remains as such in $S_F$. The dynamics of the new map can be studied very easily if we think in this geometric way. The two remaining parts of the Siegel disks are now forming a ring in the new sphere. On the left hand side (the one corresponding to $S_f$), all those points previously mapped inside the Siegel disk, are now mapped either to the Herman ring or to $S_{\widetilde{f}}$ while the opposite happens on the right hand side.

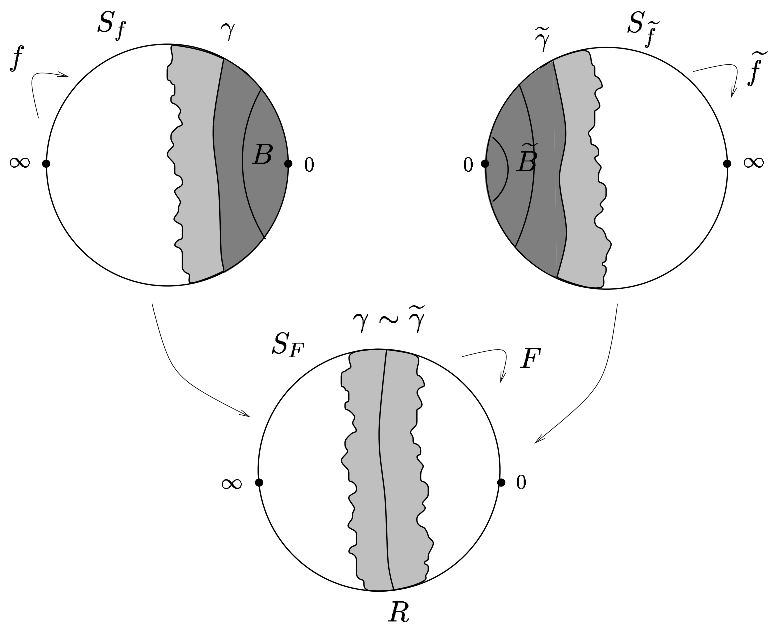


Figure 1: The geometric idea of the main construction.

We now proceed to make this construction precise.

Let $\phi : S \to \mathbb{D}$ be a conformal isomorphism which conjugates $f$ on $S$ to $R_\theta(z) := e^{i\theta}z$ on $\mathbb{D}$, and such that $\phi(0) = 0$. Likewise, let $\widetilde{\phi} : \widetilde{S} \to \mathbb{D}$ be a conformal isomorphism which conjugates $\widetilde{f}$ on $\widetilde{S}$ to $R_{-\theta}(z)$ on $\mathbb{D}$, and such that $\widetilde{\phi}(0) = 0$.

Consider the circles $C_r$ and $C_{\widetilde{r}}$, where $r$ and $\widetilde{r}$ are two arbitrary numbers in $(0, 1)$. Then, the curve $\gamma = \phi^{-1}(C_r)$ (resp. $\widetilde{\gamma} = \widetilde{\phi}^{-1}(C_{\widetilde{r}})$) is a real analytic Jordan curve in $S$ (resp. $\widetilde{S}$) invariant by $f$ (resp. $\widetilde{f}$). Observe that $\gamma$ (resp. $\widetilde{\gamma}$) cuts the Riemann sphere into two connected components which are topological discs. We denote by $B$ (resp. $\widetilde{B}$) the connected component that contains the fixed point 0.

We proceed now to define the gluing map $\Psi$, by first defining it on $\gamma$ and then extending it to the whole sphere. Consider the Möbius map $L(z) = \frac{r\widetilde{r}}{z}$. Clearly, $L$ is an orientation reversing map on $C_r$ (to $C_{\widetilde{r}}$). We then define the real analytic map $\Psi_0 : \gamma \to \widetilde{\gamma}$ to be the map that makes the following diagram commute.

$$\begin{array}{ccc} \gamma & \xrightarrow{\Psi_0} & \widetilde{\gamma} \\ \phi \big\downarrow & & \big\downarrow \widetilde{\phi} \\ C_r & \xrightarrow{L} & C_{\widetilde{r}} \end{array}$$

It then follows that, for all $z \in \gamma$,

$$\Psi_0(f(z)) = \widetilde{f}(\Psi_0(z)), \tag{1}$$

since

$$\Psi_0 \circ f = \widetilde{\phi}^{-1} \circ L \circ \phi \circ f = \widetilde{\phi}^{-1} \circ L \circ R_\theta \circ \phi = \widetilde{\phi}^{-1} \circ R_{-\theta} \circ L \circ \phi = \widetilde{f} \circ \widetilde{\phi}^{-1} \circ L \circ \phi = \widetilde{f} \circ \Psi_0,$$

where we have used that $L \circ R_\theta = R_{-\theta} \circ L$ (see Figure 2).

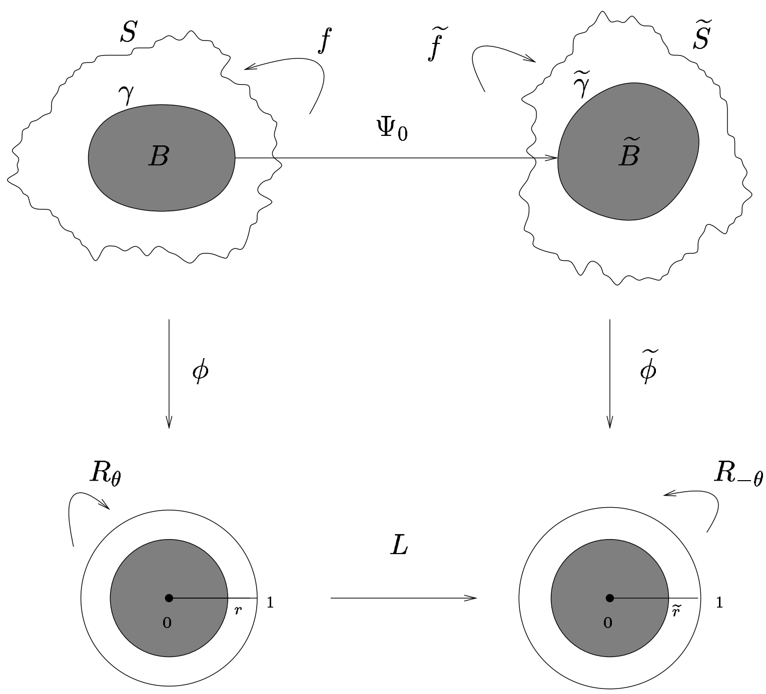


Figure 2: The definition of $\Psi_0 : \gamma \to \widetilde{\gamma}$.

