SPATIAL *p-q* RESONANT ORBITS OF THE RTBP

ESTHER BARRABÉS[1] and GERARD GÓMEZ[2]
[1]*Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Av. Lluís Santaló, s/n, 17071, Girona, Spain, e-mail: barrabes@ima.udg.es*
[2]*IEEC & Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007, Barcelona, Spain, e-mail: gomez@cerber.mat.ub.es*

(Received: 14 August 2001; accepted: 19 February 2002)

Abstract. The purpose of this paper is to extend the study of the so called *p-q* resonant orbits of the planar restricted three-body problem to the spatial case. The *p-q* resonant orbits are solutions of the restricted three-body problem which have consecutive close encounters with the smaller primary. If E, M and P denote the larger primary, the smaller one and the infinitesimal body, respectively, then p and q are the number of revolutions that P gives around M and M around E, respectively, between two consecutive close approaches. For fixed values of p and q and suitable initial conditions on a sphere of radius μ^α around the smaller primary, we will derive expressions for the final position and velocity on this sphere for the orbits under consideration.

Key words: restricted three-body problem, close encounters, resonance

1. Introduction

As it is well known, the spatial circular restricted three-body problem (RTBP) describes the motion in the three-dimensional space of a massless particle, P, under the attraction of two bodies, E and M, called primaries, which move in circular orbits around their center of mass. The motion of P does not affect the primaries. Using suitable units, the primaries can be assumed to have masses $1-\mu$ and μ with $\mu \in [0, 1]$ and to complete one inertial revolution in 2π time units. The variable μ, which is the mass of the smaller primary M, is called the *mass parameter*. If the distance between the two primaries is set equal to one, the gravitational constant equals one too.

In this study, we will consider two reference systems: the sidereal and the synodic. The *sidereal system*, $OXYZ$, is an inertial reference frame with origin at the center of masses. We will assume that the primaries E and M are moving in circular orbits on the $Z=0$ plane, of radii μ and $1-\mu$, respectively. In this reference system, Newton's equations of motion for P are

$$\ddot{\mathbf{R}} = -(1-\mu)\frac{\mathbf{R}-\mathbf{R}_E}{R_1^3} - \mu\frac{\mathbf{R}-\mathbf{R}_M}{R_2^3}, \tag{1}$$

where $\mathbf{R} = (X, Y, Z)^T$ denotes the position vector of P and R_1, R_2 are the distances from P to E and M, respectively. The *synodic system*, $Oxyz$, is a rotating

Celestial Mechanics and Dynamical Astronomy **84:** 387–407, 2002.

frame, also with the origin at the center of masses, in which the primaries are fixed on the x axis: E is located at $\mathbf{r}_E = (\mu, 0, 0)^T$ and M is at $\mathbf{r}_M = (\mu - 1, 0, 0)^T$. If $\mathbf{r} = (x, y, z)^T$ represents the position of P in the synodic system, the change from one system to the other is given by, $\mathbf{R} = G(t)\mathbf{r}$, where $G(t)$ is a planar rotation of angle t and the equations of motion are (see, for instance, Szebehely, 1967)

$$\ddot{\mathbf{r}} + A_3\dot{\mathbf{r}} = \nabla\Omega(\mathbf{r}), \tag{2}$$

where

$$A_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

and r_1, r_2 denote the distances from P to both primaries: $r_1^2 = (x - \mu)^2 + y^2 + z^2$, $r_2^2 = (x - \mu + 1)^2 + y^2 + z^2$. It is well known that the Equation (2) has the Jacobi's first integral

$$|\dot{\mathbf{r}}|^2 = 2\,\Omega(x, y, z) \; - \; C_J, \tag{3}$$

where C_J is the so called *Jacobi constant*.

The aim of the present paper is to study the spatial p-q resonant orbits. We will always assume that $\mu \ll 1 - \mu$ so, far from M, the motion will be close to the one of a two-body problem. A p-q resonant orbit is such that, between two consecutive close approaches of P to the smaller primary M, P does approximately p revolutions around E while M makes q revolutions around the origin. More precisely:

DEFINITION 1.1. Let $\mathbf{R}(t)$ be the solution of (1) with initial conditions $\mathbf{R}(t_1) = \mathbf{R}_i$, $\dot{\mathbf{R}}(t_1) = \dot{\mathbf{R}}_i$ and $\mathbf{R}_{TB}(t)$ the solution of $\ddot{\mathbf{R}} = -\mathbf{R}\,/\,R^3$ with the same initial conditions. Denoting by $B(M, \mu^\alpha)$ the sphere of center M and radius μ^α, $0 < \alpha < 1$, we will assume that $\mathbf{R}_i$ is on $B(M, \mu^\alpha)$ and that for a certain $t_2 > t_1$, $\mathbf{R}(t_2)$ belongs to $B(M, \mu^\alpha)$ too, but if $t \in (t_1, t_2)$ then $|\mathbf{R}(t) - \mathbf{R}_M(t)| > \mu^\alpha$. Under these conditions, we say that the orbit is p-q resonant if

$$t_2 - t_1 = 2\pi q + \varepsilon\mu^\alpha + O(\mu^{2\alpha}) = 2\pi p\tau + \delta\mu^\alpha + O(\mu^{2\alpha}), \tag{4}$$

where $p, q \in \mathbb{N}$ are relatively prime and $2\pi\tau$ is the period of $\mathbf{R}_{TB}$.

As a remark, the values of α will be specified later.

This kind of solutions have been considered by Yen (1985) in the framework of the analysis of a Mercury mission. Yen uses near resonant returns to Mercury, with several values for p-q, in order to reduce the orbit capture Δv requirements. Since the Mercury year lasts 90 days, the penalty in the flight time is 270 days for the 2–3 case, but a substantial reduction of the v_∞ at Mercury is achieved: from 5.7 to 4.7 km/s.

In the line of the present paper, the planar p-q resonant orbits have been studied by Font, Nunes and Simó in Font (2001). In this paper the authors give the restrictions on the initial conditions necessary to ensure that an orbit is a p-q resonant one. Our purpose is to give an extension of this result to the spatial case and derive approximated expressions (up to terms of order μ^α) of the final position and velocity at t_2 (return map).

In this paper, we will first study the behaviour of the solutions of (1) for $t \in [t_1, t_2]$. This is the so called *outer solution*. As in the outer solution P is far from M, the influence of this primary can be considered as a perturbation. In Section 2, we will give a bound of the error involved when the outer solution is approximated by a solution of the two-body problem. The main result of this section is Theorem 2.3, which states that outside a neighbourhood around the small primary of radius μ^α this error can be bounded by $O(\mu^{1-\alpha})$.

Section 3 will be devoted to the study of the restrictions on the initial conditions that ensure that an orbit is p-q resonant. For fixed values of p and q, these restrictions are for the value of the Jacobi constant and the spherical coordinates of both the initial position (on a sphere of radius μ^α around M) and the initial velocity. The result is stated precisely in Theorem 3.1.

In Section 4, the explicit expression for the outer map of the p-q resonant orbits will be given.

In what follows we introduce some of the notations and conventions used through the paper. From now on, the subscript i will be used to denote initial conditions (in both systems of coordinates) and when a position vector has the subscript 1 (resp. 2), it means that it is measured with respect to the large primary E (resp. small primary M), that is, $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_E$, $\mathbf{r}_2 = \mathbf{r} - \mathbf{r}_M$.

When we take initial conditions for P on $B(M, \mu^\alpha)$, we will always assume that P leaves the sphere B moving away from M. In synodic coordinates these conditions can be written as, $r_{2i} = |\mathbf{r}_i - \mathbf{r}_M| = \mu^\alpha$ and that $\mathbf{r}_{2i}$ and $\dot{\mathbf{r}}_i$ form an angle $a \in (-\pi/2, \pi/2)$ or, equivalently, that

$$\cos a = \frac{\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i \rangle}{v_i \mu^\alpha} \geqslant 0, \tag{5}$$

where $v_i = |\dot{\mathbf{r}}_i|$ and $\langle , \rangle$ denotes the scalar product.

Using the Jacobi integral, we can compute the norm of the initial velocity $v_i = |\dot{\mathbf{r}}_i|$ in terms of C_J and μ. If $\mathbf{r}_i = (x_i, y_i, z_i)$, taking into account that $r_1(t_1)^2 = 1 - 2(x_i - \mu + 1) + \mu^{2\alpha}$, $r_2(t_1) = \mu^\alpha$ and $x_i^2 + y_i^2 = 1 - 2(1-\mu)(x_i - \mu + 1) + O(\mu^{2\alpha})$, it follows that

$$v_i^2 = 3 - C_J + 2\mu^{1-\alpha} + O(\mu^{2\alpha}). \tag{6}$$

This equation implies that $C_J \leqslant 3$. This is a logical restriction, since it is known that there are no zero velocity curves in the planar case for $C_J \leqslant 3 - \mu(1 - \mu)$. In the spatial case, there are zero velocity surfaces for values of $C_J > -\mu(1 - \mu)$, but for values of the Jacobi constant between $-\mu(1-\mu)$ and $3-\mu(1-\mu)$ these surfaces do not intersect the $z=0$ plane.

