Regularity and uniqueness of one dimensional invariant manifolds

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1 Introduction

In this work we give sufficient conditions for the existence of differentiable or analytical one dimensional manifolds associated to an eigenvalue λ and to a corresponding eigenvector v of it. We consider first the case of local diffeomorphisms, the case of differential equations being obtained easily from it. We look for the invariant manifolds through a parametrization which gives the linearization of the map along them, that is, we look for φ defined on an interval of $\mathbb{R}$ such that

$$f(\varphi(t)) = \varphi(\lambda t).$$

We restrict ourselves to eigenvalues λ of modulus different from one, the others being arbitrary, but different from zero. First we consider the differentiable case when the eigenvalue is non resonant. In this case we get that if the map is C^m with m bigger or equal than some value k , related to the structure of the eigenvalues of $Df(0)$, there is an invariant manifold of class C^m. If m is strictly bigger that k we get uniqueness. Then, for the sake of completeness, the analytic non resonant case is considered. If the eigenvalue is resonant we construct the bifurcation equation from which we obtain the conditions for the existence of solutions of class C^m. If we allow m to be strictly bigger than k, the bifurcation equation will give the number of solutions of class C^m (which may be zero). The analytic case is also considered.

2 The differentiable case

Let $U \subset \mathbb{R}^n$ be an open set, $0 \in U$, and $f : U \longrightarrow \mathbb{R}^n$ a C^m map such that $f(0) = 0$ and $L = Df(0)$ is invertible. Let λ be an eigenvalue of L such that $\mid \lambda \mid \neq 1$ and v an eigenvector associated to λ.

Theorem 2.1. *Suppose that $\lambda^j \notin$ spec (L), for $2 \leq j \leq k-1$, and either $\mid \lambda \mid^k < \mid \mu \mid$, $\forall \mu \in$ spec (L) if $\mid \lambda \mid < 1$, or $\mid \lambda \mid^k > \mid \mu \mid$, $\forall \mu \in$ spec (L) if $\mid \lambda \mid > 1$. Then if $m = k$ there exists* *a C^k map φ defined on a neighbourhood $I = [-\rho, \rho]$ of 0 such that $\varphi(0) = 0$, $\varphi'(0) = v$ and*

$$f(\varphi(t)) = \varphi(\lambda t), \quad \forall t \in I. \tag{2.1}$$

If $m \geq k+1$ there exists a unique C^m map φ satisfying the previous conditions (This contains the case $m = \infty$).

Remark 2.2. *The integer k need not to be the smaller one such that $|\lambda|^k < |\mu|$, $\forall \mu \in$ spec (L).*

The case $|\lambda| > 1$ can be immediately obtained from the case $|\lambda| < 1$. Indeed, if $|\lambda| > 1$ and v is an eigenvector associated to λ as in the statement, $Df^{-1}(0)$ has the eigenvalue λ^{-1} with eigenvector v with $|\lambda^{-1}| < |\mu|$, $\forall \mu \in$ spec (L^{-1}). Therefore we have a parametrization φ such that $\varphi(0) = 0$, $\varphi'(0) = v$ and $f^{-1}(\varphi(t)) = \varphi(\lambda^{-1}t)$ with the stated differentiability conditions. Hence φ verifies $f(\varphi(t)) = \varphi(\lambda t)$. From now on we assume that $|\lambda| < 1$.

We write $f(x) = Lx + g(x)$ with $g(0) = Dg(0) = 0$. We take a norm in $\mathbb{R}^n$ such that in the associated linear operator norm we have $\| L^{-1} \| < |\lambda|^{-k}$. We look for φ of the form $\varphi(t) = vt + \sigma(t)$ with $\sigma(0) = 0$, $\sigma'(0) = 0$. With this notation equation (2.1) becomes

$$B\sigma(t) + g(vt + \sigma(t)) = 0 \tag{2.2}$$

with $B\sigma(t) = L\sigma(t) - \sigma(\lambda t)$.

One may think of applying the implicit function theorem to equation (2.2) to get σ in terms of g. This is indeed possible and permits to obtain that if f is of class C^{k+1} there is a unique solution σ of class C^k.

The method does not give better differentiability results because to apply the Ω-lemma [1] to the functional operator associated to (2.2) we need for g to have one more degree of differentiability than φ. The details are given in the appendix.

We shall work with the space

$$C_0^k = \{\sigma : I \longrightarrow \mathbb{R}^n;\ \sigma \text{ of class } C^k, \sigma^{(k)} \text{ bounded}, \sigma(0) = \sigma'(0) = \cdots = \sigma^{(k)}(0) = 0\},$$

with a given closed interval $I = [-\rho, \rho]$. In fact ρ will play the role of a parameter which will be made as small as necessary. In C_0^k we introduce the norm

$$\| \sigma \|_k = \sup_{t \in I} \| \sigma^{(k)}(t) \|$$

which makes it a Banach space.

If $\sigma \in C_0^k$ and $t \in I$, by Taylor's theorem we have

$$\| \sigma(t) \| = \| \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1} \sigma^{(k)}(st) t^k \, ds \| \leq \frac{\rho^k}{k!} \| \sigma \|_k,$$

and for $1 \leq j \leq k$

$$\| \sigma^{(j)}(t) \| = \| \frac{1}{(k-j-1)!} \int_0^1 (1-s)^{k-j-1} \sigma^{(k)}(st) t^{k-j} \, ds \| \leq \frac{\rho^{k-j}}{(k-j)!} \| \sigma \|_k \, .$$

By the same argument we even have that

$$\| \sigma^{(j)}(\lambda t) \| \le \frac{(\lambda\rho)^{k-j}}{(k-j)!} \| \sigma \|_k .$$

To prove theorem 2.1 we begin by proving that there exists a unique polynomial p of degree k with $p(0) = 0$, $p'(0) = v$, such that verifies equation (2.1) up to order k, that is $f(p(t)) - p(\lambda t) = o(t^k)$.