We now want to extend $\Psi_0$ to a globally defined map $\Psi$ which should be conformal in a domain as large as possible. This map $\Psi$ will be the "gluing map".

The following lemma is due to Shishikura in [21]. We include the details of his proof which were not inmediately obvious to us.

LEMMA 1. *There exists a quasi-conformal homeomorphism* $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *satisfying*

*(i)* $\Psi(\gamma) = \widetilde{\gamma}$;

*(ii)* $\Psi \circ f = \widetilde{f} \circ \Psi$ *on* $\gamma$;

*(iii)* $\Psi(B) = \widehat{\mathbb{C}} \setminus \overline{\widetilde{B}}$; $\Psi(\widehat{\mathbb{C}} \setminus \overline{B}) = \widetilde{B}$;

*(iv )* $\Psi$ *is conformal in a neighbourhood, say* $U$, *of* $\widehat{\mathbb{C}} \setminus (S \cap \Psi^{-1}(\widetilde{S}))$.

Note that $S \cap \Psi^{-1}(\widetilde{S})$ is the annulus that will become the Herman ring. Hence $\Psi$ is conformal everywhere in $\widehat{\mathbb{C}}$ except in a neighbourhood of $\gamma$, included in this rotation domain.

To prove Lemma 1 we will use the following interpolation lemma.

LEMMA 2. *Let $A$ and $\widetilde{A}$ be annular domains bounded by real analytic Jordan curves $\gamma^{(i)}, \gamma^{(o)}$, $\widetilde{\gamma}^{(i)}, \widetilde{\gamma}^{(o)}$ respectively, where (i) (resp. (o)) stands for inner boundary (resp. outer boundary). Let $f^{(i)} : \gamma^{(i)} \to \widetilde{\gamma}^{(o)}$ and $f^{(o)} : \gamma^{(o)} \to \widetilde{\gamma}^{(i)}$ be two real analytic diffeomorphisms which are orientation reversing. Then, there exists $f : A \to \widetilde{A}$ quasi-conformal such that $f|_{\gamma^{(i)}} = f^{(i)}$ and $f|_{\gamma^{(o)}} = f^{(o)}$.*

*Proof.* The proof of this lemma is very standard and we shall give only a sketch of it, leaving the details for the reader.

First observe that the condition of the maps reversing the orientation and the outer and inner boundaries is not really relevant. Indeed, we may compose the map with a reflection with respect to the core curve in the annulus and this condition would be removed.

Now, the extension is easily constructed in the universal covering space of the annuli which is a pair of vertical strips $S$ and $\widetilde{S}$. As long as we have homeomorphisms between the boundaries, we may define a map $F$ between the two strips which linearly interpolates between the two (see Figure 3). It is not hard to check that, if these boundary maps are $\mathcal{C}^1$ diffeomorphisms, then the Jacobian of $F$ is never zero. Moreover such map would project to a $\mathcal{C}^1$ map $f$ between the annuli $A$ and $\widetilde{A}$. Since any $\mathcal{C}^1$ map on a compact set is quasi-conformal, we have that $f$ is the desired extension. ☐

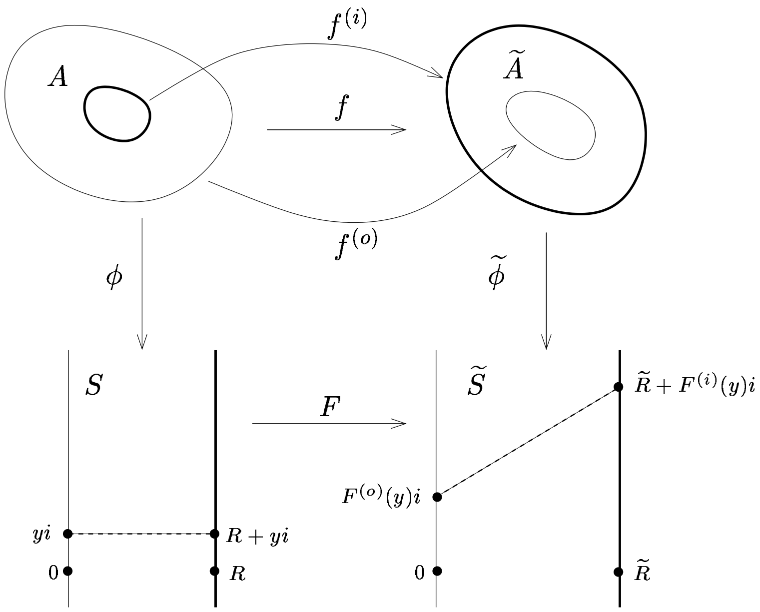


Figure 3: Sketch of the proof of Lemma 2: definition of the interpolating map $F$. The maps $\phi$ and $\widetilde{\phi}$ are the universal covering maps, one of them composed with a reflection with respect to the core curve ( $r(x + \mathrm{i}y) = -x + R + \mathrm{i}y$) to reverse the order of the boundaries.

REMARK 2.1. We observe that it is sufficient to require that the boundary functions be $\mathcal{C}^1$.

*Proof.* (of Lemma 1)

Given any set $A \subset \widehat{\mathbb{C}}$ we denote by $A^c$ the set $\widehat{\mathbb{C}} \setminus A$. Take $S$, $\widetilde{S}$, $B$, $\widetilde{B}$, $\gamma$ and $\widetilde{\gamma}$ as in the introduction of this section. Define $\widehat{\Psi}_1$ to be a conformal map such that $\widehat{\Psi}_1(B) = \left(\overline{\widetilde{B}}\right)^c$ and $\widehat{\Psi}_1(0) = \infty$. Likewise let $\widehat{\Psi}_2$ be a conformal map such that $\widehat{\Psi}_2(\overline{B})^c = \widetilde{B}$ and $\widehat{\Psi}_2(\infty) = 0$. Observe that the maps $\widehat{\Psi}_1$, $\widehat{\Psi}_2$ extend to orientation reversing diffeomorphisms on $\gamma$, like $\Psi_0$, which send $\gamma \to \widetilde{\gamma}$ (see Figure 4).