2. The Outer Solution

In this section, we will compute a bound for the error involved in the approximation of the outer solution by a Keplerian orbit. Initially, we will estimate this error using only that the distance from the third body P to the primary M is greater than μ^α. Next, using the fact that the distance between P and M grows (at least for values of t close to t_1), we will see that the bound for the error can be improved.

Assume that we have an orbit of the RTBP with initial conditions, at $t=t_1$, on the sphere $B(M,\mu^\alpha)$ and the velocity verifying the requirements mentioned at the end of the preceding section. Moreover, suppose that exists $t_2 > t_1$ for which

$$R_2(t_2) = \mu^\alpha, \qquad R_2(t) > \mu^\alpha, \qquad \forall t \in (t_1, t_2).$$

See the Figure 1.

We denote by $\mathbf{q} = (\mathbf{R}, \dot{\mathbf{R}})$ the solution of the initial value problem,

$$\begin{aligned} &\dot{\mathbf{q}} = G(\mathbf{q},\mu) + F(\mathbf{q},\mu), \\ &\mathbf{q}(t_1) = \mathbf{q}_i = \begin{pmatrix} \mathbf{R}_i \\ \dot{\mathbf{R}}_i \end{pmatrix}, \end{aligned} \tag{7}$$

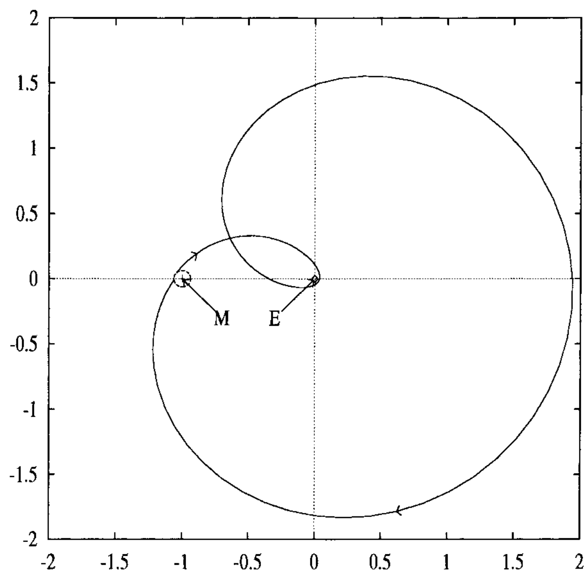

Figure 1. Example of behaviour of the outer solution in the synodic system (planar case). For this example $\mu = 10^{-3}$ and $\alpha = 0.4$.

where the equations of motion (1) have been written as a first order system, so G and F are defined by

$$G(\mathbf{q},\mu)=\begin{pmatrix}\dot{\mathbf{R}}\\ g(\mathbf{R},\mu)\end{pmatrix}=\begin{pmatrix}\dot{\mathbf{R}}\\ -(1-\mu)\dfrac{\mathbf{R}-\mathbf{R}_E}{R_1^3}\end{pmatrix},$$

$$F(\mathbf{q},\mu)=\begin{pmatrix}0\\ f(\mathbf{R},\mu)\end{pmatrix}=\begin{pmatrix}0\\ -\mu\dfrac{\mathbf{R}-\mathbf{R}_M}{R_2^3}\end{pmatrix}. \tag{8}$$

Let $\mathbf{q}_{TB}=(\mathbf{R}_{TB},\dot{\mathbf{R}}_{TB})$ be the solution of the two-body problem with the same initial conditions as above, that is, the solution of

$$\begin{aligned}&\dot{\mathbf{q}}=G(\mathbf{q},0),\\ &\mathbf{q}(t_1)=\mathbf{q}_i.\end{aligned} \tag{9}$$

Since in these equations we have removed a term which, outside the sphere $B(M,\mu^{\alpha})$, is of the order $\mu^{1-2\alpha}$, the following result holds.

THEOREM 2.1. *With the preceding notations and hypothesis and assuming, furthermore, that* $R=|\mathbf{R}|$ *cannot be arbitrarily small, we have that*

$$\mathbf{q}(t)=\mathbf{q}_{TB}(t)+O(\mu^{1-2\alpha}),\qquad \forall t\in[t_1,t_2].$$

Proof. We write the solutions of (7) and (8) as

$$\mathbf{q}(t)=\mathbf{q}_i+\int_{t_1}^{t}(G(\mathbf{q}(\tau),\mu)+F(\mathbf{q}(\tau),\mu))\,\mathrm{d}\tau,$$

$$\mathbf{q}_{TB}(t)=\mathbf{q}_i+\int_{t_1}^{t}G(\mathbf{q}_{TB}(\tau),0)\,\mathrm{d}\tau.$$

Subtracting both expressions, it follows that

$$|\mathbf{q}(t)-\mathbf{q}_{TB}(t)| \leqslant \underbrace{\int_{t_1}^{t}|G(\mathbf{q}(\tau),\mu)-G(\mathbf{q}_{TB}(\tau),0)|\,\mathrm{d}\tau}_{(a)}+\underbrace{\int_{t_1}^{t}|F(\mathbf{q}(\tau),\mu)|\,\mathrm{d}\tau}_{(b)}.$$

Let us find a bound for the first term (a). The distance from P to the body E can be written as

$$\begin{aligned}R_1^2&=(X-\mu\cos t)^2+(Y-\mu\sin t)^2+Z^2\\ &=R^2-2\mu(X\cos t+Y\sin t)+\mu^2,\end{aligned}$$

were $R^2 = X^2 + Y^2 + Z^2$. Since P must come back to the sphere $B(M, \mu^\alpha)$ at $t = t_2 < \infty$, its distance to the origin is bounded from above. By hypothesis it is also bounded from below, so $0 < m < R < M$, and we can write

$$R_1 = R\left(1 - \frac{X\cos t + Y\sin t}{R^2}\mu + O(\mu^2)\right).$$

Using this equality, the fact that R is bounded and the expression for $\mathbf{R}_E = \mu\,(\cos\, t, \sin t, 0)^T$, we can expand the function $g(\mathbf{q}, \mu)$ appearing in Equation (8) in terms of μ to get

$$g(\mathbf{q}, \mu) = -(1-\mu)\frac{\mathbf{R} - \mathbf{R}_E}{R_1^3} = -\frac{\mathbf{R}}{R^3} + O(\mu) = g(\mathbf{q}, 0) + O(\mu).$$

Using this expression, the mean value theorem and that $|DG|$ is bounded, the integrand in (a) can be bounded as

$$\begin{aligned} |G(\mathbf{q}(\tau), \mu) - G(\mathbf{q}_{TB}(\tau), 0)| &\leqslant |DG(\eta)|\,|\mathbf{q}(\tau) - \mathbf{q}_{TB}(\tau)| + C_1\mu \\ &\leqslant C_2\,|\mathbf{q}(\tau) - \mathbf{q}_{TB}(\tau)| + C_1\mu. \end{aligned}$$

Now, let us bound the integrand in (b). From the definition of F in (8) and just using that the distance from P to M, $R_2(t)$, is greater than μ^α for $t \in (t_1, t_2)$, it is clear that

$$|F(\mathbf{q}(\tau), \mu)| = \frac{\mu}{R_2^2} \leqslant \frac{\mu}{\mu^{2\alpha}} = \mu^{1-2\alpha}.$$

Therefore, we can set that

$$|\mathbf{q}(t) - \mathbf{q}_{TB}(t)| \leqslant C_0\mu^{1-2\alpha} + C_2\int_{t_1}^{t} |\mathbf{q}(\tau) - \mathbf{q}_{TB}(\tau)|\,\mathrm{d}\tau,$$

where C_0 and C_2 are suitable constants. Using Gronwal's lemma (see, for instance, Perko, 1996) we get

$$|\mathbf{q}(t) - \mathbf{q}_{TB}(t)| \leqslant C_0\mu^{1-2\alpha}e^{C_2(t-t_1)}, \qquad \forall\, t \in [t_1, t_2],$$

which ends the proof. □

We want to remark two things. First, to ensure that $\mathbf{q}_{TB}$ is an approximation of $\mathbf{q}$ it is necessary that $\alpha < 1/2$, so we will assume this in what follows. Second, the bound for the error in the approximation of the outer solution is due to the fact that $R_2(t) > \mu^\alpha$ so, if we can improve this last bound then we will be able to improve the error for the outer solution.