Indeed, we assume that the solution of (2.1) has the form $\varphi(t) = vt + a_2 t^2 + \cdots + a_k t^k + \sigma(t)$, with $a_j \in \mathbb{R}^n$, $2 \le j \le k$, and $\sigma(0) = \sigma'(0) = \cdots = \sigma^{(k)}(0) = 0$. Then (2.1) becomes

$$L[vt + \sum_{j=2}^{k} a_j t^j] + L\sigma(t) + g(\varphi(t)) = v\lambda t + \sum_{j=2}^{k} a_j (\lambda t)^j + \sigma(\lambda t)$$

so that

$$\sum_{j=2}^{k} t^j (L - \lambda^j I) a_j = \sigma(\lambda t) - L\sigma(t) - g(\varphi(t)). \tag{2.3}$$

Since $\lambda^j \notin$ spec (L) for $2 \le j \le k$ then $(L - \lambda^j I)$ is invertible. Since $Dg(0) = 0$ we can solve recursively (2.3) for the orders $2 \le j \le k$ and obtain a unique solution for the coefficients a_j. In what follows we shall denote $p(t) = vt + a_2 t^2 + \cdots + a_k t^k$ the polynomial so obtained.

Therefore we have to solve

$$f(p(t) + \sigma(t)) = p(\lambda t) + \sigma(\lambda t) \tag{2.4}$$

for σ in C_0^k, which we rewrite as the fixed point equation

$$\sigma(t) = L^{-1} p(\lambda t) + L^{-1} \sigma(\lambda t) - L^{-1} g(p(t) + \sigma(t)) - p(t). \tag{2.5}$$

Since we shall encounter the same equation in the study of the resonant case we study it separately in the next section. Theorem 2.1 follows from proposition 3.1 below.

3 The fixed point equation

This section is devoted to prove

Proposition 3.1. *Under the hypothesis of theorem 2.1, let $p(t)$ be a polynomial of degree k such that $f(p(t)) - p(\lambda t) \in C_0^k$. Then, if m=k, equation (2.5) has a solution in C_0^k. If $m \ge k+1$ it has a unique solution in C_0^k of class C^m.*

We begin by assuming that $m = k$. There exists $r_0 > 0$ such that $V = B(0, r_0) \subset U$. We may assume that for $1 \le j \le k$, $\sup_{x \in V} \| D^j g(x) \|$ is bounded. Let M_j be a bound of it. Moreover, since $Dg(0) = 0$ we have that $M_1 = \sup_{x \in V} \| Dg(x) \| \le M_2 r_0$, so that it can be taken as small as necessary by taking an smaller r_0. Let $N_j = \sup_{t \in I} \| p^{(j)}(t) \|$ and $N = \max_{1 \le j \le k} N_j$. A very important role will be played by the quantity

$$a = |\lambda|^k \| L^{-1} \| < 1.$$

We take $r > 0$ and we let $\Sigma_r = \{\sigma \in C_0^k ; \parallel \sigma \parallel_k \leq r\}$.

We define $\Gamma_1 : \Sigma_r \longrightarrow \Sigma_r$ by

$$\Gamma_1(\sigma)(t) = L^{-1}p(\lambda t) + L^{-1}\sigma(\lambda t) - L^{-1}g(p(t) + \sigma(t)) - p(t).$$

First we shall see that if ρ is small enough Γ_1 is well defined. Since $\mid p(t) \mid = \mid p(t) - p(0) \mid \leq N_1 \mid t \mid$ and $\mid \sigma(t) \mid \leq \frac{\rho^k}{k!} \parallel \sigma \parallel_k$, if ρ is small $p(t) + \sigma(t) \in V$. $\Gamma_1(\sigma)$ is of class C^k with the k^{th} derivative bounded and, by construction of p, $\Gamma_1(\sigma)$ and its derivatives up to order k vanish at 0.

Let $\sigma \in \Sigma_r$. To prove that $\Gamma_1(\sigma) \in \Sigma_r$ we have to bound

$$(\Gamma_1(\sigma))^{(k)}(t) = \lambda^k L^{-1}[p^{(k)}(\lambda t) + \sigma^{(k)}(\lambda t)] - L^{-1}[g \circ (p + \sigma)]^{(k)}(t) - p^{(k)}(t).$$

By construction of p, $(\Gamma_1(\sigma))^{(k)}(0) = 0$, which implies that

$$\lambda^k L^{-1}p^{(k)}(0) - p^{(k)}(0) = L^{-1}[g \circ (p + \sigma)]^{(k)}(0),$$

but the left hand side is equal to $\lambda^k L^{-1}p^{(k)}(\lambda t) - p^{(k)}(t)$ because p is a polynomial of degree k. Then we can write

$$(\Gamma_1(\sigma))^{(k)}(t) = \lambda^k L^{-1}\sigma^{(k)}(\lambda t) + L^{-1}[g \circ (p + \sigma)]^{(k)}(0) - L^{-1}[g \circ (p + \sigma)]^{(k)}(t).$$

The first term is bounded by $a \parallel \sigma \parallel_k$. Let us call $\theta_1(\rho)$ the sum of the remaining terms. Here we recall the formula for the k^{th} derivative of the composition $g \circ h$, for $h : I \subset \mathbb{R} \longrightarrow U \subset \mathbb{R}^n$ and $g : U \longrightarrow \mathbb{R}^n$,

$$(g \circ h)^{(j)}(t) = \sum_{i=1}^{j} \sum_{*} c(j, i; j_1, \ldots, j_i) D^i g(h(x))(h^{(j_1)}(x), \ldots, h^{(j_i)}(x)),$$

where c is an integer which depends on $j, i, j_1, \ldots, j_i$ and $\sum_*$ indicates sum over the indices $j_1, \ldots, j_i$ such that $1 \leq j_1 \leq \cdots \leq j_i \leq i$ and $j_1 + \cdots + j_i = j$.