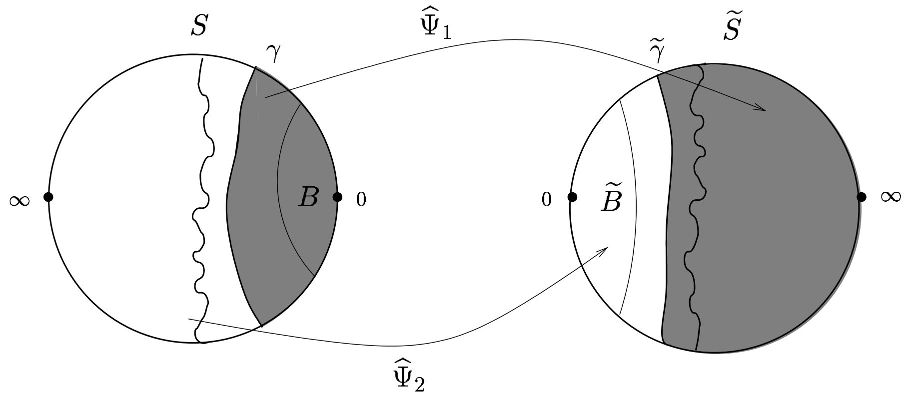


Figure 4: Definition of $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ .

We shall modify $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ so that both agree with $\Psi_0$ on $\gamma$. The map defined by

$$\Psi = \begin{cases} \Psi_1 & \text{on } \overline{B} \\ \Psi_2 & \text{on } B^c, \end{cases} \tag{2}$$

where $\Psi_1$ modifies $\widehat{\Psi}_1$ and $\Psi_2$ modifies $\widehat{\Psi}_2$, will be not only continuous but also conformal in some neighbourhood, say $U$, of $\widehat{\mathbb{C}} \setminus (S \cap \Psi^{-1}(\widetilde{S}))$.

By Lemma 2 we can modify $\widehat{\Psi}_2$ and $\widehat{\Psi}_1$. To modify $\widehat{\Psi}_2$ pick some $s$ such that $0 < r < s < 1$ and take the annulus $A_{r,s} = \{z : r < |z| < s\}$. Let $C_r$ be the circle of radius $r$ and $C_s$ be the circle of radius $s$. The annulus $A_{r,s}$ is sent to an annulus $A$ by $\phi^{-1}$, with inner boundary $\gamma = \phi^{-1}(C_r)$ and outer boundary $\eta := \phi^{-1}(C_s)$. See Figure 5. The new map $\Psi_2$ will coincide with $\widehat{\Psi}_2$ everywhere outside $A$. On $A$, we define $\Psi_2$ as the result of applying Lemma 2 to the annulus $A$ with boundary values $\Psi_0$ on $\gamma$ and $\widehat{\Psi}_2$ on $\eta$. In a similar way but in the opposite side we can modify $\widehat{\Psi}_1$, more precisely in an annulus $A' = \phi^{-1}(\{s' < |z| < r\})$, $s'$ sufficiently close to $r$ to ensure that $\widetilde{\eta}' = \widehat{\Psi}_1(\eta')$ is inside $\widetilde{S}$, where $\eta = \phi^{-1}(C_{s'})$ (see Figure 5). Thus we have new maps $\Psi_2$ and $\Psi_1$ which are equal to $\widehat{\Psi}_2$ on $B^c \setminus A$ and $\widehat{\Psi}_1$ on $\overline{B} \setminus A'$.

The new $\Psi$ (defined as in (2)) is analytic everywhere except on $A \cup A'$ , hence the region of analyticity is an open set containing $S^c \cup \Psi^{-1}(\widetilde{S})^c$. □

Now as in [21] we define a mapping $g : \mathbb{C} \to \widehat{\mathbb{C}}$ by

$$g = \begin{cases} f & \text{on } \widehat{\mathbb{C}} \setminus B \\ \Psi^{-1} \circ \widetilde{f} \circ \Psi & \text{on } B. \end{cases}$$

The map $g$ leaves $\gamma$ invariant since $\Psi|_\gamma = \Psi_0$. Note also that $g$ is continuous and quasi-regular since $\Psi$ is quasi-conformal. Observe that on the set $R := S \cap \Psi^{-1}(\widetilde{S})$, $g$ is conjugate to the rotation $R_\theta$.

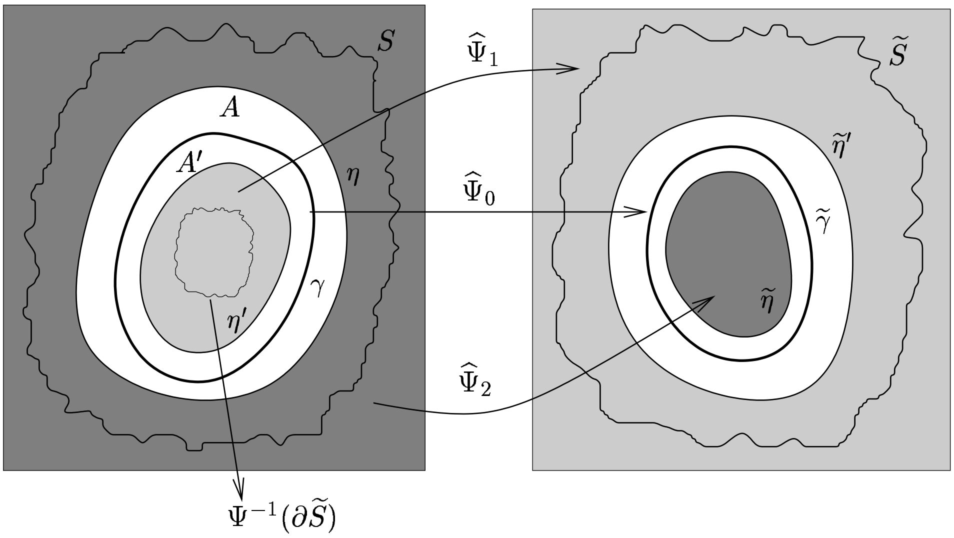


Figure 5: The global definition of $\Psi$.