Recall that we are assuming that for t close to t_1, P is moving away from M, so initially its distance grows and, for some time, it will be true that

$$R_2(t) \geqslant \mu^\alpha + k(t - t_1), \tag{10}$$

for certain k. If we use this inequality to bound (b), for the values of t for which (10) is true, we get

$$\begin{aligned}(b) = \int_{t_1}^{t} \frac{\mu}{R_2(\tau)^2}\,\mathrm{d}\tau &\leqslant \int_{t_1}^{t} \frac{\mu}{(\mu^\alpha + k(\tau - t_1))^2}\,\mathrm{d}\tau \\ &= \mu\left(\frac{-1/k}{\mu^\alpha + k(\tau - t_1)}\right)_{t_1}^{t} \\ &= \frac{\mu}{k}\left(\frac{1}{\mu^\alpha} - \frac{1}{\mu^\alpha + k(t - t_1)}\right) \\ &= \frac{\mu^{1-\alpha}}{k}\left(1 - \frac{\mu^\alpha}{\mu^\alpha + k(t - t_1)}\right) \leqslant \frac{\mu^{1-\alpha}}{k}.\end{aligned}$$

Suppose that k cannot be arbitrarily small, then this bound of (b) implies that the error involved in the approximation of $(\mathbf{R}, \dot{\mathbf{R}})$ by $(\mathbf{R}_{TB}, \dot{\mathbf{R}}_{TB})$ will be of order $\mu^{1-\alpha}$, for t verifying (10), which is smaller than the bound obtained in Theorem 2.1. By the same argument, we can obtain the same result for t close to t_2. On the other hand, if we are far enough from M, the error of the approximation will be also small. For instance, with the same hypothesis of Theorem 2.1, it is clear that for any time interval $[t_a, t'_a]$ for which $R_2(t) \geqslant \mu^{\alpha/2}$, then

$$|\mathbf{q}(t) - \mathbf{q}_{TB}(t)| \leqslant C_0\mu^{1-\alpha}e^{C_2(t-t_a)}.$$

Thus, we obtain that $\mathbf{q} = \mathbf{q}_{TB} + O(\mu^{1-\alpha})$ near and far from M. It only remains to see that $R_2(t) \geqslant \mu^{\alpha/2}$ for some time and that (10) is true, at least till P is at a distance $\mu^{\alpha/2}$ from M. We will proof the following

LEMMA 2.2. *Let* $\mathbf{q}(t)$ *the solution of* (7) *with initial conditions* $(\mathbf{R}_i, \dot{\mathbf{R}}_i)$ *such that*

$$|\mathbf{R}_i - \mathbf{R}_M(t_1)| = \mu^\alpha, \qquad \cos a \geqslant \varepsilon > 0,$$

for some ε *(see* (5)*). Then, there exists* $k \geqslant \varepsilon' > 0$ *and* $t_{\alpha/2}$ *such that* $R_2(t_{\alpha/2}) = \mu^{\alpha/2}$ *and*

$$R_2(t) \geqslant \mu^\alpha + k(t - t_1), \qquad \forall t \in [t_1, t_{\alpha/2}].$$

Using this lemma and the preceding arguments, is easy to see that under the same hypothesis as in Theorem 2.1, we have the following:

THEOREM 2.3. *If there exists* $t_{\alpha/2}$ *and* $t'_{\alpha/2}$ *such that* $t_1 < t_{\alpha/2} < t'_{\alpha/2} < t_2$,

$$R_2(t_{\alpha/2}) = R_2(t'_{\alpha/2}) = \mu^{\alpha/2}, \qquad R_2(t) > \mu^{\alpha/2} \;\; \forall\, t \in (t_{\alpha/2}, t'_{\alpha/2}),$$

and the initial conditions verify that $\cos a \geqslant \varepsilon > 0$, *then*

$$\mathbf{q}(t) = \mathbf{q}_{TB}(t) + O(\mu^{1-\alpha}), \qquad \forall\, t \in [t_1, t_2].$$

We will use this result in the following sections, such that the terms of order $\mu^{1-\alpha}$ will be neglected. For this reason, from now on we will impose that $\mu^{2\alpha} = O(\mu^{1-\alpha})$ or, equivalently, $\alpha > 1/3$. Thus, we have $\alpha \in (1/3, 1/2)$.

Let us now proof Lemma 2.2. We want to see that $\dot{R}_2(t) \geqslant k > 0$ for $t \in [t_1, t_{\alpha/2}]$. We start with the Taylor expansion of $R_2(t)$ up to the first order

$$R_2(t) = R_2(t_1) + \dot{R}_2(\tau)(t - t_1) = \mu^\alpha + \dot{R}_2(\tau)(t - t_1). \tag{11}$$

It is clear that, if at $t = t_1$, P is leaving the sphere $B(M, \mu^\alpha)$, $R_2(t)$ grows during some time and its derivative must be positive. From $R_2(t) = \langle \mathbf{R}_2(t), \mathbf{R}_2(t)\rangle^{1/2}$ we get

$$\dot{R}_2(t) = \frac{1}{R_2(t)} \langle \mathbf{R}_2(t), \dot{\mathbf{R}}_2(t)\rangle.$$

As the condition of leaving $B(M, \mu^\alpha)$ is expressed in synodic coordinates, $\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i\rangle > 0$, it will be better to rewrite the above expression as

$$\dot{R}_2(t) = \frac{1}{R_2(t)} \langle \mathbf{r}_2(t), \dot{\mathbf{r}}(t)\rangle = v(t)\cos(a(t)),$$

where $v(t) = |\dot{\mathbf{r}}(t)|$ and $a(t)$ is the angle between the vectors $\mathbf{r}_2(t)$ and $\dot{\mathbf{r}}(t)$. Then,

$$\dot{R}_2(t_1) = \frac{1}{\mu^\alpha} \langle \mathbf{r}_2(t_1), \dot{\mathbf{r}}(t_1)\rangle = v_i \cos a > 0,$$

where $a = a(t_1)$. As it has already been said, we can assure that $\dot{R}_2(t) > k > 0$ for some time so it only remains to prove that this inequality is also true while the distance from P to M grows from μ^α to $\mu^{\alpha/2}$. Taking again a Taylor development, we have that

$$\dot{R}_2(t) = v_i \cos a + \ddot{R}_2(\eta)(t - t_1). \tag{12}$$

We want a lower bound for $\dot{R}$, so let us look for a lower bound for the second derivative.

$$\begin{aligned} \ddot{R}_2(t) &= \frac{-\dot{R}_2(t)}{R_2^2(t)} \langle \mathbf{r}_2(t), \dot{\mathbf{r}}(t)\rangle + \frac{1}{R_2(t)} \left(\langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t)\rangle + \langle \mathbf{r}_2(t), \ddot{\mathbf{r}}(t)\rangle\right) \\ &= \frac{v(t)^2 - \dot{R}_2^2(t)}{R_2(t)} + \frac{\langle \mathbf{r}_2(t), \ddot{\mathbf{r}}(t)\rangle}{R_2(t)} \geqslant \frac{\langle \mathbf{r}_2(t), \ddot{\mathbf{r}}(t)\rangle}{R_2(t)}. \end{aligned} \tag{13}$$

From the equations of motion (2), we can write

$$\langle \mathbf{r}_2(t), \ddot{\mathbf{r}}(t)\rangle = \underbrace{\langle \mathbf{r}_2(t), \nabla\Omega\rangle}_{(I)} + \underbrace{\langle \mathbf{r}_2(t), -A_3\dot{\mathbf{r}}\rangle}_{(II)}. \tag{14}$$

For the second summand we have

$$\begin{aligned} (II) &= 2\langle \mathbf{r}_2(t), (\dot{y}, -\dot{x}, 0)^T\rangle = 2\langle \mathbf{r}_2(t) \times \dot{\mathbf{r}}(t), (0,0,1)^T\rangle \\ &\geqslant -2\,|\mathbf{r}_2(t) \times \dot{\mathbf{r}}(t)| \geqslant -2\,r_2(t)\,v(t). \end{aligned}$$

If we compute $\nabla\Omega$ and substitute its expression into (I), it follows that

$$(I) = \underbrace{\langle \mathbf{r}_2(t), (x, y, 0)^T \rangle}_{(I_1)} - \underbrace{\frac{1-\mu}{r_1^3} \langle \mathbf{r}_2, \mathbf{r}_1 \rangle}_{(I_2)} - \frac{\mu}{r_2}.$$

The lower bounds for the terms (I_1) and (I_2) will be:

$$\begin{aligned} (I_1) &\geqslant -(1-\mu)(x-\mu+1) \geqslant -(1-\mu)\, r_2(t), \\ (I_2) &= -\frac{1-\mu}{r_1^3}\langle \mathbf{r}_2, \mathbf{r}_1 \rangle \geqslant -(1-\mu)\,\frac{r_2(t)}{r_1^2(t)}. \end{aligned}$$