$\theta_1(\rho)$ can be written as

$$\begin{aligned} L^{-1}[&\sum_{i=1}^{k} \sum_{*} c(k, i; j_1, \ldots, j_i) D^i g(0)(p^{(j_1)}(0), \ldots, p^{(j_i)}(0)) \\ - &\sum_{i=1}^{k} \sum_{*} c(k, i; j_1, \ldots, j_i) D^i g(p(t) + \sigma(t))((p + \sigma)^{(j_1)}(t), \ldots, (p + \sigma)^{(j_i)}(t))], \end{aligned}$$

with $j_1 + \cdots + j_i = k$. We decompose the differences in telescopic form so that in any one of them there is only one different argument. There appear terms of the form

$$D^i g(0)(p^{(j_1)}(0), \ldots, p^{(j_l)}(0) - p^{(j_l)}(t) - \sigma^{(j_l)}(t), \ldots, p^{(j_i)}(t) + \sigma^{(j_i)}(t)), \quad 1 \leq i \leq k, \ 1 \leq l \leq i,$$

and

$$[D^i g(0) - D^i g(p(t) + \sigma(t))]((p + \sigma)^{(j_1)}(t), \ldots, (p + \sigma)^{(j_i)}(t)), \quad 1 \leq i \leq k.$$

The terms of the first form are bounded by

$$\| D^i g(0) \| N^{l-1}(N\rho + \frac{\rho^{k-j_l}}{(k-j_l)!} \| \sigma \|_k)(N+ \| \sigma \|_k)^{i-l}.$$

Notice that the case $j_l = k$ only can happen when $i = 1$ in which case $Dg(0) = 0$. The terms of the second form are bounded by

$$M_{i+1}(N_1\rho + \frac{\rho^k}{k!} \| \sigma \|_k) \prod_{l=1}^{i} (N + \frac{\rho^{k-j_l}}{(k-j_l)!} \| \sigma \|_k), \quad 1 \le i \le k-1,$$

but for $i = k$ we have to use the continuity of $D^k g$. If we make ρ small enough, since $\| p(t) + \sigma(t) \| \le N_1\rho + \frac{\rho^k}{k!} \| \sigma \|_k$, we get that $D^k g(0) - D^k g(p(t)+\sigma(t))$ can be as small as we need. Therefore $\lim_{\rho\to 0} \theta_1(\rho) = 0$, and hence $\| (\Gamma_1(\sigma))^{(k)}(t) \| < ar + \theta_1(\rho) < r$ if ρ is small enough.

Now we consider the sequence (σ_m) defined by

$$\begin{aligned} \sigma_0 &= 0, \\ \sigma_{m+1} &= \Gamma_1(\sigma_m). \end{aligned}$$

By the previous argument, if ρ is small enough, $\sigma_m \in \Sigma_r$ and then it is uniformly bounded. We check that it is equicontinuous. In a precise way, let

$$\eta_1(\delta) = \sup_{\|\Delta x\| \le \delta} \| D^k g(x + \Delta x) - D^k g(x) \|,$$

and

$$\eta(\delta) = \frac{c_1 \eta_1(\alpha\delta) + c_2\delta}{1-\beta},$$

with $\alpha = N_1 + \| \sigma'_m \| \le (N_1 + \frac{\rho^{k-1}}{(k-1)!} r)$, $\beta = a + \| L^{-1} \| M_1 + \theta_2(\rho)$ and $\theta_2(\rho)$ a function which tends to zero as $\rho \to 0$ and c_1, c_2 constants to be determined below. We assume that r_0 and ρ are such that $\beta < 1$. We write $\Delta\sigma_m^{(k)}(t) = \sigma_m^{(k)}(t + \Delta t) - \sigma_m^{(k)}(t)$. Then we shall prove by induction on m that if $| \Delta t | < \delta$ with $t + \Delta t \in I$ then $\| \Delta\sigma_m^{(k)}(t) \| < \eta(\delta)$. Taking into account that $[L^{-1}p(\lambda t) - p(t)]^{(k)}$ is constant and that $\| \sigma_m^{(j)}(t+\Delta t) - \sigma_m^{(j)}(t) \| \le \frac{\rho^{k-j}}{(k-j)!} \| \Delta\sigma_m^{(k)}(t) \|$ we have

$$\begin{aligned} \Delta\sigma_{m+1}^{(k)}(t) &= \lambda^k L^{-1} \Delta\sigma_m^{(k)}(t) \\ &+ L^{-1} \sum_{i=1}^{k} \sum_{*} c(k,i;j_1,\ldots,j_i) \\ &[D^i g(p(t+\Delta t) + \sigma_m(t+\Delta t))(p^{(j_1)}(t+\Delta t) + \sigma_m^{(j_1)}(t+\Delta t), \ldots) \\ &- D^i g(p(t) + \sigma_m(t))(p^{(j_1)}(t) + \sigma_m^{(j_1)}(t), \ldots)]. \end{aligned} \tag{3.1}$$

The terms in the last difference can be decomposed as

$$\begin{aligned} &[D^i g(p(t+\Delta t) + \sigma_m(t+\Delta t)) - D^i g(p(t) + \sigma_m(t))](p^{(j_1)}(t+\Delta t) + \sigma_m^{(j_1)}(t+\Delta t), \ldots) \\ &+ \sum_{l=1}^{i} D^i g(p(t) + \sigma_m(t))(\ldots, p^{(j_l)}(t+\Delta t) + \sigma_m^{(j_l)}(t+\Delta t) - p^{(j_l)}(t) - \sigma_m^{(j_l)}(t), \ldots). \end{aligned}$$

We notice that

$$\| p(t+\Delta t)+\sigma_m(t+\Delta t)-p(t)-\sigma_m(t) \| \leq (N_1+\| \sigma'_m \|) \mid \Delta t \mid \leq \alpha \mid \Delta t \mid .$$