We proceed to obtain a meromorphic map with the dynamics of $g$ by constructing a $g$-invariant bounded almost complex structure $\sigma$. Let $\sigma_0$ denote the standard complex structure defined by infinitessimal circles. We define $\sigma$ on $R$ as follows.

$$\sigma = \begin{cases} \sigma_0 & \text{on } S \setminus B \\ \Psi^*\sigma_0 & \text{on } B \cap \Psi^{-1}(\widetilde{S}) \end{cases}$$

where $\Psi^*\sigma_0$ denotes the pull back of $\sigma_0$ by the map $\Psi$, defined except on a null set. We now extend $\sigma$ to all of $\mathbb{C}$ by the dynamics of $g$, in the following recursive way:

$$\sigma = \begin{cases} (g^n)^*(\sigma) & \text{on } g^{-n}(R),\ n \geq 1 \\ \sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus \cup_{n\geq 0}\, g^{-n}(R). \end{cases}$$

The distortion of $\sigma$ with respect to $\sigma_0$ is uniformly bounded since $g$ is meromorphic outside $R$. By the Measurable Riemann Mapping Theorem [1] there exists a quasi-conformal mapping $\varphi$ of $\widehat{\mathbb{C}}$ fixing 0 and $\infty$ such that $\varphi^*\sigma_0 = \sigma$ a.e.

The final step is to define $F := \varphi \circ g \circ \varphi^{-1}$. Observe that $F$ preserves the standard complex structure everywhere except at the poles of $g$ which are the previous zeroes of $f$ (see Section 2.1), and hence a discrete set of points. Hence $F$ is meromorphic and quasi-conformally conjugate to $g$. The set $\varphi(R)$ is an invariant Herman ring of $F$.

In the case of choosing $\widetilde{f}(z) = \overline{f(\overline{z})}$ we are in fact "reflecting" with respect to $\gamma$.

## 2.1 Properties of the resulting meromorphic function

Up to this point we have constructed a new map $F$ which reflects the combined dynamics of $f$ and $\widetilde{f}$ on the Riemann sphere, and which has an invariant Herman ring. Now we would like to study the properties of $F$, in terms of its poles, critical points, zeros and essential singularities. For a map $h$ in any of the classes defined in the introduction (except for class $\mathcal{K}$), let $Z(h), P(h), CP(h), ES(h)$ and $AV(h)$ denote respectively the sets of zeros, poles, critical points, essential singularities and asymptotic values.

PROPOSITION 2.2. *Let $F$, $f$ and $\widetilde{f}$ be as in the main construction. Then*

*(i)* $\#Z(F) = \#Z(f) + \#P(\widetilde{f})$;

*(ii)* $\#P(F) = \#P(f) + \#Z(\widetilde{f}) - 1$;

*(iii)* $\#CP(F) = \#CP(f) + \#CP(\widetilde{f})$;

*(iv)* $\#ES(F) = \#ES(f) + \#ES(\widetilde{f})$ *and*

*(v)* $\#AV(F) = \#AV(f) + \#AV(\widetilde{f})$.

*Proof.* Using the geometric idea given at the begining of the main construction it is easy to check that the first two statements are true. Indeed, all points on the left hand side of the new sphere $S_F$ that were previously mapped to 0 are now still mapped to 0. On the right hand side (corresponding to $S_{\widetilde{f}}$) points that were mapped to 0 (except 0) are now mapped to infinity and hence they became poles of the new map $g$. On the contrary, the "old" poles of $\widetilde{f}$ are now zeroes of $g$ while the poles of $f$ remain as such. Since $\varphi$ is a homeomorphism, the number of zeroes and poles is the same for $g$ than for $F$ (except for one).

Alternatively, we can use the equations that define $F$,

$$F = \begin{cases} \varphi \circ f \circ \varphi^{-1} & \text{on } \varphi(\widehat{\mathbb{C}} \setminus B) \\ \varphi \circ \Psi \circ \widetilde{f} \circ \Psi^{-1} \circ \varphi^{-1} & \text{on } \varphi(B). \end{cases}$$

and the fact that $\varphi$ is a homeomorphism that fixes 0 and $\infty$ and $\Psi$ is a homeomorphism that reverses the roles of these two points, to check that $F(z) = \varphi f \varphi^{-1}(z) = 0$ if and only if $f(z) = 0$ on $\widehat{\mathbb{C}} \setminus B$. On the other hand we have that $F(z) = \varphi \Psi \widetilde{f} \Psi^{-1} \varphi^{-1}(z) = 0$ if and only if $\Psi(z)$ is a pole of $\widetilde{f}$. Thus the number of zeros of $F$ is the number of zeros of $f$ plus the number of poles of $\widetilde{f}$. Likewise, one can check that the poles of the new map $F$ are in one to one correspondence with the old poles of $f$ and the zeroes of $\widetilde{f}$.

To see statement (*iii*) one should observe that being a critical point is equivalent to having local degree larger than one, and this is a topological property. Equivalently, note that the gluing map $\Psi$ is holomorphic everywhere except in a small annulus surrounding the invariant curve $\gamma$ which is completely contained in the rotation domain. Hence, $g$ is a quasi-regular map which is holomorphic everywhere outside this annulus. Since the Siegel disks cannot contain critical points, the critical points of $g$ are exactly those points that are critical for $f$ or those that are preimages under $\Psi$ of critical points of $\widetilde{f}$. Now $\varphi$ is a homeomorphism that conjugates $g$ and $F$ and hence the critical points of $g$ and $F$ are in one-to-one correspondence.

In the case of essential singularities, recall that we allowed $f$ and $\widetilde{f}$ to have at most one essential singularity at $\infty$. Since $f$ is not modified around infinity, the new map $F$ has an essential singularity at $\infty$ if $f$ had it. In the case that $\widetilde{f}$ has such a singularity at $\infty$, now $F$ will have it at zero. Hence the resulting map $F$ can have a maximum of two essential singularities, one at 0 and one at $\infty$.

If $f$ or $\widetilde{f}$ have any asymptotic values as in (*iv*) it is easy to see that the new map will also have them (or their images under $\varphi$ or $\varphi \circ \Psi^{-1}$) as asymptotic values. □

REMARK 2.3. It is easy to check that the surgery construction we just described is commutative, up to composition by $1/z$. For convenience we will always take as $f$ the map having an essential singularity at $\infty$ (if any), so that the resulting map is already normalized.