From the fact that $\mu^\alpha \leqslant r_2(t) \leqslant \mu^{\alpha/2}$ and that the distance between the primaries is one, it follows that $r_1(t) \geqslant 1 - \mu^{\alpha/2}$ and

$$(I_2) \geqslant r_2(t)(-1 - 2\mu^{\alpha/2} - 3\mu^\alpha - 4\mu^{3\alpha/2} + O(\mu^{2\alpha})).$$

Therefore, (I) can be bounded as

$$(I) \geqslant -r_2(t)(2 + 2\mu^{\alpha/2} + 3\mu^\alpha + 4\mu^{3\alpha/2}) - \mu^{1-\alpha} + O(\mu^{2\alpha}),$$

and, coming back to (14), it follows that

$$\begin{aligned} &\langle \mathbf{r}_2(t), \ddot{\mathbf{r}}(t) \rangle \\ &\quad \geqslant -r_2(t)(2v(t) + 2 + 2\mu^{\alpha/2} + 3\mu^\alpha + 4\mu^{3\alpha/2}) - \mu^{1-\alpha} + O(\mu^{2\alpha}). \end{aligned}$$

If in (13) we take only into account terms up to μ^α, we can write

$$\ddot{R}_2(t) \geqslant -2v(t) - 2 - 2\mu^{\alpha/2} - \mu^{1-2\alpha} + O(\mu^\alpha). \tag{15}$$

The final step is find an upper bound for $v(t)$. For this, we will use the Jacobi integral (3) and the fact that $x^2 + y^2 \leqslant r^2 \leqslant (1 - \mu + \mu^{\alpha/2})^2$. Then, we will have that

$$v(t)^2 = -C_J + x^2 + y^2 + 2\frac{\mu}{r_2} + 2\frac{1-\mu}{r_1} \leqslant -C_J + 3 + 4\mu^{\alpha/2} + O(\mu^\alpha).$$

Finally, from (15) and using the above inequality, we can establish a lower bound for $\ddot{R}_2(t)$:

$$\begin{aligned} \ddot{R}_2(t) \geqslant &- 2\left(1 + \sqrt{3 - C_J}\right) - 2\left(1 + \frac{2}{\sqrt{3 - C_J}}\right)\mu^{\alpha/2} - \\ &- \mu^{1-2\alpha} + O(\mu^\alpha) = -K_\alpha, \end{aligned}$$

where $K_\alpha > 0$.

Let us use this bound in (12). Given k, we have that

$$\dot{R}_2(t) \geqslant v_i \cos a - K_\alpha(t - t_1) \geqslant k > 0,$$

only for values of t such that $t \leqslant t_1 + (v_i \cos a - k)/K_\alpha = t + \Delta t$ and, coming back to (11), this implies that

$$R_2(t) \geqslant \mu^\alpha + k(t - t_1) \quad \text{for all } t \in [t_1, t_1 + \Delta t].$$

This was the relation that we wanted to proof. It remains to show that we can choose k not arbitrarily small and such that exists $t_{\alpha/2} \leqslant t_1 + \Delta t$ verifying $R_2(t_{\alpha/2}) = \mu^{\alpha/2}$. Let be $\bar{h}(t)$ and $h(t)$ be defined as $\bar{h}(t) = R_2(t) - \mu^{\alpha/2}$, $h(t) = \mu^\alpha - \mu^{\alpha/2} + k(t - t_1)$. They verify the following properties

$$\begin{aligned} &\bar{h}(t) \geqslant h(t) \quad \text{for } t \in [t_1, t_1 + \Delta t], \\ &\bar{h}(t_1) = \mu^\alpha - \mu^{\alpha/2} < 0, \\ &h(t) = 0 \iff t = t^* = t_1 + \frac{\mu^{\alpha/2} - \mu^\alpha}{k}. \end{aligned}$$

Suppose that $t^* \leqslant t_1 + \Delta t$. Then $\bar{h}(t^*) \geqslant h(t^*) = 0$ and, using Bolzano's theorem, we can conclude that there exists $t_{\alpha/2} \leqslant t^* \leqslant t_1 + \Delta t$ such that $\bar{h}(t_{\alpha/2}) = 0$ and $R_2(t_{\alpha/2}) = \mu^{\alpha/2}$. So it is necessary to proof that exists k such that

$$t^* = t_1 + \frac{\mu^{\alpha/2} - \mu^\alpha}{k} \leqslant t_1 + \frac{v_i \cos a \, - \, k}{K_\alpha},$$

or, equivalently, that

$$p(k) = k^2 - k\, v_i \cos a + K_\alpha(\mu^{\alpha/2} - \mu^\alpha) < 0.$$

We observe that $p(k)$ is a parabola with vertex in $k = (1/2)\, v_i \cos a$, and that $p((1/2)\, v_i \cos a) < 0$ if and only if

$$v_i^2 \cos^2 a - 4K_\alpha(\mu^{\alpha/2} - \mu^\alpha) \geqslant 0.$$

Using (6), we can develop this expression in powers of μ to get

$$\begin{aligned} &v_i^2 \cos^2 a - 4K_\alpha(\mu^{\alpha/2} - \mu^\alpha) \\ &\quad = (3 - C_J)\cos^2 a - 8\left(1 + \sqrt{3 - C_J}\right)\mu^{\alpha/2} - 4\mu^{1-3\alpha/2} + O(\mu^\alpha), \end{aligned}$$

which is strictly positive because $\cos a$ cannot be arbitrarily small. Finally, we observe that $k = (1/2)\, v_i \cos a \geqslant \varepsilon' > 0$ since $v_i = \sqrt{3 - C_J} + O(\mu^{1-\alpha})$ and we are assuming that $\cos a \geqslant \varepsilon$. This concludes the proof. □

3. *p-q* Resonant Orbits

In this section, we are going to find restrictions on the initial conditions that ensure that an orbit is *p-q* resonant. Let us start with some notation. The synodic initial conditions can be written in spherical coordinates as

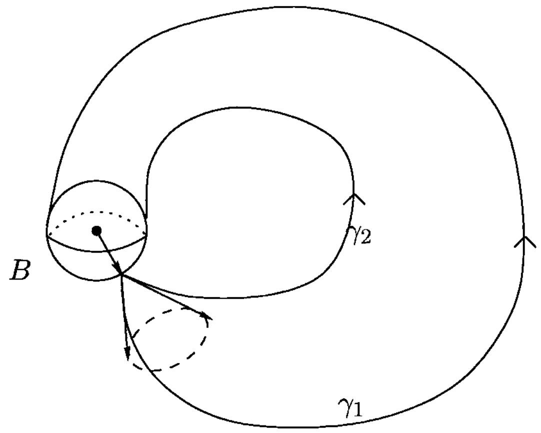

Figure 2. Fixed the initial position on the sphere B, there are different values of $\dot{\mathbf{r}}$ such that the orbit comes back to B. The curves γ_1 and γ_2 are the qualitatively representation of two solutions of the RTBP that come back to B tangently. All the orbits with initial velocity inside the cone displayed in the figure return to the sphere B after some time.

$$\mathbf{r}_i = \begin{pmatrix} \mu - 1 + \mu^\alpha \cos\varphi \cos\theta \\ \mu^\alpha \cos\varphi \sin\theta \\ \mu^\alpha \sin\varphi \end{pmatrix}, \qquad \dot{\mathbf{r}}_i = v_i \begin{pmatrix} \cos\phi \cos\psi \\ \cos\phi \sin\psi \\ \sin\phi \end{pmatrix}. \tag{16}$$

For a fixed position on $B(M, \mu^\alpha)$, defined by the values of the angle φ and θ, there are different velocities for which the orbit comes back to $B(M, \mu^\alpha)$. See Figure 2 for a qualitative representation.

We will write the spherical coordinates of the velocity, ϕ and ψ, as

$$\begin{aligned} \phi &= \phi_0 + \Delta\phi\,\mu^\alpha + O(\mu^{2\alpha}), \\ \psi &= \psi_0 + \Delta\psi\,\mu^\alpha + O(\mu^{2\alpha}). \end{aligned} \tag{17}$$

From now on, the subscript 0 will denote the zero order term of any development in terms of μ^α. For instance, if a is the angle between $\mathbf{r}_{2i}$ and $\dot{\mathbf{r}}_i$, we will have

$$\cos a = \cos a_0 + \Delta C \mu^\alpha + O(\mu^{2\alpha}), \tag{18}$$

being

$$\begin{aligned} \cos a_0 &= \cos\varphi \cos\phi_0 \cos(\theta - \psi_0) + \sin\varphi \sin\phi_0, \\ \Delta C &= \Lambda_0 \Delta\psi + \Delta\phi\,(\sin\varphi \cos\phi_0 - \cos\varphi \sin\phi_0 \cos(\psi_0 - \theta)) \end{aligned}$$

and $\Lambda_0 = \cos\varphi \cos\phi_0 \sin(\theta - \psi_0)$.