Hence the last terms of (3.2) are bounded by

$$\begin{aligned}
&\| L^{-1}[\| (M_2(N_1 \mid \Delta t \mid + \| \Delta\sigma_m(t) \|)(N_k+\| \sigma_m^{(k)} \|)+M_1 \| \Delta\sigma_m^{(k)}(t) \| \\
&+\sum_{i=2}^{k-1}\sum_{*} c(k,i;j_1,\dots,j_i) \\
&\quad [M_{i+1}(N_1 \mid \Delta t \mid + \| \Delta\sigma_m(t) \|)+\sum_{l=1}^{i} M_i(N+\| \sigma_m \|_k)^{i-1}(N \mid \Delta t \mid + \| \Delta\sigma_m^{(j_l)} \|) \\
&\quad +\eta_1(\alpha \mid \Delta t \mid)(N_1+\| \sigma'_m \|)^k+\sum_{l=1}^{i} M_k(N_1+\| \sigma'_m \|)^{k-1}(N_2 \mid \Delta t \mid + \| \Delta\sigma'_m(t) \|)].
\end{aligned}$$

Then

$$\| \Delta\sigma_{m+1}^{(k)}(t) \| \leq [a+\| L^{-1} \| M_1+\theta_2(\rho)]\eta(\delta)+c_1\eta_1(\alpha\delta)+c_2\delta<\eta(\delta).$$

Now we claim that there exist $\gamma<1$ such that

$$\| \sigma_{m+1}^{(k-1)}(t)-\sigma_m^{(k-1)}(t) \| \leq K\gamma^{m+1}, \qquad \text{for all } m\geq 0.$$

Indeed, for m=0 it is true. If it is true for m-1, we write

$$\begin{aligned}
\| \sigma_{m+1}^{(k-1)}(t)-\sigma_m^{(k-1)}(t) \| &\leq a \| \sigma_m^{(k-1)}(t)-\sigma_{m-1}^{(k-1)}(t) \| \\
&\quad + \| L^{-1} \| \| [g\circ(p+\sigma_m)]^{(k)}(t)-[g\circ(p+\sigma_{m-1})]^{(k)}(t) \| .
\end{aligned}$$

Decomposing the last term in the same way as before we get, taking ρ sufficiently small, the result for some $\gamma>a$.

From the previous claims we obtain that $(\sigma_m^{(k-1)})$ is uniformly convergent. Since all the derivatives up to order k are zero at zero, σ_m converges to some function σ of class C^{k-1}. Also we have obtained that $(\sigma_m^{(k)})$ satisfies the hypothesis of the theorem of Arzelà. Then $(\sigma_m^{(k)})$ has a uniformly convergent subsequence and therefore the limit function $\sigma\in C_0^k$.

Now we assume that f is of class C^{k+1}. To get the uniqueness of the solution of class C^{k+1} we shall use the fiber contraction lemma in the following form [2].

Lemma 3.2. *Let E,F be metric spaces and $\Gamma: E\times F \longrightarrow E\times F$ be of the form $\Gamma(x,A)=(\Gamma_1(x),\Gamma_2(x,A))$ such that*

(1) Γ_1 is a contraction,

(2) for all $A\in F$, $\Gamma_2(.,A)$ is continuous in E,

(3) for all $x\in E$, $\Gamma_2(x,.)$ is a λ-contraction with $\lambda<1$.

Then Γ has a unique fixed point, which is an attractor.

We check that Γ_1 is a contraction:

$$\begin{aligned} &\| \Gamma_1(\sigma) - \Gamma_1(\tau) \|_k \\ &= \sup_{t\in I} \| [L^{-1}(\sigma(\lambda t) - \tau(\lambda t)) - L^{-1}(g(p(t)+\sigma(t)) - g(p(t)+\tau(t)))]^{(k)} \| \end{aligned}$$

The norm of the first term is bounded by

$$| \lambda |^k \| L^{-1} \| \| \sigma - \tau \|_k .$$

To deal with the second one we make use once more of the formula for the k^{th} derivative of the composition and we decompose the differences in telescopic form so that in any term there is only one different argument. In this way we get terms of the form

$$[D^i g(p(t)+\sigma(t)) - D^i g(p(t)+\tau(t))](p^{(j_1)}(t)+\sigma^{(j_1)}(t), \dots, p^{(j_i)}(t)+\sigma^{(j_i)}(t)), \quad 1 \le i \le k,$$

and

$$D^i g(p(t)+\sigma(t))(p^{(j_1)}(t)+\sigma^{(j_1)}(t), \dots, \sigma^{(j_l)}(t) - \tau^{(j_l)}(t), \dots, p^{(j_i)}(t)+\sigma^{(j_i)}(t)), \quad 1 \le i \le k.$$

Then we need the bounds

$$\begin{aligned} &\| D^i g(p(t)+\sigma(t)) - D^i g(p(t)+\tau(t)) \| \\ &\le \| D^{i+1} g \| \| \sigma(t) - \tau(t) \| \le M_{i+1} \frac{\rho^k}{k!} \| \sigma - \tau \|_k, \end{aligned}$$

$$\| p^{(j_l)}(t) + \sigma^{(j_l)}(t) \| \le c$$

for some c, and

$$\| \sigma^{(j_l)}(t) - \tau^{(j_l)}(t) \| \le \frac{\rho^{k-j_l}}{(k-j_l)!} \| \sigma - \tau \|_k .$$

Therefore the bounds of all terms have the factor $\| \sigma - \tau \|_k$. One of them is $a \| \sigma - \tau \|_k$ and all the others explicitly have ρ as a multiplicative factor, except when $j_l = k$. In such case we have that $l = 1 = i$ and $j = k$. It occurs only in the term

$$\begin{aligned} &\| Dg(p(t)+\tau(t))(p^{(k)}(t)+\sigma^{(k)}(t)) - Dg(p(t)+\tau(t))(p^{(k)}(t)+\tau^{(k)}(t) \| \\ &\le \sup_{|x|\le r_0} \| Dg(x) \| \| \sigma - \tau \|_k = M_1 \| \sigma - \tau \|_k . \end{aligned}$$

Then Γ_1 is a contraction.