As we saw already in the proof above, the properties of the newly constructed map will completely depend on the properties of the starting maps $f$ and $\widetilde{f}$. Shishikura's construction was made with two rational maps which resulted in a new rational map. But one can see that meromorphic (transcendental) maps can be easily obtained, by choosing the pair $(f, \widetilde{f})$ (not necessarily in this order) in $(\mathcal{E}, \mathcal{R}_p)$, $(\mathcal{P}_1^{(*)}, \mathcal{R}_p)$, $(\mathcal{M}, \mathcal{R}_p)$, $(\mathcal{E}, \mathcal{R})$, $(\mathcal{P}_1^{(*)}, \mathcal{R})$ or $(\mathcal{M}, \mathcal{R})$, where $\mathcal{R}_p$ denotes a polynomial and $\mathcal{P}_1^{(*)}$ denotes a map in $\mathcal{P}_1$ modified so that the omitted pole is not at $z = 0$. Observe that all these pairs consist of a transcendental map plus a non-transcendental one. Indeed, if we "glue" two maps both having an essential singularity, we shall obtain a map in either $\mathcal{P}_2$ or $\mathcal{K}$ (see Section 4).

Finally, we observe that we excluded maps in $\mathcal{P}$ from the construction, since such maps cannot have a Siegel disk at 0. If we normalize them some other way (gluing the two spheres through some other point, or puting the second essential singularity somewhere else), the same construction goes through, and we will obtain in general a map in class $\mathcal{K}$.

# 3. Meromorphic functions

## 3.1 Proof of Theorem A

For any doubly connected set $H \in \mathbb{C}$ we recall that $B(H)$ denotes the bounded component of $\widehat{\mathbb{C}} \setminus H$. We first recall the statement of Theorem A.

THEOREM A. *Let $f \in \mathcal{R} \cup \mathcal{M}$ and $H$ be an invariant Herman ring of $f$. Suppose that $f$ has $n$ invariant Herman rings $H_1, \ldots, H_n$ in $B(H)$, where $n \in \mathbb{N}$ could possibly be 0. Then, the set $B(H) \setminus \cup_{i=1}^{n}(H_n \cup B(H_n))$ contains at least one pole of $f$.*

*Proof.* We say that $H_i$ is nested inside $H_j$ if $H_i \subset B(H_j)$. After renaming the rings if necessary, assume that $H_1, \ldots, H_k$, $k < n$, are the ones which are not nested inside any other ring than $H$.

Let $\gamma$ be an invariant Jordan curve inside $H$, and $\gamma_1, \ldots, \gamma_k$ invariant Jordan curves in $H_1, \ldots, H_k$ respectively. We denote by $U$ the bounded region whose boundary consists of $\Gamma := \gamma \cup \gamma_1 \cup \cdots \cup \gamma_k$ (see Figure 6). We will show that $U$ contains at least a pole of $f(z)$.

Suppose that $f(z)$ is analytic in $U$. Since $\Gamma$ is invariant under $f$, it follows from the Maximum Principle that $U$ is invariant under $f$. Hence $f^n(U) = U$ and so $U \subset F(f)$. But $U$ contains points of $\partial H \cup \partial H_1, \cup \cdots \cup \partial H_n$ which belong to the Julia set. Hence $f$ cannot be analytic in $U$. □

## 3.2 Construction of examples

We will construct three examples of meromorphic functions by using the main construction in Section 2.

### 3.2.1 Meromorphic map with one Herman ring and one pole which is not omitted.

The first and simplest application of the main surgery construction is to " glue" an entire transcendental map with a Siegel disc of rotation number $\theta$, to a polynomial of degree 2 with a Siegel disc of rotation number $-\theta$. The result will be a meromorphic map with one pole which is not omitted. More precisely we prove the following.

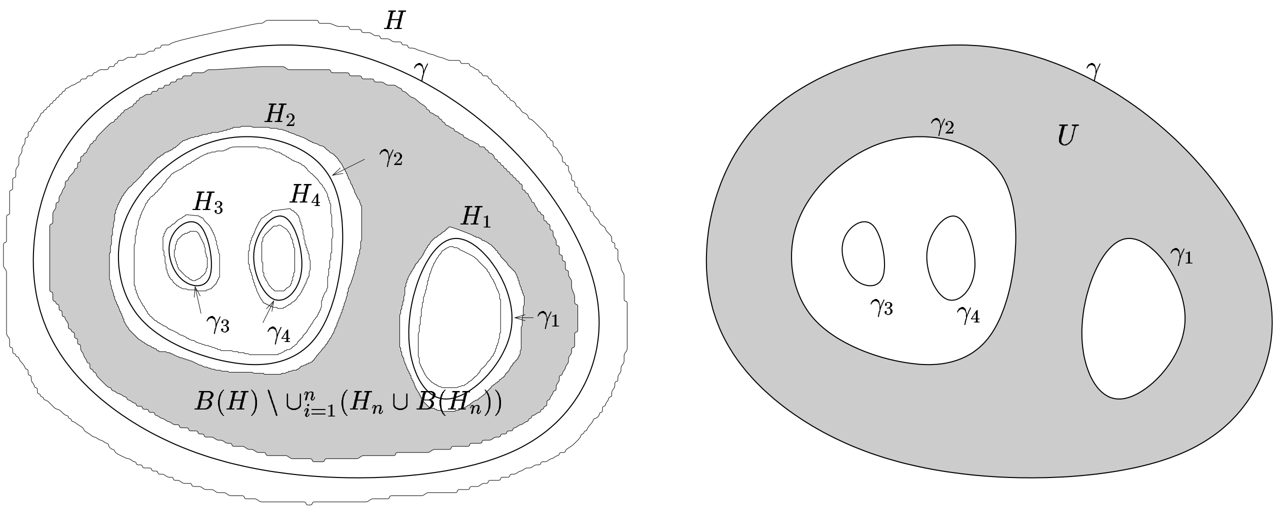


Figure 6: Ilustration of Theorem A and its proof for $k = 2$ and $n = 4$.

PROPOSITION 3.4. *There exist values $a, b \in \mathbb{C} \setminus \{0\}$ for which the meromorphic map*

$$F(z) = az^2 \frac{e^{bz}}{z+1},$$

*has a Herman ring.*

To construct such a map let

$$f(z) = e^{2\pi i\theta} z e^z \quad \text{and} \quad \widetilde{f}(z) = e^{-2\pi i\theta} z(1+z)$$

where $\theta \in \mathcal{B}$ and $\mathcal{B}$ denotes the set of Brjuno irrational numbers.

It is easy to check that $f \in \mathcal{E}$ has a fixed point at $z = 0$ of multiplier $e^{2\pi i\theta}$. Since $\theta \in \mathcal{B}$, $f$ has a Siegel disk around $z = 0$ of rotation number $\theta$. Note that $z = 0$ is also an asymptotic value and has no other preimage than itself. The point $z = -1$ is the only critical point of $f$. We will make extensive use of this function (known as the *semistandard map*) throughout the section, using it as a building block for many of the constructions.