First of all, we observe that the solution of the two-body problem, $\mathbf{R}_{TB}$, must be an elliptic solution with angular momentum, $\mathbf{c}$, different from zero. Since $\mathbf{R}(t)$ and $\mathbf{R}_{TB}(t)$ have both the same initial conditions, we can compute h and $\mathbf{c}$ in terms of φ, θ, ϕ, ψ and the Jacobi constant. Let us begin with the energy:

$$h = \frac{|\dot{\mathbf{R}}_i|^2}{2} - \frac{1}{|\mathbf{R}_i|}.$$

On one hand,

$$\begin{aligned}|\mathbf{R}_i|^2 &= |\mathbf{r}_i|^2 = r_{2i}^2 - 2(1-\mu)(x_i - \mu + 1) + (1-\mu)^2 \\ &= 1 - 2\mu^\alpha \cos\varphi \, \cos\theta + O(\mu^{2\alpha}).\end{aligned}$$

On the other hand, $|\dot{\mathbf{R}}_i|^2 = |\dot{\mathbf{r}}_i|^2 + 2(x_i \dot{y}_i - \dot{x}_i y_i) + x_i^2 + y_i^2$, and using the spherical synodic coordinates we get

$$\begin{aligned}|\dot{\mathbf{R}}_i|^2 = 4 &- C_J - 2\sqrt{3-C_J}\cos\phi_0 \, \sin\psi_0 + \\ &+ 2\,\mu^\alpha \sqrt{3-C_J}\,\Delta\phi \, \sin\phi_0 \, \sin\psi_0 + \\ &+ 2\,\mu^\alpha \sqrt{3-C_J}(-\Delta\psi \, \cos\phi_0 \, \cos\psi_0 + \\ &+ \cos\varphi \, \cos\phi_0 \, \sin(\psi_0 - \theta)) - \\ &- 2\mu^\alpha \, \cos\varphi \, \cos\theta + 2\left(1 - \frac{\cos\phi_0 \, \sin\psi_0}{\sqrt{3-C_J}}\right)\mu^{1-\alpha} + O(\mu^{2\alpha}).\end{aligned}$$

Finally, we can write that

$$h = h_0 + \Delta h \, \mu^\alpha + O(\mu^{1-\alpha}), \tag{19}$$

being

$$\begin{aligned}h_0 &= 1 - \frac{C_J}{2} - \sqrt{3-C_J}\cos\phi_0 \, \sin\psi_0, \\ \Delta h &= \sqrt{3-C_J}\,(\Delta\phi \, \sin\phi_0 \, \sin\psi_0 - \Delta\psi \, \cos\phi_0 \, \cos\psi_0 + \\ &\quad + \cos\varphi \, \cos\phi_0 \, \sin(\psi_0 - \theta)) - 2\cos\varphi \, \cos\theta.\end{aligned}$$

Since for the elliptic orbits $h < 0$, we will require that $h_0 < 0$ or, equivalently, that

$$\sqrt{3-C_J} < \cos\phi_0 \, \sin\psi_0 + \sqrt{1 + \cos^2\phi_0 \, \sin^2\psi_0}. \tag{20}$$

From this inequality we can deduce that $C_J > -2\sqrt{2}$.

The next condition to be required is $\mathbf{c} \neq 0$. As before, we can compute $\mathbf{c}$ at the initial conditions to get

$$\mathbf{c} = \mathbf{R}_i \times \dot{\mathbf{R}}_i = G(t_1)(\mathbf{r}_i \times \dot{\mathbf{r}}_i + \mathbf{w}_i),$$

where $\mathbf{w}_i = (-x_i z_i, -y_i z_i, x_i^2 + y_i^2)^T$. As $G(t_1)$ is a rotation, it will be enough to require that $\mathbf{r}_i \times \dot{\mathbf{r}}_i + \mathbf{w}_i \neq 0$. Since

$$\mathbf{w}_i = \begin{pmatrix} \mu^\alpha \sin\varphi \\ 0 \\ 1 - 2\mu^\alpha \cos\varphi \, \cos\theta \end{pmatrix} + O(\mu^{2\alpha}),$$

$$\mathbf{r}_i \times \dot{\mathbf{r}}_i = \sqrt{3-C_J}\begin{pmatrix} 0 \\ \sin\phi_0 \\ -\cos\phi_0 \, \sin\psi_0 \end{pmatrix} + O(\mu^\alpha),$$

it can be concluded that

$$
\begin{aligned}
|\mathbf{r}_i \times \dot{\mathbf{r}}_i + \mathbf{w}_i|^2 \;=\; & (3 - C_J)\sin^2\phi_0 + \\
& + \left(1 - \sqrt{3 - C_J}\cos\phi_0 \, \sin\psi_0\right)^2 + O(\mu^\alpha),
\end{aligned}
$$

and the condition that $\mathbf{c} \neq 0$ will be equivalent to

$$
(3 - C_J)\sin^2\phi_0 \,+\, \left(1 - \sqrt{3 - C_J}\cos\phi_0 \, \sin\psi_0\right)^2 \;\neq\; 0.
$$

As this expression is the sum of two squares, it will be fulfilled if one of them is different from zero, so the final restriction will be

$$
\phi_0 \neq 0, \qquad \text{or} \;\; \sin\psi_0 \;\neq\; \frac{1}{\sqrt{3 - C_J}}, \tag{21}
$$

provided that $C_J < 3$.

Next, let us ask that at $t = t_2$ the distance from P to M must be μ^α. This will give a relation between ε, δ and the coordinates of $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$.

In order to establish the final position of P we will use the approximation of $\mathbf{R}$ by the elliptic orbit $\mathbf{R}_{TM}$. Using (4) we have that

$$
\begin{aligned}
\mathbf{R}(t_2) \;=\; & \mathbf{R}_{TB}(t_2) + O(\mu^{1-\alpha}) = \mathbf{R}_{TB}(t_1 + 2\pi p\tau + \delta\mu^\alpha) + O(\mu^{1-\alpha}) \\
=\; & \mathbf{R}_{TB}(t_1 + 2\pi p\tau) + \dot{\mathbf{R}}_{TB}(t_1 + 2\pi p\tau)\mu^\alpha\delta + O(\mu^{1-\alpha}) \\
=\; & \mathbf{R}(t_1) + \dot{\mathbf{R}}(t_1)\delta\mu^\alpha + O(\mu^{1-\alpha}),
\end{aligned}
$$

where we have used that $\mu^{2\alpha} = O(\mu^{1-\alpha})$, $2\pi\tau$ is the period of the Keplerian orbit and the initial conditions for $\mathbf{R}$ and $\mathbf{R}_{TB}$ are the same. Analogously, as 2π is the period of the primaries, we have that

$$
\begin{aligned}
\mathbf{R}_M(t_2) \;=\; & \mathbf{R}_M(t_1 + 2\pi q + \varepsilon\mu^\alpha) + O(\mu^{2\alpha}) \\
=\; & \mathbf{R}_M(t_1) + \dot{\mathbf{R}}_M(t_1)\varepsilon\mu^\alpha + O(\mu^{2\alpha}).
\end{aligned}
$$

Therefore, the condition we need is

$$
\begin{aligned}
|\mathbf{R}(t_2) - \mathbf{R}_M(t_2)| \;=\; & |\mathbf{R}(t_1) - \mathbf{R}_M(t_1) + \\
& + \mu^\alpha(\dot{\mathbf{R}}(t_1)\delta - \dot{\mathbf{R}}_M(t_1)\varepsilon) + O(\mu^{1-\alpha})| = \mu^\alpha.
\end{aligned}
$$

Using (16) and the relation between the sidereal and the synodic coordinates, it can be seen (see Barrabès, 2001 for the details) that $|\mathbf{R}(t_2) - \mathbf{R}_M(t_2)| = \mu^\alpha|\mathbf{w}| + O(\mu^{1-\alpha})$, with

$$
\begin{aligned}
|\mathbf{w}|^2 \;=\; & 1 + (\varepsilon - \delta)^2 + (3 - C_J)\delta^2 + 2(\varepsilon - \delta)\delta\sqrt{3 - C_J}\,\cos\phi\,\sin\psi + \\
& + 2\delta\sqrt{3 - C_J}\,(\cos\varphi\,\cos\phi\,\cos(\psi - \theta) + \sin\varphi\,\sin\psi) + \\
& + 2(\varepsilon - \delta)\cos\varphi\,\sin\theta.
\end{aligned} \tag{22}
$$

We can use the expressions (17) for ϕ and ψ and the relation (18) to write (22) in powers of μ. Thus, the distance from P to M at t_2 can be written as $|\mathbf{R}(t_2) -$

$\mathbf{R}_M(t_2)| = \mu^\alpha|\mathbf{w_0}| + O(\mu^{1-\alpha})$ and the condition we will ask is $|\mathbf{w_0}| = 1$, or equivalently,

$$(\varepsilon-\delta)^2 + \delta^2(3-C_J) + 2(\varepsilon-\delta)\cos\varphi\,\sin\theta + 2\delta\sqrt{3-C_J}\,\cos a_0 + \\ + 2(\varepsilon-\delta)\delta\sqrt{3-C_J}\cos\phi_0\,\sin\psi_0 = 0. \quad (23)$$

This equation represents an ellipse in the $(\delta, \varepsilon-\delta)$ plane, except if the determinant of the quadratic terms vanishes, which happens if $1 = \cos^2\phi_0\,\sin^2\psi_0$. This situation will not be considered in what follows because it implies that δ and $\varepsilon-\delta$ can take any real value and then t_2 could be arbitrarily large. Therefore, we will assume that $1 \neq \cos^2\phi_0\,\sin^2\psi_0$. In Figure 3, we have represented some of these ellipses for different values of the parameters.