Now we define $\Gamma_2 : \Sigma_r \times C_0^k \longrightarrow C_0^k$ by

$$\Gamma_2(\sigma, A) = \lambda L^{-1} p'(\lambda t) + \lambda L^{-1} A(\lambda t) - L^{-1} Dg(p(t)+\sigma(t))(p'(t)+A(t)) - p'(t).$$

It is continuous and $\Gamma_2(\sigma, .)$ is a contraction. Indeed, if $A, B \in C_0^k$

$$\begin{aligned} &\| \Gamma_2(\sigma, A) - \Gamma_2(\sigma, B) \|_k \\ &\le \| \lambda^{k+1} L^{-1} [A^{(k)}(\lambda t) - B^{(k)}(\lambda t)] \| + \| L^{-1} [Dg(p(t)+\sigma(t))(A(t) - B(t))]^{(k)} \| . \end{aligned}$$

The first term is bounded by

$$| \lambda^{k+1} | \| L^{-1} \| \| A - B \|_k$$

and the second one by

$$\| L^{-1}\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)[Dg(p(t)+\sigma(t))]^{(j)}[A(t)-B(t)]^{(k-j)} \|$$

$$\leq \quad \| L^{-1} \| \| \sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)\sum_{i=0}^{j}\sum_{*} c(j,i;j_1,\dots,j_i)D^{i+1}g(p(t)+\sigma(t))(p^{(j_1)}(t)+\sigma^{(j_1)}(t),\dots$$

$$\dots,p^{(j_i)}(t)+\sigma^{(j_i)}(t)) \| \frac{\rho^j}{j!} \| A-B \|_k .$$

From this we see that all terms have a multiplicative factor ρ except the term corresponding to $j=0$. This term is bounded by

$$\| L^{-1}Dg(p(t)+\sigma(t))(A(t)-B(t))^{(k)} \| \leq \| L^{-1} \| \sup_{|x|\leq r_0} \| Dg(x) \| \| A-B \|_k$$

and $\sup_{|x|\leq r_0} \| Dg(x) \|$, as before, can be made as small as we want. Then Γ_2 is a contraction. By the fiber contraction lemma we get that Γ has a unique fixed point which is an attractor. Let $\sigma_0 \in C_0^k$ of class C^{k+1}. By the definition of Γ_2, $\Gamma(\sigma_0,\sigma_0')=(\sigma_1,\sigma_1')$ with $\sigma_1(t)= L^{-1}p(\lambda t)+L^{-1}\sigma_0(\lambda t)-L^{-1}g(p(t)+\sigma_0(t))-p(t)$. By induction $\Gamma^m(\sigma_0,\sigma_0')=(\sigma_m,\sigma_m')$ and it converges to a fixed point (σ,A) of Γ. Hence (σ_m') converges uniformly to some function A in C_0^k so that $\sigma'=A$ and we get that $\sigma\in C^{k+1}$.

If $m>k+1$ we can take $k_1=m-1$ which obviously satisfies the condition $| \lambda |^{k_1} < | \mu |$ for all $\mu \in$ spec (L) and we obtain a unique solution of class C^m.

4 The analytic case

Theorem 4.1. *Let $U\subset \mathbb{R}^n$ be an open set, $0\in U$, and $f:U\longrightarrow \mathbb{R}^n$ a analytic map such that $f(0)=0$ and $Df(0)=L$ is invertible. Let λ be an eigenvalue of L such that $|\lambda|\neq 1$ and $\lambda^j\notin$ spec (L), $\forall j\geq 2$, and v an eigenvector associated to λ. Then there exists a unique real analytic map φ defined on a neighbourhood Ω of $0\in\mathbb{C}$ such that $\varphi(0)=0$, $\varphi'(0)=v$ and*

$$f(\varphi(t))=\varphi(\lambda t), \quad \forall t\in\Omega. \tag{4.1}$$

Proof By the argument which follows remark 2.2 we may restrict ourselves to the case $|\lambda|<1$. We write $f(x)=Lx+g(x)$ with $g(0)=0$ and $Dg(0)=0$ and we look for φ of the form $\varphi(t)=vt+\sigma(t)$, with $\sigma(0)=0$ and $\sigma'(0)=0$, such that $f(\varphi(t))=\varphi(\lambda t)$. Hence the equation for σ is

$$B\sigma(t) \ +g(vt+\sigma(t))=0,$$

with $B\sigma(t)=L\sigma(t)-\sigma(\lambda t)$. g extends analytically to a complex neighbourhood $\tilde{U}$ of 0 in $\mathbb{C}^n$, being bounded on it. Let $B_r=B(0,r)\subset\tilde{U}$ be the ball of radius r centered at zero in $\mathbb{C}^n$, and $\Omega=D(0,r/(2\| v \|))\subset\mathbb{C}$. We introduce the Banach spaces

$$E=\{g:B_r\longrightarrow\mathbb{C}^n;\ \text{real analytic},\ Dg \text{ bounded},\ g(0)=0,\ Dg(0)=0\},$$

with the norm $\| g \| = \sup_{\zeta \in B_\delta} \| Dg(\zeta) \|$,

$$F = \{\sigma : \Omega \longrightarrow \mathbb{C}^n; \ \text{ real analytic, bounded in } \Omega, \ \sigma(0) = 0, \ \sigma'(0) = 0\},$$

with the norm $\| \sigma \| = \sup_{\zeta \in \Omega} \| \sigma(\zeta) \|$. We write $F_\delta = \{\sigma \in F; \ \| \sigma \| < \delta\}$, and we take $\delta = r/2$.