The map $\widetilde{f}$ is a quadratic polynomial with a fixed point at $z = 0$ of derivative $e^{-2\pi i\theta}$. Again, since $\theta \in \mathcal{B}$, the map $\widetilde{f}$ has a Siegel disk around zero of rotation number $-\theta$. The critical point of $\widetilde{f}$ is at $z = -i/2$, and the other zero is at $z = -1$.

Gluing the two maps together as in the main construction, we obtain a new map $F$ which has

(a) one essential singularity which is at $\infty$;

(b) one single pole (since $f$ has no poles and $\widetilde{f}$ has only one zero (different from 0)). This pole is not omitted because $z = -1$ is not omitted by $\widetilde{f}$;

(c) a Herman ring.

Thus $F$ is a map in class $\mathcal{M}$ which has a Herman ring.

To find an expression for $F$, we assume that we normalized the integrating map $\varphi$ to fix not only $\infty$ and 0 but also $z = -1$. Then this point is the non-omitted pole of $F$. Observe also that the origin is superattracting, and also an asymptotic value. Hence, using that any nonzero function can be written as the exponential of an entire map, we conclude that $F$ must be of the form

$$F(z) = az^2 \frac{\mathrm{e}^{k(z)}}{z+1}$$

for some value of $a \in \mathbb{C} \setminus \{0\}$ and some entire function $k(z)$. Since $\varphi$ is quasiconformal, it satisfies some Hölder condition and hence

$$\max_{|z|=r} \mathrm{e}^{k(z)} = \mathrm{e}^{Ar^N}$$

for some $A, N > 0$ as $r \to \infty$. Therefore $k(z)$ must be a polynomial. Now observe that we have not modified (topologically) the map around infinity. That is, a neighborhood of $\infty$ is still divided only into two regions: one on which $F \to 0$ and another on which $F \to \infty$. Thus $k$ must be a linear polynomial, i.e., $k(z) = bz$ for some $b \in \mathbb{C} \setminus \{0\}$.

### 3.2.2 Meromorphic map with $N$ poles and $N$ Herman rings which are not nested (Theorem B).

This case is a straight forward generalization of the example above and it is how Shishikura [21] described his construction originally (with rational maps). One needs to start with a transcendental entire map with $N$ Siegel discs of rotation numbers $\theta_1, \ldots, \theta_n$ and then 'glue' each of them to a polynomial of degree 2 (as in the example above) with a Siegel disk with the adequate rotation number. This is also a special case of the construction in [15] where they prove the existence of infinitely many Herman rings for a function in class $\mathcal{M}$, starting with a map with infinitely many Siegel disks.

Note that we obtain $N$ (not nested) Herman rings, each one with a single pole inside. The map has no other poles.

### 3.2.3 Meromorphic map with $N$ poles and $N$ Herman rings which are nested (Theorem C).

The construction of this example is slightly more involved than the previous ones. We shall make an inductive construction which, at each step, adds a new Herman ring and a new pole.

To obtain nested Herman rings we need to choose as $\widetilde{f}$ a map having more than one Siegel disc, so that, after converting one of them into a Herman ring, there is still one left (inside) in order to repeat the main construction again. That would be acomplished, choosing a polynomial of degree greater than 2. However, such a choice would add too many poles to the resulting map; precisely at least two simple poles (or one double) for every Herman ring.

A possible solution consists of taking as $\widetilde{f}$ a degree two map which adds, one Herman ring plus the leftover Siegel disk to continue the construction. In this way, at each step we add one pole but also one ring. Of course such a map cannot be a polynomial; it has to be a rational map. More precisely, we take as $\widetilde{f}$ a degree two map with two Siegel discs. Such a map is

$$\widetilde{f}_{a,b}(z) = z\frac{z+a}{1+bz},$$

where $a$ and $b$ are in the unit circle. A simple computation shows that $\widetilde{f}_{a,b}(0) = a$ and $\widetilde{f}_{a,b}(\infty) = b$. Hence if we chose $a = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ and $b = \mathrm{e}^{2\pi\mathrm{i}\beta}$ with $\alpha$ and $\beta$ Brjuno numbers, then $\widetilde{f}_{a,b}$ has two Siegel discs at 0 and $\infty$ of rotation number $\alpha$ and $\beta$ respectively. For later counting, observe that $\widetilde{f}$ has two zeroes and one pole.

As usual, on the left we take the semistandard map

$$f(z) = e^{2\pi i\theta} z e^{z},$$

where $\theta \in \mathcal{B}$ and $\mathcal{B}$ denotes the set of Brjuno irrational numbers. As it was mentioned in the example of Section 3.2.1, $f$ is an entire transcendental map which has a Siegel disc around the fixed point $z = 0$. This point is also an asymptotic value and has no other preimage than itself.

We proceed now to glue $f$ and $\widetilde{f}$. Observe that the two Siegel discs at $z = 0$ melt to form a Herman ring. The Siegel disc of $\widetilde{f}$ at $\infty$ remains as a Siegel disc at $z = 0$ for the new map. Hence, the resulting map $F_1$ has one ring plus one Sigel disc at $z = 0$ (see Figure 7). Moreover, $F_1$ has only one pole, coming from the nonzero root of $\widetilde{f}$.

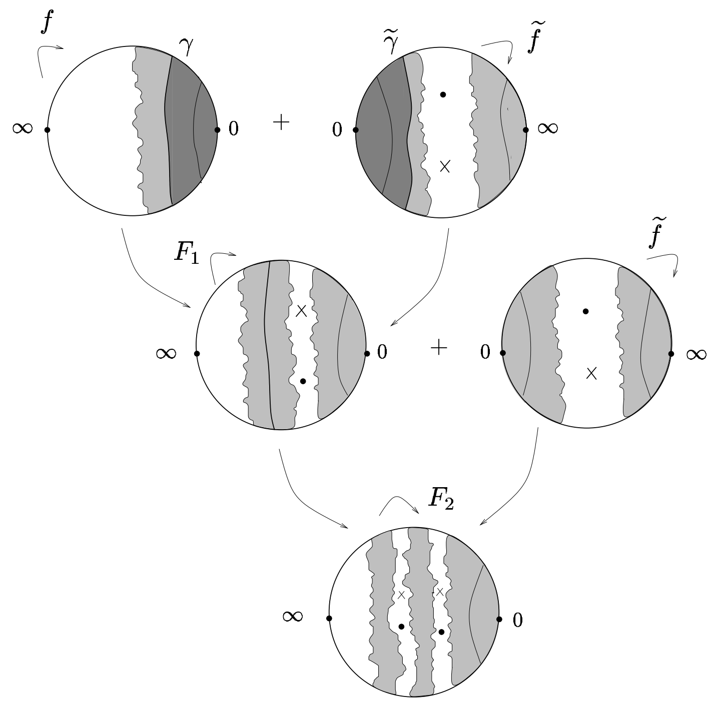


Figure 7: The recursive construction in Theorem C. Dotes and crosses denote the zeroes and poles respectively.