Fixed $\varphi, \theta, \phi_0, \psi_0$ and C_J, Equation (23) restricts the rang of values for ε and δ. From the definition of t_2 in Equation (4), it follows that there is a relation between $\varepsilon-\delta$ and the period τ (modulus 2π) of the elliptic orbit which, at the same time, can be written in terms of the initial conditions using

$$\tau^{2/3} = \frac{1}{2|h|},$$

where h is the energy of the elliptic orbit. Thus, we will have another relation to take into account. Let us write it explicitly. On one hand, from (19), we get that

$$\frac{1}{2|h|} = \frac{1}{2|h_0|}\left(1 + \frac{\Delta h}{|h_0|}\mu^\alpha + O(\mu^{1-\alpha})\right). \quad (24)$$

On the other hand, from Definition 1.1, it follows that

$$\tau = \frac{q}{p} + \frac{\varepsilon-\delta}{2\pi p}\mu^\alpha + O(\mu^{2\alpha}). \quad (25)$$

Therefore, equating terms of the same order in Equations (24) and (25), we get the following results:

1. $\left(\frac{q}{p}\right)^{2/3} = \frac{1}{2|h_0|}$ or, using the expression for $2|h_0|$,

$$C_J - 2 + 2\sqrt{3-C_J}\cos\phi_0\,\sin\psi_0 = \left(\frac{p}{q}\right)^{2/3}. \quad (26)$$

This relation implies, in particular, (20). From it, we can get several information. First of all, $(p/q)^{2/3} \leqslant C_J - 2 + 2\sqrt{3-C_J} \leqslant 2$, so p and q must verify

$$\frac{p}{q} \leqslant 2\sqrt{2}.$$

Next, we observe that $\left|2 - C_J + \left(\frac{p}{q}\right)^{2/3}\right| \leqslant 2\sqrt{3-C_J}$, from which we get that

$$C_{J1} = \left(\frac{p}{q}\right)^{2/3} - 2\sqrt{2-\left(\frac{p}{q}\right)^{2/3}} \leqslant C_J \leqslant \left(\frac{p}{q}\right)^{2/3} + 2\sqrt{2-\left(\frac{p}{q}\right)^{2/3}} \\ = C_{J2}. \quad (27)$$

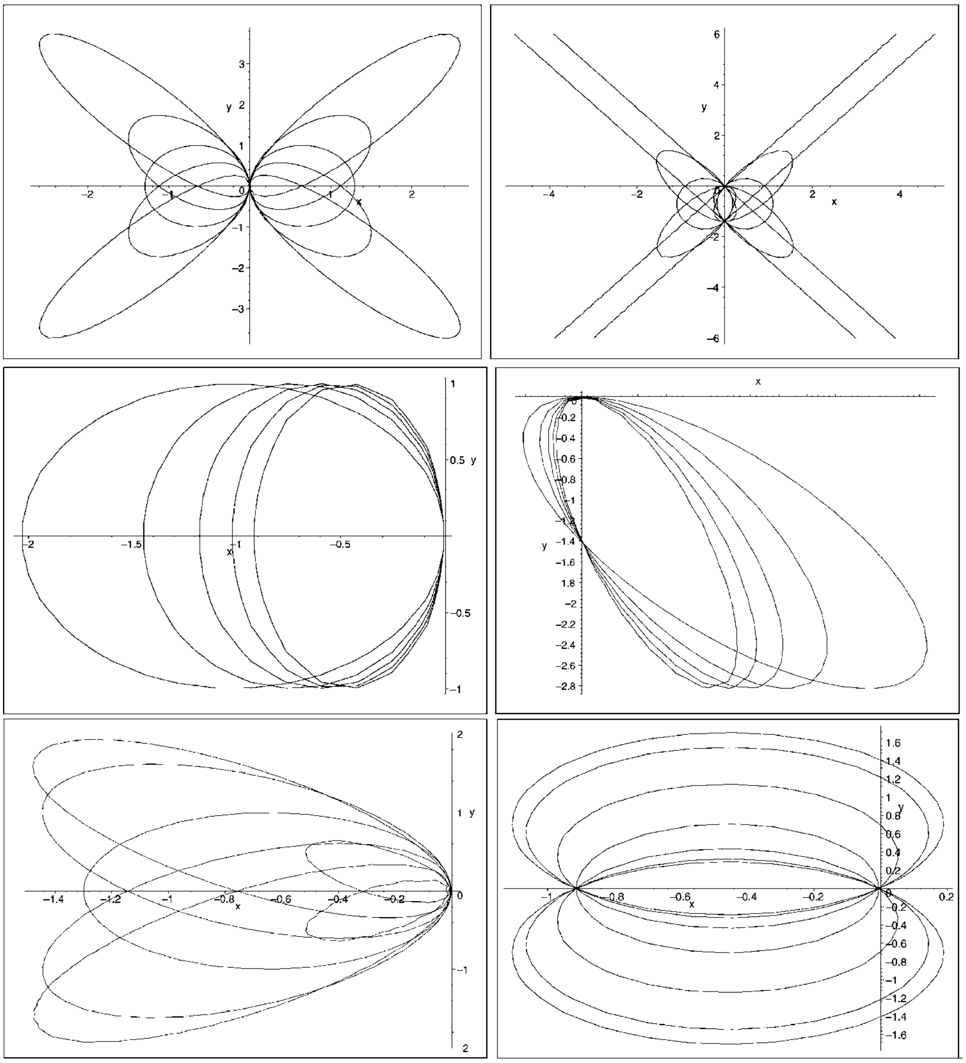

Figure 3. Examples of ellipses represented by Equation (23) for different values of the parameters C_J, φ, ϕ_0, θ and ψ_0. For $p=1$ and $q=2$, C_J can take any value on the interval $(-1.711013183, 2.970934233)$. The figures on the top represent the ellipses in the planar case $\varphi = \phi_0 = 0$, for $C_J = 0.6299605255$, θ fixed and varying ψ_0. The figures on the middle represent the ellipses also in the planar case but with fixed values of θ and ψ_0 and varying C_J. The figures on the bottom represent the ellipses for $C_J = 0.6299605255$. In the figure on the left ψ_0 is variable and in the figure on the right θ is variable.

Thus, for fixed values of p and q, the range of admissible values for C_J is $[C_{J1}, C_{J2}] \subset (-2\sqrt{2}, 3)$. On another hand, using (26), the restrictions given by (21) can be written as

$$\phi_0 \neq 0 \quad \text{or} \quad C_J \neq \left(\frac{p}{q}\right)^{2/3}. \tag{28}$$

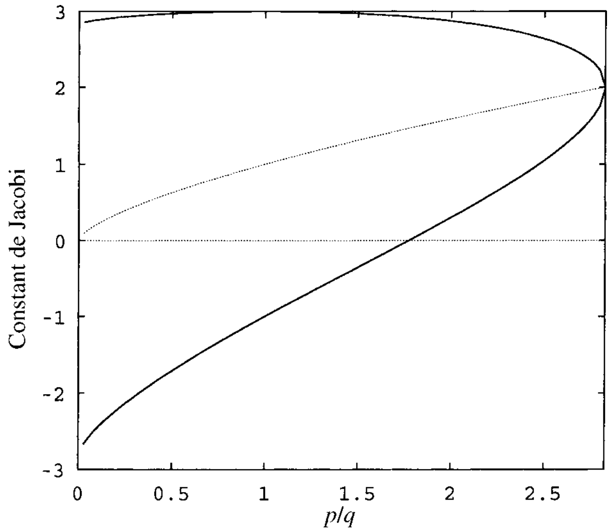

Figure 4. The area limited by the continuous curve defines the range of variation of the Jacobi constant, (C_{J1}, C_{J2}), as a function of p/q. The dotted curve represents $C_J = (p/q)^{2/3}$, and these values must be excluded if $\phi_0 = 0$.