We define $\Gamma : E \times F_\delta \longrightarrow F$ by

$$\Gamma(g, \sigma)(t) = B\sigma(t) + g(vt + \sigma(t)).$$

It is easy to check that Γ is of class C^1, $\Gamma(0,0) = 0$ and that $D_\sigma \Gamma(0, \sigma)\Delta\sigma = B\Delta\sigma$. Now we prove that $D_\sigma \Gamma(0, \sigma)$ is invertible. Let $\psi \in F$, $\psi(t) = \sum_{k=2}^{\infty} b_k t^k$. Writing $\Delta\sigma(t) = \sum_{k=2}^{\infty} a_k t^k$, the condition $D_\sigma \Gamma(0, \sigma)\Delta\sigma = \psi$ implies,

$$L \sum_{k \geq 2} a_k t^k - \sum_{k \geq 2} a_k \lambda^k t^k = \sum_{k \geq 2} b_k t^k,$$

so that

$$a_k = (L - \lambda^k I)^{-1} b_k, \quad k \geq 2.$$

This provides the uniqueness of $\Delta\sigma$. On the other hand, since $| \lambda | < 1$ and the inverse operator on linear maps is continuous we have $(L - \lambda^k I)^{-1} \to L^{-1}$. Therefore the radius of convergence of $\sum a_k t^k$ is the same as the one of $\sum b_k t^k$. Furthermore on Ω we have $\Delta\sigma(t) = L^{-1}(\psi(t) + \Delta\sigma(\lambda t))$ which permits to extend $\Delta\sigma$ to some open set which contains the boundary of Ω intersected by the domain of ψ and hence $\Delta\sigma$ is bounded on Ω. Then we can apply the implicit function theorem to Γ in a neighbourhood of $(0, 0)$ and get $\sigma = \sigma^*(g)$. Scaling the variables, if necessary, to have $\| g \|$ sufficiently small, we get a unique solution of (4.1).

□

5 The resonant case

Now we consider the case when there exist indices $p \geq 2$ such that $\lambda^p \in \text{spec } (L)$. We shall refer to such a λ as a resonant eigenvalue and to such p as a resonant index. Now, in the equation

$$Bu(t) - g(vt + u(t)) = 0, \tag{5.1}$$

the linear operator B is not invertible but we can apply the Lyapunov-Schmidt method.

Let k be such that $| \lambda |^k < | \mu |, \forall \mu \in \text{spec } (L)$. Let $N_j = \text{Nuc } (L - \lambda^j I)$ and N_j' an arbitrary complementary subspace in $\mathbb{R}^n$. Let $R_j = \text{Im} \, (L - \lambda^j I)$ and R_j' an arbitrary complementary subspace of it.

Remark 5.1. *If we have $Nuc(L - \lambda^p) = Nuc(L - \lambda^p)^2$ (that is, the Jordan box associated to λ^p is diagonal) for the resonant indices then we have $\mathbb{R}^n = Nuc(L - \lambda^p) \oplus Im(L - \lambda^p)$ so that we can take $N_p' = R_p$ and $R_p' = N_p$.*

We consider the space F of lemma 9.1 and we introduce the following subspaces

$$\begin{aligned}
P_1 &= \{\sum_{j=2}^{k} a_j t^j;\ a_j \in \text{Nuc}\ (L - \lambda^j I)\},\\
P_2 &= \{\sum_{j=2}^{k} a_j t^j;\ a_j \in N'_j\},\\
Q_1 &= \{\sum_{j=2}^{k} a_j t^j;\ a_j \in R_j\},\\
Q_2 &= \{\sum_{j=2}^{k} a_j t^j;\ a_j \in R'_j\},\\
G &= \{\sigma \in C^k;\ \sigma(0) = \cdots = \sigma^{(k)}(0) = 0\}.
\end{aligned}$$

We have the decompositions $F = P_1 \oplus P_2 \oplus G$ and $F = Q_1 \oplus Q_2 \oplus G$. Clearly $B_{|P_1} = 0$ and $B : P_2 \longrightarrow Q_1$ and $B : G \longrightarrow G$ are isomorphisms.

Let Π_{Q_1}, Π_{Q_2} and Π_G be the projections to the subspaces Q_1, Q_2, G. We have $\Pi_{Q_1} B = B\Pi_{P_2}$, $\Pi_{Q_2} B = 0$ and $\Pi_G B = B\Pi_G$. For $u \in F$ we write $u(t) = p_1(t) + p_2(t) + \sigma(t)$ according to the first decomposition of F. By means of these projections, equation (5.1) can be rewritten as

$$Bp_2(t) + \Pi_{Q_1}[g(vt + p_1(t) + p_2(t) + \sigma(t))] = 0, \tag{5.2}$$

$$\Pi_{Q_2}[g(vt + p_1(t) + p_2(t) + \sigma(t))] = 0, \tag{5.3}$$

$$B\sigma(t) + \Pi_G[g(vt + p_1(t) + p_2(t) + \sigma(t))] = 0. \tag{5.4}$$

Notice that actually equations (5.2) and (5.3) do not depend on σ due to the projections Π_{Q_1} and Π_{Q_2} In equation (5.2) we can apply the implicit function theorem to obtain $p_2 = p_2^*(p_1, g)$, since $\Pi_{|Q_1} \circ g$ is a polynomial and therefore of class C^∞.

We substitute p_2 into equation (5.3) and we get

$$\Pi_{Q_2}[g(vt + p_1(t) + p_2^*(p_1, g)(t))] = 0. \tag{5.5}$$

which is called the bifurcation equation. It is on a finite dimensional space. If it has solutions $p_1 = p_1^*(g)$ we substitute them in equation (5.4) which becomes

$$B\sigma(t) + \Pi_G[g(vt + p_1^*(g)(t) + p_2^*(p_1^*(g), g)(t) + \sigma(t))] = 0, \tag{5.6}$$

and can be rewritten as the fixed point equation

$$\sigma(t) = L^{-1}\sigma(\lambda t) - L^{-1}\Pi_G[g(vt + p_1^*(g)(t) + p_2^*(p_1^*(g), g)(t) + \sigma(t))] = 0, \tag{5.7}$$

which is of the form (2.5) so that proposition 3.1 applies to it and we have that if f is of class C^k equation (5.7) has a C^k solution if f is of class C^k and a unique solution of class C^{k+1} if f is of class C^{k+1}.