We may now repeat the construction again. That is, we can glue $F_1$ on the left to the same map $\widetilde{f}$ on the right. It is easy to check that we obtain a new map $F_2$ with 2 Herman rings, 2 poles and a Siegel disk at $z = 0$ which may be used to continue the construction if necessary (see Figure 7). Thus after $n$ steps we obtain a meromorphic function with $n$ Herman rings which are nested, $n$ poles and still a Siegel disc around zero.

REMARK 3.5. If we take $\widetilde{f}$ as in Proposition E ($\widetilde{f}$ has two Siegel discs and one Herman ring) and $f$ as above, afer $n$ steps of the construction we obtain a meromorphic function $F_n$ with $2n$ Herman rings which are nested, $2n$ poles and still a Siegel disc around zero. If we desire an odd number of rings, say $N = 2n + 1$, we would take the map $F_n$ from above and glue it to a polynomial of degree 2 as in the first example. This final step would add one new ring nested inside the others, and one new pole. Observe that no Siegel disc would be left to continue the procedure.

## 4. Functions in class $\mathcal{K}$

Up to this point we have always obtained maps that were rational, entire or meromorphic. As we already mentioned in Section 2.1, this is a result of carefully choosing the pairs of starting maps.

For clarity's sake, we wanted to separate maps in class $\mathcal{K}$ from the main construction, both as starting or resulting maps. However, once we are familiar with the constructions in the sections above, it is not difficult to check that the same main construction would go thorugh. Even if one or both of the starting maps have a countable infinity of essential singularities.

Clearly, combining a map in class $\mathcal{K}$ with any other map would always result in a map in the same class since the essential singularities would remain. It is interesting however to see which combinations of other types would give as a result a map in class $\mathcal{K}$. We summarize these observations in the following table, leaving the details for the reader.

| $f \setminus \tilde{f}$ | $Pol$ | $\mathcal{R}$ | $\mathcal{E}$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{M}$ | $\mathcal{K}$ |
|---|---|---|---|---|---|---|---|
| $Pol$ | $\mathcal{R}$ | | | | | | |
| $\mathcal{R}$ | $\mathcal{R}$ | $\mathcal{R}$ | | | | | |
| $\mathcal{E}$ | $\mathcal{M}$ | $\mathcal{M}$ | $\mathcal{P}_2, \mathcal{K}$ | | | | |
| $\mathcal{P}_1$ | $\mathcal{M}$ | $\mathcal{M}$ | $\mathcal{K}$ | $\mathcal{K}$ | | | |
| $\mathcal{P}_2$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | | |
| $\mathcal{M}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | |
| $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ | $\mathcal{K}$ |

REMARK 4.6. By the commutativity of the construction (see Remark 2.3) it is only necessary to fill up half of the table. As usual, we take the "most complicated" map to be on the left, so that the result is already normalized. Notice also that maps in $\mathcal{P}_1$ and $\mathcal{P}_2$ need to be composed with a translation that moves the pole or essential singularity away from 0, before making the construction. If necessary, the result must be normalized again.

## 5. Unbounded Herman rings. Proof of Theorem D

Recall that a Herman ring $H$ is unbounded if there is an essential singularity contained in one of the components of $\partial H$. If only one component contains such a point, we call $H$ unbounded on one side, while $H$ is double side unbounded (or doubly unbounded) if this occurs to both.

As we mentioned in the introduction, there is no known example of a map in $\mathcal{P}_2$ (i.e., maps with no preimages of the essential singularities) with a Herman ring which is unbounded in either side. However, the construction above can be used to construct examples of meromorphic maps which have Herman rings unbounded on one side (double unboundness would be impossible), or maps in $\mathcal{K}$ with doubly unbounded rings. This is exactly the statement of Theorem D. In what follows we shall prove this theorem and give an expression for such maps.

The building block for both constructions is the map

$$f(z) = e^{2\pi i\theta}(e^z - 1)$$

which, for suitable values of $\theta$, has a Siegel disc around 0 which is unbounded. Indeed, it was shown in [20] that this is the case for a set of $\theta \in S^1$ of full measure, and later in [9]

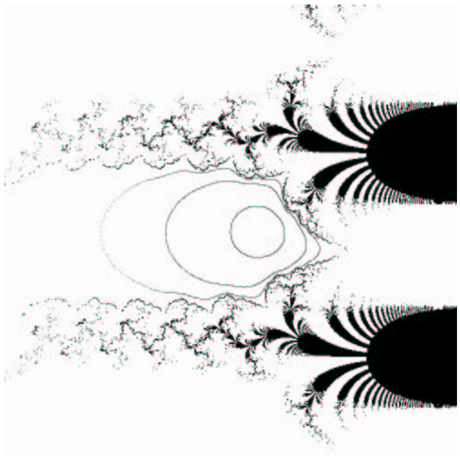

Figure 8: An unbounded Siegel disc of the function $f(z) = \lambda(\mathrm{e}^z - 1)$ where $\lambda = \mathrm{e}^{2\pi\mathrm{i}\frac{\sqrt{5}+1}{2}}$.

that these angles contain the set $\mathcal{H}$ of irrational numbers (which in particular includes all diophantines and is included in $\mathcal{B}$, see [19] for a precise definition). See Figure 8.

Observe that $f$ has no critical points but has one asymptotic value at $-\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$ which is an omitted value.

## 5.1 Example of a meromorphic map with an unbounded Herman ring

The following proposition proves the first part of Theorem D.