A summary of the restrictions on C_J are represented in Figure 4. Finally, $|h_0|$ must not be too close to zero. For this, it will be enough that $\frac{1}{2}\mu^{2\alpha} / h_0 = O(\mu^{1-\alpha})$ or, equivalently,

$$\left(\frac{p}{q}\right)^{2/3} > \mu^{3\alpha-1} \qquad \Leftrightarrow \qquad p > q\mu^{(3\alpha-1)3/2}.$$

This relation restricts the range of values of p and q as a function of μ. For example, if we take $\alpha = 0.4$, the maxim values for q when $p = 1$ are

μ	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$q_{\max}$	7.943	15.84	31.6	63.09

2. Equating terms of order μ^α in Equations (24) and (25), we get

$$\frac{1}{2|h_0|}\frac{\Delta h}{|h_0|} = \frac{1}{3}\left(\frac{p}{q}\right)^{1/3}\frac{1}{\pi p}(\varepsilon - \delta),$$

and isolating $\varepsilon - \delta$, we get that

$$\begin{aligned}\varepsilon - \delta \;=\; & 6\pi q\left(\frac{q}{p}\right)^{2/3}\Big(\sqrt{3 - C_J}(\cos\varphi\,\cos\phi_0\,\sin(\psi_0 - \theta) + \\ & + \Delta\phi\,\sin\phi_0\,\sin\psi_0 - \Delta\psi\,\cos\phi_0\,\cos\psi_0) - 2\cos\varphi\,\cos\theta\Big). \qquad (29)\end{aligned}$$

Recall that $\varepsilon - \delta$ must verify Equation (23), too. Since this is an ellipse in the $(\delta, \varepsilon - \delta)$ plane, there exist k_1 and k_2 such that any point on the ellipse verifies that $k_1 < \varepsilon - \delta < k_2$. This is the condition that we will impose on the value of $\varepsilon - \delta$ given by (29). The values of k_i, $i = 1, 2$ can be obtained from Equation (23) and are given by

$$k_1 = \frac{\cos a_0 \cos\phi_0 \sin\psi_0 - \cos\varphi \sin\theta - \sqrt{\Delta k}}{1 - \cos^2\phi_0 \sin^2\psi_0},$$

$$k_2 = \frac{\cos a_0 \cos\phi_0 \sin\psi_0 - \cos\varphi \sin\theta + \sqrt{\Delta k}}{1 - \cos^2\phi_0 \sin^2\psi_0},$$

being $\Delta k = (\cos a_0 - 2\cos\varphi \cos\phi_0 \sin\theta \sin\psi_0)\cos a_0 + \cos^2\varphi \sin^2\theta$. Observe that $k_1 < 0$ and $k_2 > 0$.

So let us obtain the final condition. Denoting by

$$\begin{aligned} & A(\varphi, \theta, \phi_0, \psi_0, \Delta\phi, \Delta\psi) \\ &= \lambda\sqrt{3 - C_J}(\cos\varphi \cos\phi_0 \sin(\psi_0 - \theta) + \Delta\phi \sin\phi_0 \sin\psi_0 - \\ &\quad -\Delta\psi \cos\phi_0 \cos\psi_0) - 2\lambda\cos\varphi\cos\theta, \end{aligned}$$

$$\begin{aligned} & B(\varphi, \theta, \phi_0, \psi_0) \\ &= \frac{\cos a_0 \cos\phi_0 \sin\psi_0 - \cos\varphi \sin\theta}{1 - \cos^2\phi_0 \sin^2\psi_0}, \end{aligned}$$

$$\begin{aligned} & C(\varphi, \theta, \phi_0, \psi_0) \\ &= \frac{\sqrt{(\cos a_0 - 2\cos\varphi \cos\phi_0 \sin\theta \sin\psi_0)\cos a_0 + \cos^2\varphi \sin^2\theta}}{1 - \cos^2\phi_0 \sin^2\psi_0}, \end{aligned}$$

where $\lambda = 6\pi q \left(\frac{q}{p}\right)^{2/3}$, the condition $k_1 < \varepsilon - \delta < k_2$ becomes

$$|A(\varphi, \theta, \phi_0, \psi_0, \Delta\phi, \Delta\psi) - B(\varphi, \theta, \phi_0, \psi_0)| \leqslant C(\varphi, \theta, \phi_0, \psi_0), \tag{30}$$

where the variables are restricted by (26) and (28).

As a conclusion we can establish the following result:

THEOREM 3.1. *Let* $\mathbf{r}(t)$ *be a solution of* (2) *with initial conditions* $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ *on the sphere* $B(M, \mu^\alpha)$. *If it is a p-q resonant orbit, then the Jacobi constant* $C_J \in (C_{j1}, C_{J2})$ *and the variables* φ, θ, ϕ *and* ψ, *defined by* (16) *and* (17), *must verify*

$$C_J - 2 + 2\sqrt{3 - C_J}\cos\phi_0 \sin\psi_0 = \left(\frac{p}{q}\right)^{2/3},$$

$$\phi_0 \neq 0 \quad or \quad C_J \neq \left(\frac{p}{q}\right)^{2/3},$$

$$|A(\varphi, \theta, \phi_0, \psi_0, \Delta\phi, \Delta\psi) - B(\varphi, \theta, \phi_0, \psi_0)| \leqslant C(\varphi, \theta, \phi_0, \psi_0).$$

4. The Outer Map

In this section, we will compute the position and the velocity $(\mathbf{r}_e, \dot{\mathbf{r}}_e)$ at the return time t_2 on the sphere $B(M, \mu^\alpha)$. As we have seen in the previous section, we can write $\mathbf{R}(t_2)$ in terms of the initial coordinates as

$$\mathbf{R}(t_2) = \mathbf{R}_i + \dot{\mathbf{R}}_i \,\delta\, \mu^\alpha + O(\mu^{1-\alpha}).$$

To derive an expression for the velocity at t_2, we will use again that $\mathbf{R}_{_{TB}}$ approaches the solution of the RTBP and the definition of t_2:

$$\begin{aligned}\dot{\mathbf{R}}(t_2) &= \dot{\mathbf{R}}_{_{TB}}(t_1 + 2\pi p\tau + \delta\mu^\alpha) + O(\mu^{1-\alpha})\\ &= \dot{\mathbf{R}}_{_{TB}}(t_1) + \ddot{\mathbf{R}}_{_{TB}}(t_1)\delta\,\mu^\alpha + O(\mu^{1-\alpha})\\ &= \dot{\mathbf{R}}(t_1) - \frac{\mathbf{R}(t_1)}{R(t_1)^3}\,\delta\,\mu^\alpha + O(\mu^{1-\alpha}).\end{aligned}$$

Using the fact that $|\mathbf{R}| = |\mathbf{r}|$ and the Jacobi integral (3), it can be seen that $\mathbf{R}_i = 1 + O(\mu^\alpha)$. Therefore, the out-map in sidereal coordinates is

$$\begin{aligned}\mathbf{R}(t_2) &= \mathbf{R}_i + \dot{\mathbf{R}}_i\delta\,\mu^\alpha + O(\mu^{1-\alpha}),\\ \dot{\mathbf{R}}(t_2) &= \dot{\mathbf{R}}_i - \mathbf{R}_i\delta\,\mu^\alpha + O(\mu^{1-\alpha}).\end{aligned} \tag{31}$$

Now, we will write the outer map in synodic coordinates (the relation between sidereal and synodic coordinates has been given in the first section). From the first equation in (31), it can be obtained that

$$\begin{aligned}\mathbf{r}(t_2) &= G^T(t_2)(G(t_1)\mathbf{r}_i + (G(t_1)\dot{\mathbf{r}}_i + \dot{G}(t_1)\mathbf{r}_i)\delta\,\mu^\alpha) + O(\mu^{1-\alpha})\\ &= \begin{pmatrix} 1 & (\varepsilon-\delta)\mu^\alpha & 0\\ -(\varepsilon-\delta)\mu^\alpha & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{r}_i + \delta\,\mu^\alpha\dot{\mathbf{r}}_i + O(\mu^{1-\alpha}).\end{aligned}$$

From the second equation in (31) it follows that

$$\begin{aligned}\dot{\mathbf{r}}(t_2) + G^T(t_2)\dot{G}(t_2)\mathbf{r}(t_2) &= G^T(t_2)(G(t_1)\dot{\mathbf{r}}_i + \dot{G}(t_1)\mathbf{r}_i - \delta\,\mu^\alpha\, G(t_1)\mathbf{r}_i) +\\ &\quad + O(\mu^{1-\alpha}),\end{aligned}$$

and, simplifying, we get

$$\dot{\mathbf{r}}(t_2) = \begin{pmatrix} 1 & (\varepsilon+\delta)\mu^\alpha & 0\\ -(\varepsilon+\delta)\mu^\alpha & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\dot{\mathbf{r}}_i + O(\mu^{1-\alpha}).$$