In the analytical case we consider equation (5.6) for $\sigma \in G$, and we solve it by applying the implicit function theorem in a similar way as in section 4. We define $E_k = \{g \in E;\ D^j g(0) = 0, 0 \leq j \leq k\}$, and $F_k = \{\sigma \in F;\ \sigma^{(j)}(0) = 0, 0 \leq j \leq k\}$, and $\Gamma : E_k \times F_k \longrightarrow F_k$ by

$$\Gamma(g,\sigma)(t) = B\sigma(t) + \Pi_G[g(vt + p_1^*(g)(t) + p_2^*(p_1^*(g), g)(t) + \sigma(t))] = 0.$$

As in section 4, Γ is C^1, $\Gamma(0,0) = 0$ and $D_\sigma\Gamma(0,0) = B$ but this time we have to consider $B : G \longrightarrow G$. The argument used in the proof of theorem 4.1, taking into account that $L - \lambda^j I$ is invertible for $j \geq k$, works here and we have that B is invertible.

Summarizing we have proved

Theorem 5.2. *Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^m map such that $f(0) = 0$ and $Df(0) = L$ is invertible. Let $\lambda \in$ spec (L) and v be an eigenvector associated to it. Let $k \in \mathbb{Z}$ be such that $|\lambda|^k < |\mu|$ for all $\mu \in$ spec (L), if $|\lambda| < 1$, or $|\lambda|^k > |\mu|$ for all $\mu \in$ spec (L), if $|\lambda| > 1$. Assume that $m \geq k$. Then there is a C^m solution of*

$$f(\varphi(t)) = \varphi(\lambda t), \quad \varphi(0) = 0, \quad \varphi'(0) = v,$$

if and only if equation 5.5 has a solution. Furthermore, if $m \geq k+1$ or if f is analytic there will be as many C^m solutions or analytic solutions as equation 5.5 has.

6 The general resonant case in $\mathbb{R}^2$

In $\mathbb{R}^2$ an eigenvalue λ is resonant if the eigenvalues are of the form λ, λ^p with $p \in \mathbb{N}$, $p \geq 2$. In order to deal comfortably with the bifurcation equation first we put the map in normal form. The normal form is

$$f(x,y) = (\lambda x, \lambda^p y + cx^p) + O_{p+1}.$$

We look for the invariant manifold associated to λ and $v = (1,0)$. In this case $P_1 = \{\begin{pmatrix} 0 \\ \beta \end{pmatrix} t^p;\ \beta \in \mathbb{R}\} = Q_2$ and $Q_1 = \{\sum_{j=2}^{p-1} a_j t^j + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} t^p;\ a_j \in \mathbb{R}^2,\ \alpha \in \mathbb{R}\} = P_2$. Now equations (5.2), (5.3), (5.4) become

$$Bp_2 + \Pi_{Q_1}[\begin{pmatrix} O(t^{p+1}) \\ c(t^p + \dots)^p \end{pmatrix} = Bp_2 = 0, \tag{6.1}$$

$$\Pi_{Q_2}[\begin{pmatrix} O(t^{p+1}) \\ c(t^p + \dots)^p \end{pmatrix} = \begin{pmatrix} 0 \\ ct^p \end{pmatrix} = 0, \tag{6.2}$$

$$B\sigma(t) + \Pi_G[g(vt + p_1(t) + p_2(t) + \sigma(t))] = 0. \tag{6.3}$$

We see that the bifurcation equation is independent of p_1 and p_2. Therefore we have a solution if and only if $c = 0$. If $c = 0$ the invariant manifold depends on an arbitrary $p_1 \in P_1$, that is, it depends on a real parameter. From the previous section we have that if f is of class C^{p+1} (analytic), for any $\beta \in \mathbb{R}$ we have a unique C^{p+1} (analytic) invariant manifold of the form

$$\sigma(t) = (t, \beta t^p) + O(t^{p+1}),$$

if $c = 0$, and none of this class of differentiability if $c \neq 0$.

7 One example in $\mathbb{R}^3$

Let f be a map such that spec $(Df(0)) = \{\lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 = \lambda$, $\lambda_2 = \lambda^2$ and $\lambda_3 = \lambda^6$. Here we have the resonances $\lambda_2 = \lambda_1^2$, $\lambda_3 = \lambda_1^6$, $\lambda_3 = \lambda_1^4\lambda_2$, $\lambda_3 = \lambda_1^2\lambda_2^2$ and $\lambda_3 = \lambda_2^3$. We assume that f is of class C^m, $m \geq 6$. In this case the normal form of f up to terms of order 6 is

$$\begin{aligned} f_1(x,y,z) &= \lambda x, \\ f_2(x,y,z) &= \lambda^2 y + cx^2, \\ f_3(x,y,z) &= \lambda^6 z + d_1 y^3 + d_2 x^2 y^2 + d_3 x^4 y + d_4 x^6. \end{aligned}$$

We look for the invariant manifolds associated to λ and $v = (1,0,0)$.

We have that

$$P_1 = Q_2 = \{(0, \delta t^2, \eta t^6)\},$$

$$P_2 = Q_1 = \{(\sum_{j=2}^{6} \alpha_j t^j, \sum_{j=3}^{6} \beta_j t^j, \sum_{j=2}^{5} \gamma_j t^j);\ \alpha_j, \beta_j, \gamma_j \in \mathbb{R}\}.$$

Now equation 5.2 has the form

$$Bp_2(t) + \Pi_{Q_1}[\begin{pmatrix} 0 \\ cx^2 \\ d_1 y^3 + d_2 x^2 y^2 + d_3 x^4 y + d_4 x^6 \end{pmatrix}(vt + p_1(t) + p_2(t))] = 0. \tag{7.1}$$

Solving the 3 components of (7.1) for p_2 in order we get that it is zero, so that $p^*(p_1, g) = 0$. Therefore the bifurcation equation is

$$\begin{pmatrix} 0 \\ ct^2 \\ d_1\delta 3 t^6 + d_2\delta^2 t^6 + d_3\delta t^6 + d_4 t^6 \end{pmatrix} = 0. \tag{7.2}$$

Then in order to have C^6 solutions it is necessary and sufficient that

$$c = 0, \tag{7.3}$$

$$d_1\delta 3 + d_2\delta^2 + d_3\delta + d_4 = 0. \tag{7.4}$$

Notice that equation (7.4) has one, three or ∞ real solutions for δ. Hence if $c = 0$ there will be C^m (analytic) solutions of the form

$$\varphi(t) = (t, \delta t^2, \eta t^6) + o(t^6).$$

with $\eta \in \mathbb{R}$ and δ being a solution of (7.4). The solution of this form will be unique if f is of class C^m, $m \geq 7$ (analytic).