Proposition 5.7. *Given any $\theta \in \mathcal{H}$, there exist constants $c, p \in \mathbb{C}$ such that if $\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$, the meromorphic function*

$$F(z) = -\lambda\left(-1 + \frac{(1+cp)z - p}{z-p}\mathrm{e}^{cz}\right),$$

*has an unbounded Herman ring.*

*Proof.* Let us consider the maps

$$f(z) = \lambda(\mathrm{e}^z - 1) \quad \text{and} \quad \widetilde{f}(z) = \lambda^{-1}z(1+z).$$

The choice of $\theta \in \mathcal{H}$ assures that both maps have a Siegel disc around 0, and moreover, the Siegel disk of $f$ is unbounded.

If we perform the surgery construction to glue $f$ and $\widetilde{f}$ we obtain a new map $F$ which has a Herman ring. Notice that the Herman ring must be unbounded since the original Siegel disc of $f$ was. Indeed, since $\varphi$ is a homeomorphism that fixes $\infty$, the point at infinity must belong to the "outer" boundary of the ring $H$.

We now proceed to find an expression for the map $F$. Assume that we normalized the integrating map $\varphi$ to fix not only $0, \infty$ but also $-\lambda$. Observe that $-\lambda$ is an asymptotic value for $f$ which is omitted. Thus, its image $-\lambda$ is an asymptotic value for $F$, although in this case it is not omitted. Clearly it has no preimage on the "left side" (i.e., on $\varphi(\mathbb{C} \setminus B)$) but now it has exactly one preimage on the "right side" (i.e., on $\varphi(B)$), more precisely at the unique point $\varphi(z)$ where $\widetilde{f}\Psi(z) = \Psi(\underset{\sim}{-\lambda}) \in \widetilde{B}$ (see page 8 for the definition of $\Psi$). We note also that $F$ has one single pole (since $\widetilde{f}$ has one single zero apart from $z = 0$). Thus $F$ must be of the form

$$F(z) = -\lambda + \frac{a + bz}{z - p} e^{k(z)},$$

for some constant $a, b, p \in \mathbb{C}$ and some entire function $k(z)$. Arguing as in Proposition 3.4 we deduce that $k(z)$ must be a linear polynomial, i.e., $k(z) = cz$ for some $c \in \mathbb{C}$. Providing that $F(0) = F'(0) = 0$ we obtain that $a = -\lambda p$ and $acp + bp + a = 0$. Hence

$$F(z) = -\lambda \left( -1 + \frac{(1 + cp)z - p}{z - p} e^{cz} \right).$$

□

### 5.2 Example of a function in class $\mathcal{K}$ with a doubly unbounded Herman ring

To construct this example we shall glue the maps

$$f(z) = \lambda(e^z - 1) \qquad \text{with} \qquad \widetilde{f} = \lambda^{-1}(e^z - 1),$$

where as before $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathcal{H}$ so that both maps have an unbounded Siegel disc. This can be thought as gluing a reflection of $f$ inside its Siegel disc.

After performing the main surgery construction, we obtain a map $F$ with a Herman ring which is unbounded in both sides, since 0 and $\infty$ are now both essential singularities and they lie respectively on the inner and outer boundaries of the new ring.

To see the type of map $F$ is we observe that $F$ has two essential singularities at 0 and $\infty$ together with infinitely many preimages of both points. Indeed, $f$ and $\widetilde{f}$ have infinitely many zeroes (exactly $\{z_k = 2k\pi i\}_{k\in\mathbb{Z}}$) which will correspond under $\varphi$ or $\varphi \circ \Psi^{-1}$ to zeroes and poles of $F$.

## 6. Proof of Proposition E

PROPOSITION E. *Let* $F_{\alpha,a,b}(z) = e^{i\alpha} z \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z}$. *Then, for any* $\rho \in \mathcal{B}$, *the parameters* $\alpha$, $a$ *and* $b$ *can be adjusted so that* $F$ *has two Siegel discs and one Herman ring, all of rotation number* $\rho$.

*Proof.* To obtain a rational map that is symetric with respect to the unit circle, we shall glue a rational map to itself. Since we want the final result to have a Siegel disc at 0 and $\infty$, naturally the initial map must have a Siegel disc at $\infty$. Hence we consider the maps $f_\rho$ and $f_{-\rho}$ where

$$f_\rho(z) = e^{2\pi i\rho} z \frac{z + 1}{z - 1},$$

and $\rho \in \mathcal{B}$. Observe that $f_\rho$ has fixed points at $z = 0, \infty$, critical points at $z = 1 \pm \sqrt{2}$, zeros at $z = 0, -1$ and a pole at $z = 1$. The multiplier of both fixed points 0 and $\infty$ is $e^{2\pi i\rho}$. Hence taking $\rho \in \mathcal{B}$ implies that $f$ has two Siegel discs one around each fixed point. Now by using the main construction in Section 2 we can "glue" $f_\rho$ to $f_{-\rho}$ (see Figure 7). The resulting map is a rational function $F$ (since $F$ cannot have any essential singularity) which has two Siegel discs, one around zero and the other around $\infty$, and one Herman ring, all with rotation number $\rho$. Clearly, $F$ is symetric, hence the unit circle is invariant and the Herman ring must be around $S^{-1}$. By Proposition 2.2, the new function has two poles, two zeros and four critical points (see Figure 9). A map with these properties must necessarily be a Blaschke product of degree three, hence of the form $\widetilde{f}_{\alpha,a,b}$, for some values of $\alpha$, $a$ and $b$. □

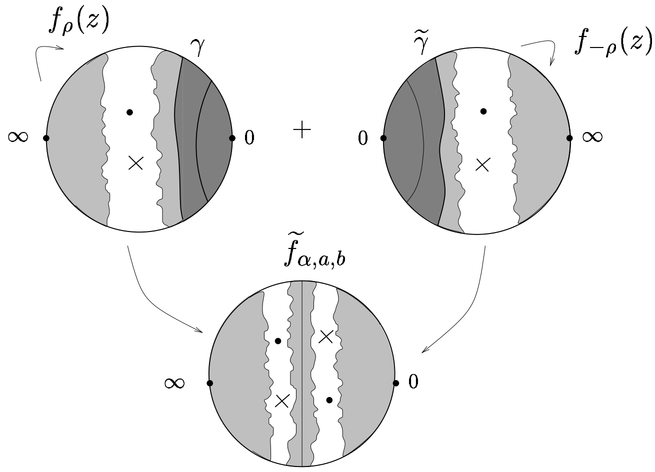


Figure 9: The construction of $\widetilde{f}_{\alpha,a,b}$.

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