In conclusion, the out-map in synodic coordinates is

$$\mathbf{r}_e = \left(I - \frac{(\varepsilon-\delta)}{2}\,\mu^\alpha A_3\right)\mathbf{r}_i + \delta\,\mu^\alpha\dot{\mathbf{r}}_i, \tag{32a}$$

$$\dot{\mathbf{r}}_e = \left(I - \frac{(\varepsilon+\delta)}{2}\,\mu^\alpha A_3\right)\dot{\mathbf{r}}_i, \tag{32b}$$

being I the 3×3 identity matrix. Next, let us specify each coordinate of the out-map. We define φ_e, θ_e, ϕ_e, ψ_e and v_e such that

$$\mathbf{r}_e = \begin{pmatrix} \mu - 1 + r_{2e}\cos\varphi_e\,\cos\theta_e \\ r_{2e}\cos\varphi_e\,\sin\theta_e \\ r_{2e}\sin\varphi_e \end{pmatrix}, \qquad \dot{\mathbf{r}}_e = v_e \begin{pmatrix} \cos\phi_e\,\cos\psi_e \\ \cos\phi_e\,\sin\psi_e \\ \sin\phi_e \end{pmatrix},$$

where r_{2e} is the distance from P to M. It can be computed in terms of μ^α from (32a). In fact,

$$\begin{aligned} r_{2e}^2 &= \mu^{2\alpha} - (\varepsilon-\delta)\mu^\alpha\langle \mathbf{r}_{2i}, A_3\mathbf{r}_i\rangle + 2\delta\mu^\alpha\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i\rangle + \\ &\quad + \frac{(\varepsilon-\delta)^2}{4}\mu^{2\alpha}\langle A_3\mathbf{r}_i, A_3\mathbf{r}_i\rangle - \delta(\varepsilon-\delta)\mu^{2\alpha}\langle A_3\mathbf{r}_i, \dot{\mathbf{r}}_i\rangle + \delta^2\mu^{2\alpha}v_i^2. \end{aligned}$$

Using (16), (18) and (6), the following identities can be obtained

$$\begin{aligned} \langle A_3\mathbf{r}_i, A_3\mathbf{r}_i\rangle &= 4(1 - 2\mu^\alpha\cos\varphi\,\cos\theta + O(\mu^{2\alpha})), \\ \langle \mathbf{r}_{2i}, A_3\mathbf{r}_i\rangle &= -2\mu^\alpha\cos\varphi\,\sin\theta + O(\mu^{1+\alpha}), \\ \langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i\rangle &= \mu^\alpha\sqrt{3-C_J}(\cos a_0 + \mu^\alpha\Delta C + O(\mu^{1-\alpha})), \\ \langle A_3\mathbf{r}_i, \dot{\mathbf{r}}_i\rangle &= -2\sqrt{3-C_J}[\cos\phi_0\,\sin\psi_0 + \\ &\quad +\mu^\alpha(\Delta\psi\,\cos\phi_0\,\cos\psi_0 - \Delta\phi\sin\phi_0\,\sin\psi_0 + \Lambda_0)] + \\ &\quad + O(\mu^{1-\alpha}). \end{aligned}$$

From them, it follows that

$$\begin{aligned} r_{2e}^2 &= \mu^{2\alpha}[1 - 2\mu^\alpha(\varepsilon-\delta)^2\cos\varphi\,\cos\theta + 2\mu^\alpha\delta\sqrt{3-C_J}(\Delta C + \\ &\quad +(\varepsilon-\delta)(\Delta\psi\,\cos\phi_0\,\cos\psi_0 - \\ &\quad - \Delta\phi\sin\phi_0\,\sin\psi_0 + \Lambda_0)) + O(\mu^{1-\alpha})]. \end{aligned}$$

Thus we get that $r_{2e} = \mu^\alpha[1 + \mu^\alpha\Delta r_e + O(\mu^{1-\alpha})]$ where

$$\begin{aligned} \Delta r_e &= \delta\sqrt{3-C_J}[\Lambda_0(\varepsilon-\delta) + \Delta\psi\,(\Lambda_0 + (\varepsilon-\delta)\cos\phi_0\,\cos\psi_0) + \\ &\quad + \Delta\phi\,(\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\,\cos(\psi_0-\theta) - \\ &\quad - (\varepsilon-\delta)\sin\phi_0\,\sin\psi_0)] - (\varepsilon-\delta)^2\cos\varphi\,\cos\theta, \end{aligned} \tag{33}$$

where $\Lambda_0 = \cos\varphi\,\cos\phi_0\,\sin(\theta-\psi_0)$.

Therefore, from (32a) and the expression for r_{2e}, it can be shown (see Barrabés, 2001) that

$$\begin{aligned} \cos\varphi_e\,\cos\theta_e &= \cos\varphi\,\cos\theta + \delta\sqrt{3-C_J}\,\cos\phi_0\cos\psi_0 - \\ &\quad - \mu^\alpha\Bigg[\delta\sqrt{3-C_J}(\Delta\phi\,\sin\phi_0\cos\psi_0 + \Delta\psi\,\sin\psi_0\,\cos\phi_0) - \\ &\quad - (\varepsilon-\delta)\cos\varphi\,\sin\theta + \Delta r_e\Bigg(\cos\varphi\,\cos\theta + \\ &\quad + \delta\sqrt{3-C_J}\,\cos\phi_0\cos\psi_0\Bigg)\Bigg] + O(\mu^{1-\alpha}), \end{aligned} \tag{34a}$$

$$\begin{aligned}\cos\varphi_e \ \sin\theta_e \ = \ & \cos\varphi \ \sin\theta + \delta\sqrt{3-C_J} \ \cos\phi_0 \ \sin\psi_0 + (\varepsilon-\delta) + \\ & + \mu^\alpha \Bigg[\delta\sqrt{3-C_J}(\Delta\psi \ \cos\phi_0 \ \cos\psi_0 - \Delta\phi \ \sin\psi_0 \ \sin\phi_0) - \\ & - (\varepsilon-\delta)\cos\varphi \ \cos\theta - \Delta r_e \Bigg(\cos\varphi \ \sin\theta + (\varepsilon-\delta) + \\ & + \delta\sqrt{3-C_J} \ \cos\phi_0 \ \sin\psi_0 \Bigg) \Bigg] + O(\mu^{1-\alpha}), \qquad (34b)\end{aligned}$$

$$\begin{aligned}\sin\varphi_e \ = \ & \sin\varphi + \delta\sqrt{3-C_J}\sin\phi_0 + \mu^\alpha \Bigg[\delta\sqrt{3-C_J}\Delta\phi \ \cos\phi_0 - \\ & - \Delta r_e \Bigg(\sin\varphi + \delta\sqrt{3-C_J}\sin\phi_0 \Bigg) \Bigg] + O(\mu^{1-\alpha}). \qquad (34c)\end{aligned}$$

Now, from (32b), it is easy to see that $v_e = v_i + O(\mu^{2\alpha})$ and, then, if $\cos\varphi \neq 0$, it follows that

$$\phi_e \ = \ \phi + O(\mu^{2\alpha}), \qquad (35a)$$

$$\psi_e \ = \ \psi_0 + (\Delta\psi - (\varepsilon+\delta))\mu^\alpha + O(\mu^{2\alpha}). \qquad (35b)$$

5. Conclusions

In this paper, we have studied the behaviour of the *p*-*q* resonant orbits in the spatial case. Both the analytical restrictions on the initial conditions and the explicit expressions of the outer map have been given up to terms of order μ^α. In a forthcoming paper this study will be used to get solutions of the RTBP close to periodic second species solutions.

Acknowledgements

The authors are indebted with J. Font for many fruitful discussions. The second author has been partially supported by the Catalan grants CIRIT 1998S0GR-00042 and 2000 SGR-27 and by the spanish DGICYT grant BFM2000-0805-C02.

References

Barrabés, E.: 2001, 'Òrbites de segona espcie del problema espacial de 3 cossos', PhD Thesis, Universitat Autònoma de Barcelona.

Font, J., Nunes, A. and Simó, C.: 2001, 'Successive quasi-collisions in the planar circular RTBP', *Nonlinearity* (to appear).

Perko, L.: 1996, *Differential Equations and Dynamical Systems*, 2nd edn, Springer-Verlag, New York.

Szebehely, V.: 1967, *Theory of Orbits. The Restricted Problem of Three Bodies*, Academic Press.

Yen, C.: 1985, 'Ballistic mercury orbiter mission via Venus and Mercury gravity assists', *AAS/AIAA Astrodynamics Specialist Conference, Paper AAS 85–346*.