8 Differential equations

The analogous results for differential equations are immediately obtained considering the time 1 map.

9 Appendix

Here we show how to apply the implicit function theorem to get a C^k solution of equation (2.2). We assume that $\mid \lambda \mid^k < \mid \mu \mid$ for all $\mu \in$ spec (L). Let $r_0 > 0$ be such that $V = B(0, r_0) \subset U$. We introduce the spaces

$$E = \{g \in C^{k+1}(V, \mathbb{R}^n);\ D^j g \text{ bounded }, 0 \leq j \leq k+1,\ g(0) = 0, Dg(0) = 0\}$$

with the norm

$$\| g \| = \max_{0 \leq j \leq k+1} \sup_{x \in B_r} \| D^j g(x) \|,$$

and

$$F = \{\sigma \in C^k([-\rho, \rho], \mathbb{R}^n);\ \sigma(0) = 0, \sigma'(0) = 0\}$$

with $\rho = r/(2 \| v \|)$ and the norm

$$\| \sigma \| = \max_{0 \leq j \leq k} \sup_{t \in I} \| \sigma^{(j)}(t) \| .$$

Let F_δ be the ball of radius δ in F.

We define $\Gamma : E \times F_{r/2} \longrightarrow F$ by

$$\Gamma(g, \sigma)(t) = B\sigma(t) + g(vt + \sigma(t)).$$

Γ is well defined, is of class C^1 by the Ω-lemma , $\Gamma(0,0) = 0$ and $D_\sigma \Gamma(0, \sigma)\Delta\sigma = L\Delta\sigma - \Delta\sigma \circ \lambda$. Let $B = D_\sigma \Gamma(0, \sigma)$. In the next lema we shall prove that B is invertible.

Now we can apply the implicit function theorem to Γ to obtain that there exists a C^1 function $\sigma = \sigma^*(g)$ such that $\Gamma(g, \sigma^*(g)) = 0$. It provides a unique C^k solution σ of $\Gamma(g, \sigma) = 0$ in a neighbourhood of $(0, 0)$. Finally we recall that given a function f as in the statement, $g = f - L$ can be made as small as we want by scaling variables in the usual way.

Lemma 9.1. *The linear map $B : F \longrightarrow F$ is invertible.*

Proof Let $P^k = \{a_2 t^2 + \cdots + a_k t^k;\ a_i \in \mathbb{R}^n\} \subset F$ and $F^k = \{\sigma \in C^k;\ \sigma(0) = \cdots = \sigma^{(k)}(0) = 0\}$. Clearly $F = P^k \oplus F^k$ is an invariant decomposition of F. Let $B_1 = B_{|P^k} : P^k \longrightarrow P^k$ and $B_2 = B_{|F^k} : F^k \longrightarrow F^k$. Let $\sigma = a_2 t^2 + \cdots + a_k t^k$ and $B_1(\sigma) = \sum_{j=2}^k [La_j - \lambda^j a_j] t^j = 0$. This means that $La_j - \lambda^j a_j = 0$ for $2 \leq j \leq k$ but since λ^j is not an eigenvalue we have $a_j = 0, \forall j$. Then B_1 is one to one and hence invertible since P^k has finite dimension. Now we are going to prove that B_2 is invertible. To show that $B_2 : F^k \longrightarrow F^k$ is onto we check that $C : F^k \longrightarrow F^k$ defined by

$$C(\psi)(t) = \sum_{j=1}^{\infty} L^{-j} \psi(\lambda^{j-1} t).$$

is well defined and satisfies $B_2 C = I$. First we define $\psi_j(t) = L^{-j}\psi(\lambda^{j-1}t)$. The series of the k^{th} derivatives is

$$\sum_{j \geq 1} L^{-j} \lambda^{k(j-1)} \psi^{(k)}(\lambda^{j-1} t).$$

It is uniformly convergent since it is majorated by

$$\sum_{j\geq 1} \mid \lambda \mid^{k(j-1)} \| L^{-j} \| \| \psi \| = \sum_{j\geq 1} \mid \lambda \mid^{-k} \| \psi \| (\mid \lambda \mid^{k} \| L^{-j} \|^{1/j})^{j}$$

and $\mid \lambda \mid^k \| L^{-j} \|^{1/j}$ tends to a limit less than one. Furthermore $\sum_{j\geq 1} \psi_j^{(s)}(t)$ converges at $t = 0$ for $0 \leq s \leq k$ because $\psi_j^{(s)}(0) = 0$ for $0 \leq s \leq k$. Therefore $\sum_{j\geq 1} \psi_j^{(s)}(t)$ converges uniformly and so $\sum \psi_j(t)$ converges to some function of F^k. $B_2(\sum_{j\geq 1} \psi_j(t)) = \sum_{j\geq 1} L^{-(j-1)} \psi(\lambda^{j-1} t) - \sum_{j\geq 1} L^{-j} \psi(\lambda^j t) = \psi(t)$. To show that B_2 is one to one let $\sigma \in F^k$ such that $B_2 \sigma = 0$. Then, derivating k times this relation we have

$$L\sigma^{(k)}(t) = \lambda^k \sigma^{(k)}(\lambda t).$$

It follows by induction that for all $i \geq 0$

$$\sigma^{(k)}(t) = \lambda^{ik} L^{-i} \sigma^{(k)}(\lambda^i t),$$

so that $\sigma^{(k)}(t) = \lim_{i\to\infty} \lambda^{ik} L^{-i} \sigma^{(k)}(\lambda^i t)$, but $\| \lambda^{ik} L^{-i} \| \leq (\mid \lambda \mid^k \| L^{-1} \|)^i$ tends to zero and therefore $\sigma^{(k)}(t) = 0$. This implies that $\sigma \in P^k \cap F^k = \{0\}$. □

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